This is a linear differential equation with a constant coefficient a and a constant term b. Thus, by formula (15.5′), we have

$$z(t) = \left[z(0) - \frac{s}{\lambda}\right]e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}$$

The substitution of $z = k^{1-\alpha}$ will then yield the final solution

$$k^{1-\alpha} = \left[k(0)^{1-\alpha} - \frac{s}{\lambda}\right]e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}$$

where $k(0)$ is the initial value of the capital–labor ratio k.

This solution is what determines the time path of k. Recalling that $(1-\alpha)$ and λ are both positive, we see that as $t \to \infty$ the exponential expression will approach zero; consequently,

$$k^{1-\alpha} \to \frac{s}{\lambda} \qquad \text{or} \qquad k \to \left(\frac{s}{\lambda}\right)^{1/(1-\alpha)} \qquad \text{as } t \to \infty$$

Therefore, the capital–labor ratio will approach a constant as its equilibrium value. This equilibrium or steady-state value, $(s/\lambda)^{1/(1-\alpha)}$, varies directly with the propensity to save s, and inversely with the rate of growth of labor λ.

EXERCISE 15.7

1. Divide (15.30) through by k, and interpret the resulting equation in terms of the growth rates of k, K, and L.
2. Show that, if capital is growing at the rate λ (that is, $K = Ae^{\lambda t}$), net investment I must also be growing at the rate λ.
3. The original input variables of the Solow model are K and L, but the fundamental equation (15.30) focuses on the capital–labor ratio k instead. What assumption(s) in the model is(are) responsible for (and make possible) this shift of focus? Explain.
4. Draw a phase diagram for each of the following, and discuss the qualitative aspects of the time path $y(t)$:

   (a) $\dot{y} = 3 - y - \ln y$ (b) $\dot{y} = e^y - (y + 2)$

Chapter 16

Higher-Order Differential Equations

In Chap. 15, we discussed the methods of solving a *first-order* differential equation, one in which there appears no derivative (or differential) of orders higher than 1. At times, however, the specification of a model may involve the second derivative or a derivative of an even higher order. We may, for instance, be given a function describing "the rate of change of the rate of change" of the income variable Y, say,

$$\frac{d^2Y}{dt^2} = kY$$

from which we are supposed to find the time path of Y. In this event, the given function constitutes a *second-order* differential equation, and the task of finding the time path $Y(t)$ is that of *solving* the second-order differential equation. The present chapter is concerned with the methods of solution and the economic applications of such higher-order differential equations, but we shall confine our discussion to the *linear* case only.

A simple variety of linear differential equations of order n is of the following form:

$$\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}\frac{dy}{dt} + a_ny = b \tag{16.1}$$

or, in an alternative notation,

$$y^{(n)}(t) + a_1y^{(n-1)}(t) + \cdots + a_{n-1}y'(t) + a_ny = b \tag{16.1'}$$

This equation is of *order n*, because the nth derivative (the first term on the left) is the highest derivative present. It is *linear*, since all the derivatives, as well as the dependent variable y, appear only in the first degree, and moreover, no product term occurs in which y and any of its derivatives are multiplied together. You will note, in addition, that this differential equation is characterized by *constant coefficients* (the a's) and a *constant term* (b). The constancy of the coefficients is an assumption we shall retain throughout this chapter. The constant term b, on the other hand, is adopted here as a first approach; later, in Sec. 16.5, we shall drop it in favor of a variable term.

16.1 Second-Order Linear Differential Equations with Constant Coefficients and Constant Term

For pedagogic reasons, let us first discuss the method of solution for the *second-order* case ($n = 2$). The relevant differential equation is then the simple one

$$y''(t) + a_1 y'(t) + a_2 y = b \qquad \textbf{(16.2)}$$

where a_1, a_2, and b are all constants. If the term b is identically zero, we have a *homogeneous* equation, but if b is a nonzero constant, the equation is *nonhomogeneous*. Our discussion will proceed on the assumption that (16.2) is nonhomogeneous; in solving the nonhomogeneous version of (16.2), the solution of the homogeneous version will emerge automatically as a by-product.

In this connection, we recall a proposition introduced in Sec. 15.1 which is equally applicable here: If y_c is the *complementary function*, i.e., the general solution (containing arbitrary constants) of the reduced equation of (16.2) and if y_p is the *particular integral*, i.e., any particular solution (containing no arbitrary constants) of the complete equation (16.2), then $y(t) = y_c + y_p$ will be the general solution of the complete equation. As was explained previously, the y_p component provides us with the equilibrium value of the variable y in the intertemporal sense of the term, whereas the y_c component reveals, for each point of time, the deviation of the time path $y(t)$ from the equilibrium.

The Particular Integral

For the case of constant coefficients and constant term, the particular integral is relatively easy to find. Since the particular integral can be *any* solution of (16.2), i.e., any value of y that satisfies this nonhomogeneous equation, we should always try the simplest possible type: namely, y = a constant. If y = a constant, it follows that

$$y'(t) = y''(t) = 0$$

so that (16.2) in effect becomes $a_2 y = b$, with the solution $y = b/a_2$. Thus, the desired particular integral is

$$y_p = \frac{b}{a_2} \qquad \text{(case of } a_2 \neq 0\text{)} \qquad \textbf{(16.3)}$$

Since the process of finding the value of y_p involves the condition $y'(t) = 0$, the rationale for considering that value as an intertemporal equilibrium becomes self-evident.

Example 1 Find the particular integral of the equation

$$y''(t) + y'(t) - 2y = -10$$

The relevant coefficients here are $a_2 = -2$ and $b = -10$. Therefore, the particular integral is $y_p = -10/(-2) = 5$.

What if $a_2 = 0$—so that the expression b/a_2 is not defined? In such a situation, since the constant solution for y_p fails to work, we must try some *nonconstant* form of solution. Taking the simplest possibility, we may try $y = kt$. Since $a_2 = 0$, the differential equation is now

$$y''(t) + a_1 y'(t) = b$$

but if $y = kt$, which implies $y'(t) = k$ and $y''(t) = 0$, this equation reduces to $a_1 k = b$. This determines the value of k as b/a_1, thereby giving us the particular integral

$$y_p = \frac{b}{a_1}t \qquad (\text{case of } a_2 = 0;\ a_1 \neq 0) \tag{16.3'}$$

Inasmuch as y_p is in this case a nonconstant function of time, we shall regard it as a moving equilibrium.

Example 2 Find the y_p of the equation $y''(t) + y'(t) = -10$. Here, we have $a_2 = 0$, $a_1 = 1$, and $b = -10$. Thus, by (16.3′), we can write

$$y_p = -10t$$

If it happens that a_1 is also zero, then the solution form of $y = kt$ will also break down, because the expression bt/a_1 will now be undefined. We ought, then, to try a solution of the form $y = kt^2$. With $a_1 = a_2 = 0$, the differential equation now reduces to the extremely simple form

$$y''(t) = b$$

and if $y = kt^2$, which implies $y'(t) = 2kt$ and $y''(t) = 2k$, the differential equation can be written as $2k = b$. Thus, we find $k = b/2$, and the particular integral is

$$y_p = \frac{b}{2}t^2 \qquad (\text{case of } a_1 = a_2 = 0) \tag{16.3''}$$

The equilibrium represented by this particular integral is again a moving equilibrium.

Example 3 Find the y_p of the equation $y''(t) = -10$. Since the coefficients are $a_1 = a_2 = 0$ and $b = -10$, formula (16.3″) is applicable. The desired answer is $y_p = -5t^2$.

The Complementary Function

The complementary function of (16.2) is defined to be the general solution of its reduced (homogeneous) equation

$$y''(t) + a_1 y'(t) + a_2 y = 0 \tag{16.4}$$

This is why we stated that the solution of a homogeneous equation will always be a *by-product* in the process of solving a complete equation.

Even though we have never tackled such an equation before, our experience with the complementary function of the first-order differential equations can supply us with a useful hint. From the solutions (15.3), (15.3′), (15.5), and (15.5′), it is clear that exponential expressions of the form Ae^{rt} figure very prominently in the complementary functions of first-order differential equations with constant coefficients. Then why not try a solution of the form $y = Ae^{rt}$ in the second-order equation, too?

If we adopt the trial solution $y = Ae^{rt}$, we must also accept

$$y'(t) = rAe^{rt} \qquad \text{and} \qquad y''(t) = r^2Ae^{rt}$$

as the derivatives of y. On the basis of these expressions for y, $y'(t)$, and $y''(t)$, the reduced differential equation (16.4) can be transformed into

$$Ae^{rt}(r^2 + a_1 r + a_2) = 0 \tag{16.4'}$$

As long as we choose those values of A and r that satisfy (16.4′), the trial solution $y = Ae^{rt}$ should work. Since e^{rt} can never be zero, we must either let $A = 0$ or see to it that r satisfies the equation

$$r^2 + a_1 r + a_2 = 0 \tag{16.4''}$$

Since the value of the (arbitrary) constant A is to be definitized by use of the initial conditions of the problem, however, we cannot simply set $A = 0$ at will. Therefore, it is essential to look for values of r that satisfy (16.4″).

Equation (16.4″) is known as the *characteristic equation* (or *auxiliary equation*) of the homogeneous equation (16.4), or of the complete equation (16.2). Because it is a quadratic equation in r, it yields two roots (solutions), referred to in the present context as *characteristic roots*, as follows:†

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \tag{16.5}$$

These two roots bear a simple but interesting relationship to each other, which can serve as a convenient means of checking our calculation: The *sum* of the two roots is always equal to $-a_1$, and their *product* is always equal to a_2. The proof of this statement is straightforward:

$$\begin{aligned} r_1 + r_2 &= \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} + \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} = \frac{-2a_1}{2} = -a_1 \\ r_1 r_2 &= \frac{(-a_1)^2 - \left(a_1^2 - 4a_2\right)}{4} = \frac{4a_2}{4} = a_2 \end{aligned} \tag{16.6}$$

The values of these two roots are the only values we may assign to r in the solution $y = Ae^{rt}$. But this means that, in effect, there are *two* solutions which will work, namely,

$$y_1 = A_1 e^{r_1 t} \qquad \text{and} \qquad y_2 = A_2 e^{r_2 t}$$

where A_1 and A_2 are two arbitrary constants, and r_1 and r_2 are the characteristic roots found from (16.5). Since we want only *one* general solution, however, there seems to be one too many. Two alternatives are now open to us: (1) pick either y_1 or y_2 at random, or (2) combine them in some fashion.

The first alternative, though simpler, is unacceptable. There is only one arbitrary constant in y_1 or y_2, but to qualify as a general solution of a *second-order* differential equation, the expression must contain *two* arbitrary constants. This requirement stems from the fact that, in proceeding from a function $y(t)$ to its second derivative $y''(t)$, we "lose" two constants during the two rounds of differentiation; therefore, to revert from a second-order differential equation to the primitive function $y(t)$, two constants should be reinstated. That leaves us only the alternative of combining y_1 and y_2, so as to include both constants

† Note that the quadratic equation (16.4″) is in the normalized form; the coefficient of the r^2 term is 1. In applying formula (16.5) to find the characteristic roots of a differential equation, we must first make sure that the characteristic equation is indeed in the normalized form.

A_1 and A_2. As it turns out, we can simply take their *sum*, $y_1 + y_2$, as the general solution of (16.4). Let us demonstrate that, if y_1 and y_2, respectively, satisfy (16.4), then the sum $(y_1 + y_2)$ will also do so. If y_1 and y_2 are indeed solutions of (16.4), then by substituting each of these into (16.4), we must find that the following two equations hold:

$$y_1''(t) + a_1 y_1'(t) + a_2 y_1 = 0$$
$$y_2''(t) + a_1 y_2'(t) + a_2 y_2 = 0$$

By adding these equations, however, we find that

$$\underbrace{[y_1''(t) + y_2''(t)]}_{=\frac{d^2}{dt^2}(y_1+y_2)} + a_1 \underbrace{[y_1'(t) + y_2'(t)]}_{=\frac{d}{dt}(y_1+y_2)} + a_2(y_1 + y_2) = 0$$

Thus, like y_1 or y_2, the sum $(y_1 + y_2)$ satisfies the equation (16.4) as well. Accordingly, the general solution of the homogeneous equation (16.4) or the complementary function of the complete equation (16.2) can, in general, be written as $y_c = y_1 + y_2$.

A more careful examination of the characteristic-root formula (16.5) indicates, however, that as far as the values of r_1 and r_2 are concerned, three possible cases can arise, some of which may necessitate a modification of our result $y_c = y_1 + y_2$.

Case 1 (distinct real roots) When $a_1^2 > 4a_2$, the square root in (16.5) is a real number, and the two roots r_1 and r_2 will take *distinct* real values, because the square root is added to $-a_1$ for r_1, but subtracted from $-a_1$ for r_2. In this case, we can indeed write

$$y_c = y_1 + y_2 = A_1 e^{r_1 t} + A_2 e^{r_2 t} \qquad (r_1 \neq r_2) \qquad \textbf{(16.7)}$$

Because the two roots are distinct, the two exponential expressions must be linearly independent (neither is a multiple of the other); consequently, A_1 and A_2 will always remain as separate entities and provide us with two constants, as required.

Example 4

Solve the differential equation

$$y''(t) + y'(t) - 2y = -10$$

The particular integral of this equation has already been found to be $y_p = 5$, in Example 1. Let us find the complementary function. Since the coefficients of the equation are $a_1 = 1$ and $a_2 = -2$, the characteristic roots are, by (16.5),

$$r_1, r_2 = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = 1, -2$$

(*Check:* $r_1 + r_2 = -1 = -a_1$; $r_1 r_2 = -2 = a_2$.) Since the roots are distinct real numbers, the complementary function is $y_c = A_1 e^t + A_2 e^{2t}$. Therefore, the general solution can be written as

$$y(t) = y_c + y_p = A_1 e^t + A_2 e^{-2t} + 5 \qquad \textbf{(16.8)}$$

In order to definitize the constants A_1 and A_2, there is need now for *two* initial conditions. Let these conditions be $y(0) = 12$ and $y'(0) = -2$. That is, when $t = 0$, $y(t)$ and $y'(t)$ are, respectively, 12 and -2. Setting $t = 0$ in (16.8), we find that

$$y(0) = A_1 + A_2 + 5$$

Differentiating (16.8) with respect to t and then setting $t = 0$ in the derivative, we find that

$$y'(t) = A_1 e^t - 2A_2 e^{-2t} \qquad \text{and} \qquad y'(0) = A_1 - 2A_2$$

To satisfy the two initial conditions, therefore, we must set $y(0) = 12$ and $y'(0) = -2$, which results in the following pair of simultaneous equations:

$$A_1 + A_2 = 7$$
$$A_1 - 2A_2 = -2$$

with solutions $A_1 = 4$ and $A_2 = 3$. Thus the definite solution of the differential equation is

$$y(t) = 4e^t + 3e^{-2t} + 5 \qquad \textbf{(16.8')}$$

As before, we can check the validity of this solution by differentiation. The first and second derivatives of (16.8') are

$$y'(t) = 4e^t - 6e^{-2t} \qquad \text{and} \qquad y''(t) = 4e^t + 12e^{-2t}$$

When these are substituted into the given differential equation along with (16.8'), the result is an identity $-10 = -10$. Thus the solution is correct. As you can easily verify, (16.8') also satisfies both of the initial conditions.

Case 2 (repeated real roots) When the coefficients in the differential equation are such that $a_1^2 = 4a_2$, the square root in (16.5) will vanish, and the two characteristic roots take an identical value:

$$r(= r_1 = r_2) = -\frac{a_1}{2}$$

Such roots are known as *repeated roots*, or *multiple* (here, *double*) *roots*.

If we attempt to write the complementary function as $y_c = y_1 + y_2$, the sum will in this case collapse into a single expression

$$y_c = A_1 e^{rt} + A_2 e^{rt} = (A_1 + A_2)e^{rt} = A_3 e^{rt}$$

leaving us with only one constant. This is not sufficient to lead us from a second-order differential equation back to its primitive function. The only way out is to find another eligible component term for the sum—a term which satisfies (16.4) and yet which is linearly independent of the term $A_3 e^{rt}$, so as to preclude such "collapsing."

An expression that will satisfy these requirements is $A_4 t e^{rt}$. Since the variable t has entered into it multiplicatively, this component term is obviously linearly independent of the $A_3 e^{rt}$ term; thus it will enable us to introduce another constant, A_4. But does $A_4 t e^{rt}$ qualify as a solution of (16.4)? If we try $y = A_4 t e^{rt}$, then, by the product rule, we can find its first and second derivatives to be

$$y'(t) = (rt + 1)A_4 e^{rt} \qquad \text{and} \qquad y''(t) = (r^2 t + 2r)A_4 e^{rt}$$

Substituting these expressions of y, y', and y'' into the left side of (16.4), we get the expression

$$[(r^2 t + 2r) + a_1(rt + 1) + a_2 t]A_4 e^{rt}$$

Inasmuch as, in the present context, we have $a_1^2 = 4a_2$ and $r = -a_1/2$, this last expression vanishes identically and thus is always equal to the right side of (16.4); this shows that A_4te^{rt} does indeed qualify as a solution.

Hence, the complementary function of the double-root case can be written as

$$y_c = A_3e^{rt} + A_4te^{rt} \tag{16.9}$$

Example 5

Solve the differential equation

$$y''(t) + 6y'(t) + 9y = 27$$

Here, the coefficients are $a_1 = 6$ and $a_2 = 9$; since $a_1^2 = 4a_2$, the roots will be repeated. According to formula (16.5), we have $r = -a_1/2 = -3$. Thus, in line with the result in (16.9), the complementary function may be written as

$$y_c = A_3e^{-3t} + A_4te^{-3t}$$

The general solution of the given differential equation is now also readily obtainable. Trying a constant solution for the particular integral, we get $y_p = 3$. It follows that the general solution of the complete equation is

$$y(t) = y_c + y_p = A_3e^{-3t} + A_4te^{-3t} + 3$$

The two arbitrary constants can again be definitized with two initial conditions. Suppose that the initial conditions are $y(0) = 5$ and $y'(0) = -5$. By setting $t = 0$ in the preceding general solution, we should find $y(0) = 5$; that is,

$$y(0) = A_3 + 3 = 5$$

This yields $A_3 = 2$. Next, by differentiating the general solution and then setting $t = 0$ and also $A_3 = 2$, we must have $y'(0) = -5$. That is,

$$y'(t) = -3A_3e^{-3t} - 3A_4te^{-3t} + A_4e^{-3t}$$

and $$y'(0) = -6 + A_4 = -5$$

This yields $A_4 = 1$. Thus we can finally write the definite solution of the given equation as

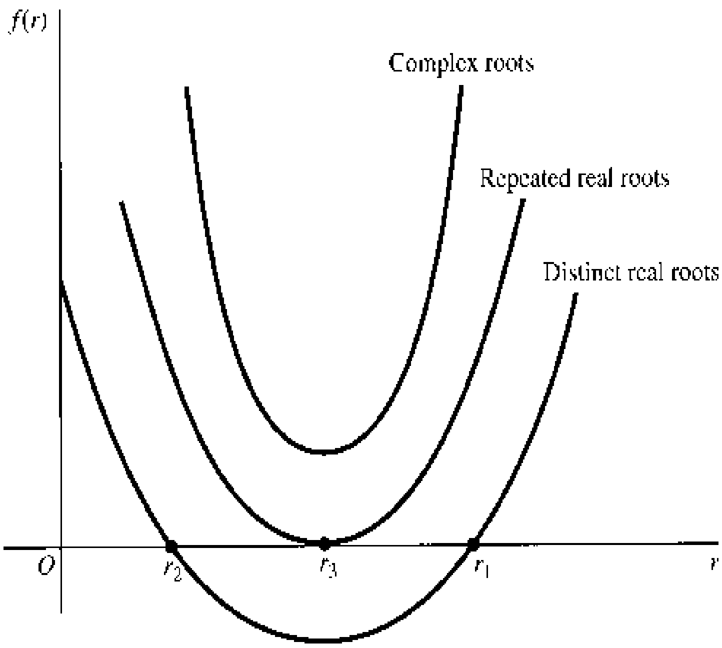
$$y(t) = 2e^{-3t} + te^{-3t} + 3$$

Case 3 (complex roots) There remains a third possibility regarding the relative magnitude of the coefficients a_1 and a_2, namely, $a_1^2 < 4a_2$. When this eventuality occurs, formula (16.5) will involve the square root of a *negative* number, which cannot be handled before we are properly introduced to the concepts of *imaginary* and *complex* numbers. For the time being, therefore, we shall be content with the mere cataloging of this case and shall leave the full discussion of it to Secs. 16.2 and 16.3.

The three cases cited can be illustrated by the three curves in Fig. 16.1, each of which represents a different version of the quadratic function $f(r) = r^2 + a_1r + a_2$. As we learned earlier, when such a function is set equal to zero, the result is a quadratic *equation* $f(r) = 0$, and to solve the latter equation is merely to "find the zeros of the quadratic *function*." Graphically, this means that the roots of the equation are to be found on the horizontal axis, where $f(r) = 0$.

The position of the lowest curve in Fig. 16.1, is such that the curve intersects the horizontal axis twice; thus we can find two distinct roots r_1 and r_2, both of which satisfy the

FIGURE 16.1



quadratic equation $f(r) = 0$ and both of which, of course, are real-valued. Thus the lowest curve illustrates Case 1. Turning to the middle curve, we note that it meets the horizontal axis only once, at r_3. This latter is the only value of r that can satisfy the equation $f(r) = 0$. Therefore, the middle curve illustrates Case 2. Last, we note that the top curve does not meet the horizontal axis at all, and there is thus no real-valued root to the equation $f(r) = 0$. While there exist no real roots in such a case, there are nevertheless two complex numbers that can satisfy the equation, as will be shown in Sec. 16.2.

The Dynamic Stability of Equilibrium

For Cases 1 and 2, the condition for dynamic stability of equilibrium again depends on the algebraic signs of the characteristic roots.

For Case 1, the complementary function (16.7) consists of the two exponential expressions $A_1e^{r_1t}$ and $A_2e^{r_2t}$. The coefficients A_1 and A_2 are arbitrary constants; their values hinge on the initial conditions of the problem. Thus we can be sure of a dynamically stable equilibrium ($y_c \to 0$ as $t \to \infty$), regardless of what the initial conditions happen to be, if and only if the roots r_1 and r_2 are *both* negative. We emphasize the word *both* here, because the condition for dynamic stability does *not* permit even *one* of the roots to be positive or zero. If $r_1 = 2$ and $r_2 = -5$, for instance, it might appear at first glance that the second root, being larger in absolute value, can outweigh the first. In actuality, however, it is the *positive* root that must eventually dominate, because as t increases, e^{2t} will grow increasingly larger, but e^{-5t} will steadily dwindle away.

For Case 2, with repeated roots, the complementary function (16.9) contains not only the familiar e^{rt} expression, but also a multiplicative expression te^{rt}. For the former term to approach zero whatever the initial conditions may be, it is necessary-and-sufficient to have $r < 0$. But would that also ensure the vanishing of te^{rt}? As it turns out, the expression te^{rt} (or, more generally, t^ke^{rt}) possesses the same general type of time path as does e^{rt} ($r \neq 0$). Thus the condition $r < 0$ is indeed necessary-and-sufficient for the entire complementary function to approach zero as $t \to \infty$, yielding a dynamically stable intertemporal equilibrium.

EXERCISE 16.1

1. Find the particular integral of each equation:
   (*a*) $y''(t) - 2y'(t) + 5y = 2$ (*d*) $y''(t) + 2y'(t) - y = -4$
   (*b*) $y''(t) + y'(t) = 7$ (*e*) $y''(t) = 12$
   (*c*) $y''(t) + 3y = 9$
2. Find the complementary function of each equation:
   (*a*) $y''(t) + 3y'(t) - 4y = 12$ (*c*) $y''(t) - 2y'(t) + y = 3$
   (*b*) $y''(t) + 6y'(t) + 5y = 10$ (*d*) $y''(t) + 8y'(t) + 16y = 0$
3. Find the general solution of each differential equation in Prob. 2, and then definitize the solution with the initial conditions $y(0) = 4$ and $y'(0) = 2$.
4. Are the intertemporal equilibriums found in Prob. 3 dynamically stable?
5. Verify that the definite solution in Example 5 indeed (*a*) satisfies the two initial conditions and (*b*) has first and second derivatives that conform to the given differential equation.
6. Show that, as $t \to \infty$, the limit of te^{rt} is zero if $r < 0$, but is infinite if $r \geq 0$.

16.2 Complex Numbers and Circular Functions

When the coefficients of a second-order linear differential equation, $y''(t) + a_1 y'(t) + a_2 y = b$, are such that $a_1^2 < 4a_2$, the characteristic-root formula (16.5) would call for taking the square root of a *negative* number. Since the square of any positive or negative real number is invariably positive, whereas the square of zero is zero, only a *nonnegative* real number can ever yield a real-valued square root. Thus, if we confine our attention to the real number system, as we have so far, no characteristic roots are available for this case (Case 3). This fact motivates us to consider numbers outside of the real-number system.

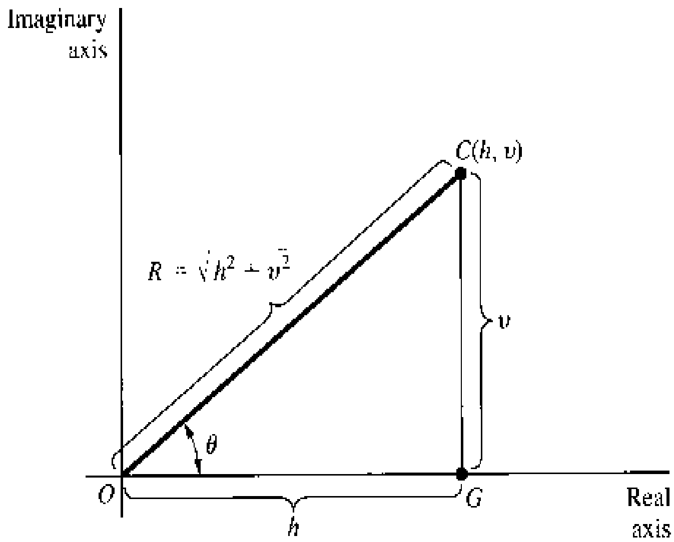
Imaginary and Complex Numbers

Conceptually, it is possible to define a number $i \equiv \sqrt{-1}$, which when squared will equal -1. Because i is the square root of a negative number, it is obviously not real-valued; it is therefore referred to as an *imaginary number*. With it at our disposal, we may write a host of other imaginary numbers, such as $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$ and $\sqrt{-2} = \sqrt{2}i$.

Extending its application a step further, we may construct yet another type of number—one that contains a *real* part as well as an *imaginary* part, such as $(8 - i)$ and $(3 + 5i)$. Known as *complex numbers*, these can be represented generally in the form $(h + vi)$, where h and v are two real numbers.[†] Of course, in case $v = 0$, the complex number will reduce to a real number, whereas if $h = 0$, it will become an imaginary number. Thus the *set of all real numbers* (call it **R**) constitutes a subset of the *set of all complex numbers* (call it **C**). Similarly, the *set of all imaginary numbers* (call it **I**) also constitutes a subset of **C**. That is, $\mathbf{R} \subset \mathbf{C}$, and $\mathbf{I} \subset \mathbf{C}$. Furthermore, since the terms *real* and *imaginary* are mutually exclusive, the sets **R** and **I** must be disjoint; that is $\mathbf{R} \cap \mathbf{I} = \varnothing$.

[†] We employ the symbols h (for horizontal) and v (for vertical) in the general complex-number notation, because we shall presently plot the values of h and v, respectively, on the horizontal and vertical axes of a two-dimensional diagram.

FIGURE 16.2



A complex number $(h + vi)$ can be represented graphically in what is called an *Argand diagram*, as illustrated in Fig. 16.2. By plotting h horizontally on the *real axis* and v vertically on the *imaginary axis*, the number $(h + vi)$ can be specified by the point (h, v), which we have alternatively labeled C. The values of h and v are algebraically signed, of course, so that if $h < 0$, the point C will be to the left of the point of origin; similarly, a negative v will mean a location below the horizontal axis.

Given the values of h and v, we can also calculate the length of the line OC by applying Pythagoras's theorem, which states that the square of the hypotenuse of a right-angled triangle is the sum of the squares of the other two sides. Denoting the length of OC by R (for radius vector), we have

$$R^2 = h^2 + v^2 \qquad \text{and} \qquad R = \sqrt{h^2 + v^2} \tag{16.10}$$

where the square root is always taken to be positive. The value of R is sometimes called the *absolute value*, or *modulus*, of the complex number $(h + vi)$. (Note that changing the signs of h and v will produce no effect on the absolute value of the complex number, R.) Like h and v, then, R is real-valued, but unlike these other values, R is always positive. We shall find the number R to be of great importance in the ensuing discussion.

Complex Roots

Meanwhile, let us return to formula (16.5) and examine the case of complex characteristic roots. When the coefficients of a second-order differential equation are such that $a_1^2 < 4a_2$, the square-root expression in (16.5) can be written as

$$\sqrt{a_1^2 - 4a_2} = \sqrt{4a_2 - a_1^2}\sqrt{-1} = \sqrt{4a_2 - a_1^2}\,i$$

Hence, if we adopt the shorthand

$$h = \frac{-a_1}{2} \qquad \text{and} \qquad v = \frac{\sqrt{4a_2 - a_1^2}}{2}$$

the two roots can be denoted by a pair of *conjugate complex numbers:*

$$r_1, r_2 = h \pm vi$$

These two complex roots are said to be "conjugate" because they always appear together, one being the *sum* of h and vi, and the other being the *difference* between h and vi. Note that they share the same absolute value R.

Example 1 Find the roots of the characteristic equation $r^2 + r + 4 = 0$. Applying the familiar formula, we have

$$r_1, r_2 = \frac{-1 \pm \sqrt{-15}}{2} = \frac{-1 \pm \sqrt{15}\sqrt{-1}}{2} = \frac{-1}{2} \pm \frac{\sqrt{15}}{2}i$$

which constitute a pair of conjugate complex numbers.

As before, we can use (16.6) to check our calculations. If correct, we should have $r_1 + r_2 = -a_1 (= -1)$ and $r_1 r_2 = a_2 (= 4)$. Since we do find

$$r_1 + r_2 = \left(\frac{-1}{2} + \frac{\sqrt{15}i}{2}\right) + \left(\frac{-1}{2} - \frac{\sqrt{15}i}{2}\right)$$
$$= \frac{-1}{2} + \frac{-1}{2} = -1$$

and

$$r_1 r_2 = \left(\frac{-1}{2} + \frac{\sqrt{15}i}{2}\right)\left(\frac{-1}{2} - \frac{\sqrt{15}i}{2}\right)$$
$$= \left(\frac{-1}{2}\right)^2 - \left(\frac{\sqrt{15}i}{2}\right)^2 = \frac{1}{4} - \frac{-15}{4} = 4$$

our calculation is indeed validated.

Even in the complex-root case (Case 3), we may express the complementary function of a differential equation according to (16.7); that is,

$$y_c = A_1 e^{(h+vi)t} + A_2 e^{(h-vi)t} = e^{ht}(A_1 e^{vit} + A_2 e^{-vit}) \qquad \textbf{(16.11)}$$

But a new feature has been introduced: the number i now appears in the exponents of the two expressions in parentheses. How do we interpret such imaginary exponential functions?

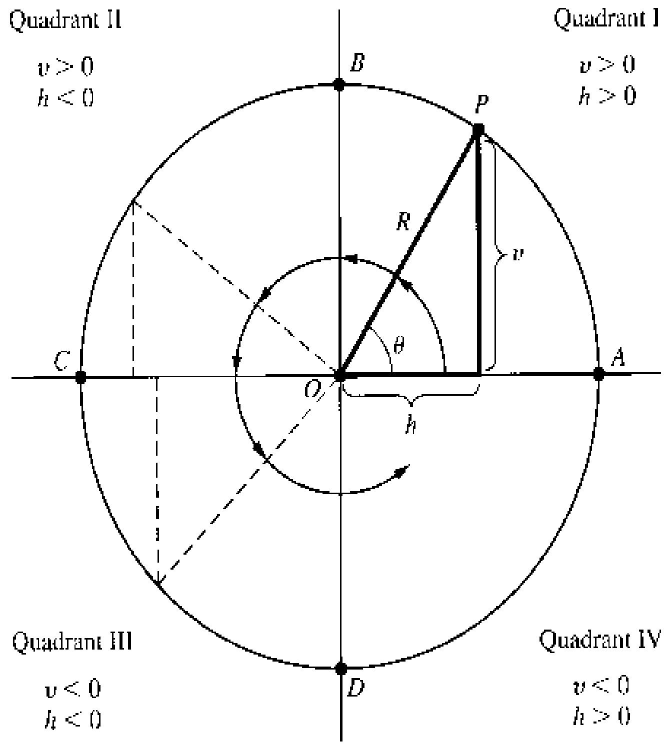
To facilitate their interpretation, it will prove helpful first to transform these expressions into equivalent *circular-function* forms. As we shall presently see, the latter functions characteristically involve periodic fluctuations of a variable. Consequently, the complementary function (16.11), being translatable into circular-function forms, can also be expected to generate a cyclical type of time path.

Circular Functions

Consider a circle with its center at the point of origin and with a radius of length R, as shown in Fig. 16.3. Let the radius, like the hand of a clock, rotate in the counterclockwise direction. Starting from the position OA, it will gradually move into the position OP, followed successively by such positions as OB, OC, and OD; and at the end of a cycle, it will return to OA. Thereafter, the cycle will simply repeat itself.

When in a specific position—say, OP—the clock hand will make a definite angle θ with line OA, and the tip of the hand (P) will determine a vertical distance v and a horizontal distance h. As the angle θ changes during the process of rotation, v and h will vary, although

FIGURE 16.3



R will not. Thus the ratios v/R and h/R must change with θ; that is, these two ratios are both functions of the angle θ. Specifically, v/R and h/R are called, respectively, the *sine* (function) of θ and the *cosine* (function) of θ:

$$\sin\theta \equiv \frac{v}{R} \tag{16.12}$$

$$\cos\theta \equiv \frac{h}{R} \tag{16.13}$$

In view of their connection with a circle, these functions are referred to as *circular functions*. Since they are also associated with a triangle, however, they are alternatively called *trigonometric functions*. Another (and fancier) name for them is *sinusoidal functions*. The sine and cosine functions are not the only circular functions; another frequently encountered one is the *tangent* function, defined as

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{v}{h} \qquad (h \neq 0)$$

Our major concern here, however, will be with the sine and cosine functions.

The independent variable in a circular function is the angle θ, so the mapping involved here is from an *angle* to a *ratio of two distances*. Usually, angles are measured in *degrees* (for example, 30, 45, and 90°); in analytical work, however, it is more convenient to measure angles in *radians* instead. The advantage of the radian measure stems from the fact that, when θ is so measured, the derivatives of circular functions will come out in neater expressions—much as the base e gives us neater derivatives for exponential and logarithmic functions. But just how much is a radian? To explain this, let us return to Fig. 16.3, where we have drawn the point P so that the length of the *arc AP* is exactly equal to the radius R. A *radian* (abbreviated as *rad*) can then be defined as the size of the angle θ

(in Fig. 16.3) formed by such an R-length arc. Since the circumference of the circle has a total length of $2\pi R$ (where $\pi = 3.14159\ldots$), a complete circle must involve an angle of 2π rad altogether. In terms of degrees, however, a complete circle makes an angle of 360°; thus, by equating 360° to 2π rad, we can arrive at the following conversion table:

Degrees	360	270	180	90	45	0
Radians	2π	$\frac{3\pi}{2}$	π	$\frac{\pi}{2}$	$\frac{\pi}{4}$	0

Properties of the Sine and Cosine Functions

Given the length of R, the value of $\sin\theta$ hinges upon the way the value of v changes in response to changes in the angle θ. In the starting position OA, we have $v = 0$. As the clock hand moves counterclockwise, v starts to assume an increasing positive value, culminating in the maximum value of $v = R$ when the hand coincides with OB, that is, when $\theta = \pi/2$ rad ($= 90°$). Further movement will gradually shorten v, until its value becomes zero when the hand is in the position OC, i.e., when $\theta = \pi$ rad ($= 180°$). As the hand enters the third quadrant, v begins to assume negative values; in the position OD, we have $v = -R$. In the fourth quadrant, v is still negative, but it will increase from the value of $-R$ toward the value of $v = 0$, which is attained when the hand returns to OA—that is, when $\theta = 2\pi$ rad ($= 360°$). The cycle then repeats itself.

When these illustrative values of v are substituted into (16.12), we can obtain the results shown in the "$\sin\theta$" row of Table 16.1. For a more complete description of the sine function, however, see the graph in Fig. 16.4*a*, where the values of $\sin\theta$ are plotted against those of θ (expressed in radians).

The value of $\cos\theta$, in contrast, depends instead upon the way that h changes in response to changes in θ. In the starting position OA, we have $h = R$. Then h gradually shrinks, till $h = 0$ when $\theta = \pi/2$ (position OB). In the second quadrant, h turns negative, and when $\theta = \pi$ (position OC), $h = -R$. The value of h gradually increases from $-R$ to zero in the third quadrant, and when $\theta = 3\pi/2$ (position OD), we find that $h = 0$. In the fourth quadrant, h turns positive again, and when the hand returns to position OA ($\theta = 2\pi$), we again have $h = R$. The cycle then repeats itself.

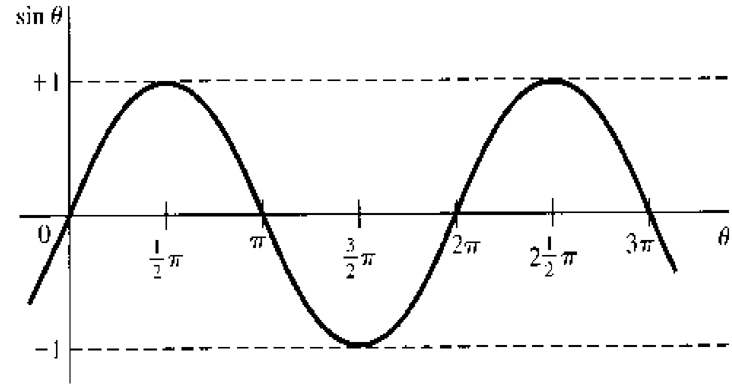
The substitution of these illustrative values of h into (16.13) yields the results in the bottom row of Table 16.1, but Fig. 16.4*b* gives a more complete depiction of the cosine function.

The $\sin\theta$ and $\cos\theta$ functions share the same domain, namely, the set of all real numbers (radian measures of θ). In this connection, it may be pointed out that a *negative* angle simply refers to the reverse rotation of the clock hand; for instance, a clockwise movement

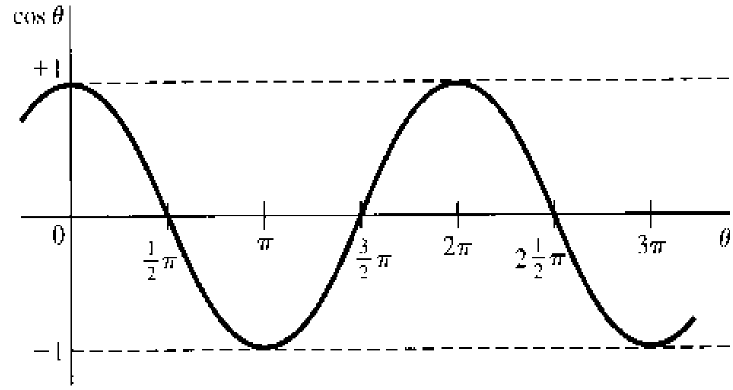
TABLE 16.1

θ	0	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$	2π
$\sin\theta$	0	1	0	-1	0
$\cos\theta$	1	0	-1	0	1

FIGURE 16.4



(a)



(b)

from OA to OD in Fig. 16.3 generates an angle of $-\pi/2$ rad $(= -90°)$. There is also a common range for the two functions, namely, the closed interval $[-1, 1]$. For this reason, the graphs of $\sin\theta$ and $\cos\theta$ are, in Fig. 16.4, confined to a definite horizontal band.

A major distinguishing property of the sine and cosine functions is that both are *periodic:* their values will repeat themselves for every 2π rad (a complete circle) the angle θ travels through. Each function is therefore said to have a *period* of 2π. In view of this periodicity feature, the following equations hold (for any integer n):

$$\sin(\theta + 2n\pi) = \sin\theta \qquad \cos(\theta + 2n\pi) = \cos\theta$$

That is, adding (or subtracting) any integer multiple of 2π to any angle θ will affect neither the value of $\sin\theta$ nor that of $\cos\theta$.

The graphs of the sine and cosine functions indicate a constant range of fluctuation in each period, namely, ± 1. This is sometimes alternatively described by saying that the *amplitude* of fluctuation is 1. By virtue of the identical period and the identical amplitude, we see that the $\cos\theta$ curve, if shifted rightward by $\pi/2$, will be exactly coincident with the $\sin\theta$ curve. These two curves are therefore said to differ only in *phase,* i.e., to differ only in the location of the peak in each period. Symbolically, this fact may be stated by the equation

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

The sine and cosine functions obey certain identities. Among these, the more frequently used are

$$\begin{aligned} \sin(-\theta) &\equiv -\sin\theta \\ \cos(-\theta) &\equiv \cos\theta \end{aligned} \tag{16.14}$$

$$\sin^2\theta + \cos^2\theta \equiv 1 \qquad [\text{where } \sin^2\theta \equiv (\sin\theta)^2, \text{etc.}] \tag{16.15}$$

$$\begin{aligned} \sin(\theta_1 \pm \theta_2) &\equiv \sin\theta_1\cos\theta_2 \pm \cos\theta_1\sin\theta_2 \\ \cos(\theta_1 \pm \theta_2) &\equiv \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2 \end{aligned} \tag{16.16}$$

The pair of identities (16.14) serves to underscore the fact that the cosine function is symmetrical with respect to the vertical axis (that is, θ and $-\theta$ always yield the same cosine value), while the sine function is not. Shown in (16.15) is the fact that, for any magnitude of θ, the sum of the squares of its sine and cosine is always unity. And the set of identities in (16.16) gives the sine and cosine of the sum and difference of two angles θ_1 and θ_2.

Finally, a word about derivatives. Being continuous and smooth, both $\sin\theta$ and $\cos\theta$ are differentiable. The derivatives, $d(\sin\theta)/d\theta$ and $d(\cos\theta)/d\theta$, are obtainable by taking the limits, respectively, of the difference quotients $\Delta(\sin\theta)/\Delta\theta$ and $\Delta(\cos\theta)/\Delta\theta$ as $\Delta\theta \to 0$. The results, stated here without proof, are

$$\frac{d}{d\theta}\sin\theta = \cos\theta \tag{16.17}$$

$$\frac{d}{d\theta}\cos\theta = -\sin\theta \tag{16.18}$$

It should be emphasized, however, that these derivative formulas are valid only when θ is measured in radians; if measured in degrees, for instance, (16.17) will become $d(\sin\theta)/d\theta = (\pi/180)\cos\theta$ instead. It is for the sake of getting rid of the factor $(\pi/180)$ that radian measures are preferred to degree measures in analytical work.

Example 2 Find the slope of the $\sin\theta$ curve at $\theta = \pi/2$. The slope of the sine curve is given by its derivative ($= \cos\theta$). Thus, at $\theta = \pi/2$, the slope should be $\cos(\pi/2) = 0$. You may refer to Fig. 16.4 for verification of this result.

Example 3 Find the second derivative of $\sin\theta$. From (16.17), we know that the first derivative of $\sin\theta$ is $\cos\theta$, therefore the desired second derivative is

$$\frac{d^2}{d\theta^2}\sin\theta = \frac{d}{d\theta}\cos\theta = -\sin\theta$$

Euler Relations

In Sec. 9.5, it was shown that any function which has finite, continuous derivatives up to the desired order can be expanded into a polynomial function. Moreover, if the remainder term R_n in the resulting Taylor series (expansion at any point x_0) or Maclaurin series (expansion at $x_0 = 0$) happens to approach zero as the number of terms n becomes infinite, the polynomial may be written as an infinite series. We shall now expand the sine and cosine functions and then attempt to show how the imaginary exponential expressions encountered in (16.11) can be transformed into circular functions having equivalent expansions.

For the sine function, write $\phi(\theta) = \sin\theta$; it then follows that $\phi(0) = \sin 0 = 0$. By successive derivation, we can get

$$\left.\begin{aligned} \phi'(\theta) &= \cos\theta \\ \phi''(\theta) &= -\sin\theta \\ \phi'''(\theta) &= -\cos\theta \\ \phi^{(4)}(\theta) &= \sin\theta \\ \phi^{(5)}(\theta) &= \cos\theta \\ \vdots \quad & \quad \vdots \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \phi'(0) &= \cos 0 = 1 \\ \phi''(0) &= -\sin 0 = 0 \\ \phi'''(0) &= -\cos 0 = -1 \\ \phi^{(4)}(0) &= \sin 0 = 0 \\ \phi^{(5)}(0) &= \cos 0 = 1 \\ \vdots \quad & \quad \vdots \end{aligned}\right.$$

When substituted into (9.14), where θ now replaces x, these will give us the following Maclaurin series with remainder:

$$\sin\theta = 0 + \theta + 0 - \frac{\theta^3}{3!} + 0 + \frac{\theta^5}{5!} + \cdots + \frac{\phi^{(n+1)}(p)}{(n+1)!}\theta^{n+1}$$

Now, the expression $\phi^{(n+1)}(p)$ in the last (remainder) term, which represents the $(n+1)$st derivative evaluated at $\theta = p$, can only take the form of $\pm\cos p$ or $\pm\sin p$ and, as such, can only take a value in the interval $[-1, 1]$, regardless of how large n is. On the other hand, $(n+1)!$ will grow rapidly as $n \to \infty$—in fact, much more rapidly than θ^{n+1} as n increases. Hence, the remainder term will approach zero as $n \to \infty$, and we can therefore express the Maclaurin series as an infinite series:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \tag{16.19}$$

Similarly, if we write $\psi(\theta) = \cos\theta$, then $\psi(0) = \cos 0 = 1$, and the successive derivatives will be

$$\left.\begin{aligned} \psi'(\theta) &= -\sin\theta \\ \psi''(\theta) &= -\cos\theta \\ \psi'''(\theta) &= \sin\theta \\ \psi^{(4)}(\theta) &= \cos\theta \\ \psi^{(5)}(\theta) &= -\sin\theta \\ \vdots \quad & \quad \vdots \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \psi'(0) &= -\sin 0 = 0 \\ \psi''(0) &= -\cos 0 = -1 \\ \psi'''(0) &= \sin 0 = 0 \\ \psi^{(4)}(0) &= \cos 0 = 1 \\ \psi^{(5)}(0) &= -\sin 0 = 0 \\ \vdots \quad & \quad \vdots \end{aligned}\right.$$

On the basis of these derivatives, we can expand $\cos\theta$ as follows:

$$\cos\theta = 1 + 0 - \frac{\theta^2}{2!} + 0 + \frac{\theta^4}{4!} + \cdots + \frac{\psi^{(n+1)}(p)}{(n+1)!}\theta^{n+1}$$

Since the remainder term will again tend toward zero as $n \to \infty$, the cosine function is also expressible as an infinite series, as follows:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \tag{16.20}$$

You must have noticed that, with (16.19) and (16.20) at hand, we are now capable of constructing a table of sine and cosine values for all possible values of θ (in radians). However, our immediate interest lies in finding the relationship between imaginary exponential expressions and circular functions. To this end, let us now expand the two exponential expressions $e^{i\theta}$ and $e^{-i\theta}$. The reader will recognize that these are but special cases of the expression e^x, which has previously been shown, in (10.6), to have the expansion

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

Letting $x = i\theta$, therefore, we can immediately obtain

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \end{aligned}$$

Similarly, by setting $x = -i\theta$, the following result will emerge:

$$\begin{aligned} e^{-i\theta} &= 1 - i\theta + \frac{(-i\theta)^2}{2!} + \frac{(-i\theta)^3}{3!} + \frac{(-i\theta)^4}{4!} + \frac{(-i\theta)^5}{5!} + \cdots \\ &= 1 - i\theta - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} - \frac{i\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) - i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \end{aligned}$$

By substituting (16.19) and (16.20) into these two results, the following pair of identities—known as the *Euler relations*—can readily be established:

$$e^{i\theta} \equiv \cos\theta + i\sin\theta \qquad \textbf{(16.21)}$$

$$e^{-i\theta} \equiv \cos\theta - i\sin\theta \qquad \textbf{(16.21')}$$

These will enable us to translate any imaginary exponential function into an equivalent linear combination of sine and cosine functions, and vice versa.

Example 4 Find the value of $e^{i\pi}$. First let us convert this expression into a trigonometric expression. By setting $\theta = \pi$ in (16.21), it is found that $e^{i\pi} = \cos\pi + i\sin\pi$. Since $\cos\pi = -1$ and $\sin\pi = 0$, it follows that $e^{i\pi} = -1$.

Example 5 Show that $e^{-i\pi/2} = -i$. Setting $\theta = \pi/2$ in (16.21'), we have

$$e^{-i\pi/2} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = 0 - i(1) = -i$$

Alternative Representations of Complex Numbers

So far, we have represented a pair of conjugate complex numbers in the general form $(h \pm vi)$. Since h and v refer to the abscissa and ordinate in the Cartesian coordinate system of an Argand diagram, the expression $(h \pm vi)$ represents the *Cartesian form* of a pair of conjugate complex numbers. As a by-product of the discussion of circular functions and Euler relations, we can now express $(h \pm vi)$ in two other ways.

Referring to Fig. 16.2, we see that as soon as h and v are specified, the angle θ and the value of R also become determinate. Since a given θ and a given R can together identify a *unique point in the Argand diagram*, we may employ θ and R to specify the particular pair of complex numbers. By rewriting the definitions of the sine and cosine functions in (16.12) and (16.13) as

$$v = R\sin\theta \quad \text{and} \quad h = R\cos\theta \tag{16.22}$$

the conjugate complex numbers $(h \pm vi)$ can be transformed as follows:

$$h \pm vi = R\cos\theta \pm Ri\sin\theta = R(\cos\theta \pm i\sin\theta)$$

In so doing, we have in effect switched from the Cartesian coordinates of the complex numbers (h and v) to what are called their *polar coordinates* (R and θ). The right-hand expression in the preceding equation, accordingly, exemplifies the *polar form* of a pair of conjugate complex numbers.

Furthermore, in view of the Euler relations, the polar form may also be rewritten into the *exponential form* as follows: $R(\cos\theta \pm i\sin\theta) = Re^{\pm i\theta}$. Hence, we have a total of three alternative representations of the conjugate complex numbers:

$$h \pm vi = R(\cos\theta \pm i\sin\theta) = Re^{\pm i\theta} \tag{16.23}$$

If we are given the values of R and θ, the transformation to h and v is straightforward: we use the two equations in (16.22). What about the reverse transformation? With given values of h and v, no difficulty arises in finding the corresponding value of R, which is equal to $\sqrt{h^2 + v^2}$. But a slight ambiguity arises in regard to θ: the desired value of θ (in radians) is that which satisfies the two conditions $\cos\theta = h/R$ and $\sin\theta = v/R$; but for given values of h and v, θ is not unique! (Why?) Fortunately, the problem is not serious, for by confining our attention to the interval $[0, 2\pi)$ in the domain, the indeterminancy is quickly resolved.

Example 6

Find the Cartesian form of the complex number $5e^{3i\pi/2}$. Here we have $R = 5$ and $\theta = 3\pi/2$; hence, by (16.22) and Table 16.1,

$$h = 5\cos\frac{3\pi}{2} = 0 \qquad \text{and} \qquad v = 5\sin\frac{3\pi}{2} = -5$$

The Cartesian form is thus simply $h - vi = -5i$.

Example 7

Find the polar and exponential forms of $(1 + \sqrt{3}i)$. In this case, we have $h = 1$ and $v = \sqrt{3}$; thus $R = \sqrt{1+3} = 2$. Table 16.1 is of no use in locating the value of θ this time, but Table 16.2, which lists some additional selected values of $\sin\theta$ and $\cos\theta$, will help. Specifically,

TABLE 16.2

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{4}$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}\left(=\frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}\left(=\frac{\sqrt{2}}{2}\right)$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}\left(=\frac{\sqrt{2}}{2}\right)$	$\frac{1}{2}$	$\frac{-1}{\sqrt{2}}\left(=\frac{-\sqrt{2}}{2}\right)$

we are seeking the value of θ such that $\cos\theta = h/R = 1/2$ and $\sin\theta = v/R = \sqrt{3}/2$. The value $\theta = \pi/3$ meets the requirements. Thus, according to (16.23), the desired transformation is

$$1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\pi/3}$$

Before leaving this topic, let us note an important extension of the result in (16.23). Supposing that we have the nth power of a complex number—say, $(h + vi)^n$—how do we write its polar and exponential forms? The exponential form is the easier to derive. Since $h + vi = Re^{i\theta}$, it follows that

$$(h + vi)^n = (Re^{i\theta})^n = R^n e^{in\theta}$$

Similarly, we can write

$$(h - vi)^n = (Re^{-i\theta})^n = R^n e^{-in\theta}$$

Note that the power n has brought about two changes: (1) R now becomes R^n, and (2) θ now becomes $n\theta$. When these two changes are inserted into the polar form in (16.23), we find that

$$(h \pm vi)^n = R^n(\cos n\theta \pm i\sin n\theta) \qquad \textbf{(16.23')}$$

That is,

$$[R(\cos\theta \pm i\sin\theta)]^n = R^n(\cos n\theta \pm i\sin n\theta)$$

Known as *De Moivre's theorem*, this result indicates that, to raise a complex number to the nth power, one must simply modify its polar coordinates by raising R to the nth power and multiplying θ by n.

EXERCISE 16.2

1. Find the roots of the following quadratic equations:
   (*a*) $r^2 - 3r + 9 = 0$ (*c*) $2x^2 + x + 8 = 0$
   (*b*) $r^2 + 2r + 17 = 0$ (*d*) $2x^2 - x + 1 = 0$
2. (*a*) How many degrees are there in a radian?
   (*b*) How many radians are there in a degree?
3. With reference to Fig. 16.3, and by using Pythagoras's theorem, prove that
   (*a*) $\sin^2\theta + \cos^2\theta \equiv 1$ (*b*) $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$
4. By means of the identities (16.14), (16.15), and (16.16), show that:
   (*a*) $\sin 2\theta \equiv 2\sin\theta\cos\theta$
   (*b*) $\cos 2\theta \equiv 1 - 2\sin^2\theta$
   (*c*) $\sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2) \equiv 2\sin\theta_1\cos\theta_2$
   (*d*) $1 + \tan^2\theta \equiv \frac{1}{\cos^2\theta}$
   (*e*) $\sin\left(\frac{\pi}{2} - \theta\right) \equiv \cos\theta$ (*f*) $\cos\left(\frac{\pi}{2} - \theta\right) \equiv \sin\theta$
5. By applying the chain rule:
   (*a*) Write out the derivative formulas for $\frac{d}{d\theta}\sin f(\theta)$ and $\frac{d}{d\theta}\cos f(\theta)$, where $f(\theta)$ is a function of θ.
   (*b*) Find the derivatives of $\cos\theta^3$, $\sin(\theta^2 + 3\theta)$, $\cos e^\theta$, and $\sin(1/\theta)$.

6. From the Euler relations, deduce that:

   (*a*) $e^{-i\pi} = -1$ (*c*) $e^{i\pi/4} = \frac{\sqrt{2}}{2}(1+i)$

   (*b*) $e^{i\pi/3} = \frac{1}{2}(1+\sqrt{3}i)$ (*d*) $e^{-3i\pi/4} = -\frac{\sqrt{2}}{2}(1+i)$

7. Find the Cartesian form of each complex number:

   (*a*) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ (*b*) $4e^{i\pi/3}$ (*c*) $\sqrt{2}e^{-i\pi/4}$

8. Find the polar and exponential forms of the following complex numbers:

   (*a*) $\frac{3}{2} + \frac{3\sqrt{3}}{2}i$ (*b*) $4(\sqrt{3}+i)$

16.3 Analysis of the Complex-Root Case

With the concepts of complex numbers and circular functions at our disposal, we are now prepared to approach the complex-root case (Case 3), referred to in Sec. 16.1. You will recall that the classification of the three cases, according to the nature of the characteristic roots, is concerned only with the complementary function of a differential equation. Thus, we can continue to focus our attention on the reduced equation

$$y''(t) + a_1y'(t) + a_2y = 0 \qquad \text{[reproduced from (16.4)]}$$

The Complementary Function

When the values of the coefficients a_1 and a_2 are such that $a_1^2 < 4a_2$, the characteristic roots will be the pair of conjugate complex numbers

$$r_1, r_2 = h \pm vi$$

where
$$h = -\frac{1}{2}a_1 \qquad \text{and} \qquad v = \frac{1}{2}\sqrt{4a_2 - a_1^2}$$

The complementary function, as was already previewed, will thus be in the form

$$y_c = e^{ht}(A_1e^{vit} + A_2e^{-vit}) \qquad \text{[reproduced from (16.11)]}$$

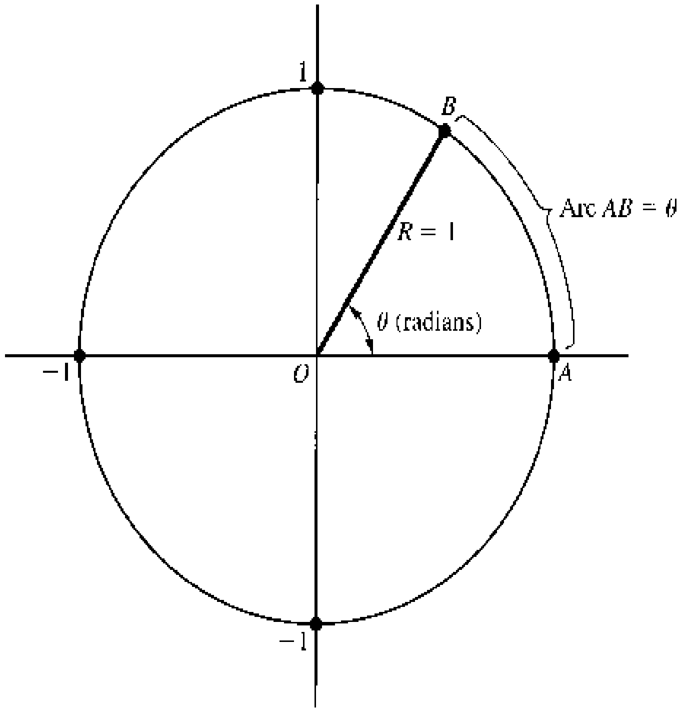
Let us first transform the imaginary exponential expressions in the parentheses into equivalent trigonometric expressions, so that we may interpret the complementary function as a circular function. This may be accomplished by using the Euler relations. Letting $\theta = vt$ in (16.21) and (16.21′), we find that

$$e^{vit} = \cos vt + i\sin vt \quad \text{and} \quad e^{-vit} = \cos vt - i\sin vt$$

From these, it follows that the complementary function in (16.11) can be rewritten as

$$\begin{aligned} y_c &= e^{ht}[A_1(\cos vt + i\sin vt) + A_2(\cos vt - i\sin vt)] \\ &= e^{ht}[(A_1 + A_2)\cos vt + (A_1 - A_2)i\sin vt] \end{aligned} \tag{16.24}$$

FIGURE 16.5



Furthermore, if we employ the shorthand symbols

$$A_5 \equiv A_1 + A_2 \quad \text{and} \quad A_6 \equiv (A_1 - A_2)i$$

it is possible to simplify (16.24) into†

$$y_c = e^{ht}(A_5 \cos vt + A_6 \sin vt) \tag{16.24'}$$

where the new arbitrary constants A_5 and A_6 are later to be definitized.

If you are meticulous, you may feel somewhat uneasy about the substitution of θ by vt in the foregoing procedure. The variable θ measures an angle, but vt is a magnitude in units of t (in our context, time). Therefore, how can we make the substitution $\theta = vt$? The answer to this question can best be explained with reference to the *unit circle* (a circle with radius $R = 1$) in Fig. 16.5. True, we have been using θ to designate an angle; but since the angle is measured in radian units, the value of θ is always the ratio of the length of arc AB to the radius R. When $R = 1$, we have specifically

$$\theta \equiv \frac{\text{arc}\, AB}{R} \equiv \frac{\text{arc}\, AB}{1} \equiv \text{arc}\, AB$$

In other words, θ is not only the radian measure of the angle, but also the length of the arc AB, which is a number rather than an angle. If the passing of time is charted on the circumference of the unit circle (counterclockwise), rather than on a straight line as we do in plotting a time series, it really makes no difference whatsoever whether we consider the

† The fact that in defining A_6, we include in it the imaginary number i is by no means an attempt to "sweep the dirt under the rug." Because A_6 is an arbitrary constant, it can take an imaginary as well as a real value. Nor is it true that, as defined, A_6 will necessarily turn out to be imaginary. Actually, if A_1 and A_2 are a pair of conjugate complex numbers, say, $m \pm ni$, then A_5 and A_6 will both be real: $A_5 = A_1 + A_2 = (m + ni) + (m - ni) = 2m$, and $A_6 = (A_1 - A_2)i = [(m + ni) - (m - ni)]i = (2ni)i = -2n$.

lapse of time as an increase in the radian measure of the angle θ or as a lengthening of the arc AB. Even if $R \neq 1$, moreover, the same line of reasoning can apply, except that in that case θ will be equal to (arc $AB)/R$ instead; i.e., the angle θ and the arc AB will bear a fixed proportion to each other, instead of being equal. Thus, the substitution $\theta = vt$ is indeed legitimate.

An Example of Solution

Let us find the solution of the differential equation

$$y''(t) + 2y'(t) + 17y = 34$$

with the initial conditions $y(0) = 3$ and $y'(0) = 11$.

Since $a_1 = 2$, $a_2 = 17$, and $b = 34$, we can immediately find the particular integral to be

$$y_p = \frac{b}{a_2} = \frac{34}{17} = 2 \qquad \text{[by (16.3)]}$$

Moreover, since $a_1^2 = 4 < 4a_2 = 68$, the characteristic roots will be the pair of conjugate complex numbers $(h \pm vi)$, where

$$h = -\frac{1}{2}a_1 = -1 \qquad \text{and} \qquad v = \frac{1}{2}\sqrt{4a_2 - a_1^2} = \frac{1}{2}\sqrt{64} = 4$$

Hence, by (16.24′), the complementary function is

$$y_c = e^{-t}(A_5 \cos 4t + A_6 \sin 4t)$$

Combining y_c and y_p, the general solution can be expressed as

$$y(t) = e^{-t}(A_5 \cos 4t + A_6 \sin 4t) + 2$$

To definitize the constants A_5 and A_6, we utilize the two initial conditions. First, by setting $t = 0$ in the general solution, we find that

$$\begin{aligned} y(0) &= e^0(A_5 \cos 0 + A_6 \sin 0) + 2 \\ &= (A_5 + 0) + 2 = A_5 + 2 \qquad [\cos 0 = 1; \sin 0 = 0] \end{aligned}$$

By the initial condition $y(0) = 3$, we can thus specify $A_5 = 1$. Next, let us differentiate the general solution with respect to t—using the product rule and the derivative formulas (16.17) and (16.18) while bearing in mind the chain rule [Exercise 16.2-5]—to find $y'(t)$ and then $y'(0)$:

$$y'(t) = -e^{-t}(A_5 \cos 4t + A_6 \sin 4t) + e^{-t}[A_5(-4 \sin 4t) + 4A_6 \cos 4t]$$

so that

$$\begin{aligned} y'(0) &= -(A_5 \cos 0 + A_6 \sin 0) + (-4A_5 \sin 0 + 4A_6 \cos 0) \\ &= -(A_5 + 0) + (0 + 4A_6) = 4A_6 - A_5 \end{aligned}$$

By the second initial condition $y'(0) = 11$, and in view that $A_5 = 1$, it then becomes clear that $A_6 = 3$.† The definite solution is, therefore,

$$y(t) = e^{-t}(\cos 4t + 3 \sin 4t) + 2 \qquad \textbf{(16.25)}$$

† Note that, here, A_6 indeed turns out to be a real number, even though we have included the imaginary number i in its definition.

As before, the y_p component $(= 2)$ can be interpreted as the intertemporal equilibrium level of y, whereas the y_c component represents the deviation from equilibrium. Because of the presence of circular functions in y_c, the time path (16.25) may be expected to exhibit a fluctuating pattern. But what specific pattern will it involve?

The Time Path

We are familiar with the paths of a simple sine or cosine function, as shown in Fig. 16.4. Now we must study the paths of certain variants and combinations of sine and cosine functions so that we can interpret, in general, the complementary function (16.24′)

$$y_c = e^{ht}(A_5 \cos vt + A_6 \sin vt)$$

and, in particular, the y_c component of (16.25).

Let us first examine the term $(A_5 \cos vt)$. By itself, the expression $(\cos vt)$ is a circular function of (vt), with period 2π $(= 6.2832)$ and amplitude 1. The period of 2π means that the graph will repeat its configuration every time that (vt) increases by 2π. When t alone is taken as the independent variable, however, repetition will occur every time t increases by $2\pi/v$, so that with reference to t as is appropriate in dynamic economic analysis we shall consider the period of $(\cos vt)$ to be $2\pi/v$. (The amplitude, however, remains at 1.) Now, when a multiplicative constant A_5 is attached to $(\cos vt)$, it causes the range of fluctuation to change from ± 1 to $\pm A_5$. Thus the amplitude now becomes A_5, though the period is unaffected by this constant. In short, $(A_5 \cos vt)$ is a cosine function of t, with period $2\pi/v$ and amplitude A_5. By the same token, $(A_6 \sin vt)$ is a sine function of t, with period $2\pi/v$ and amplitude A_6.

There being a common period, the sum $(A_5 \cos vt + A_6 \sin vt)$ will also display a repeating cycle every time t increases by $2\pi/v$. To show this more rigorously, let us note that for given values of A_5 and A_6 we can always find two constants A and ε, such that

$$A_5 = A \cos \varepsilon \qquad \text{and} \qquad A_6 = -A \sin \varepsilon$$

Thus we may express the said sum as

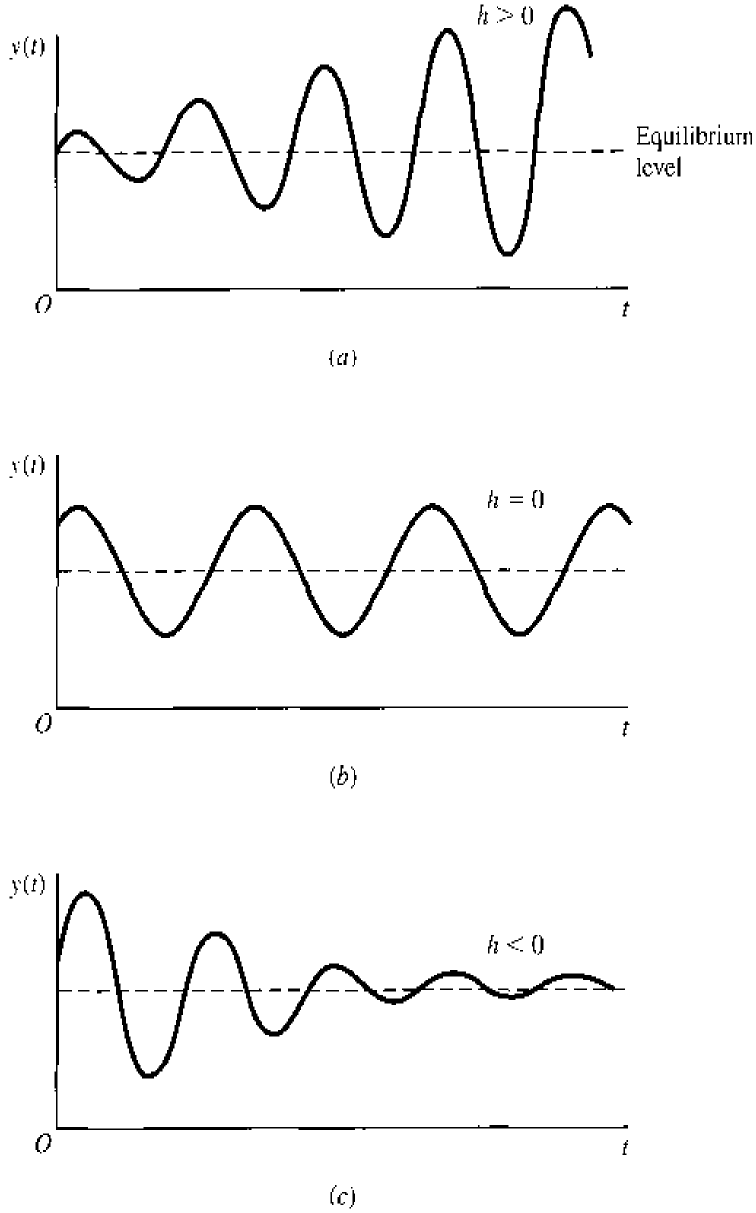
$$\begin{aligned} A_5 \cos vt + A_6 \sin vt &= A \cos \varepsilon \cos vt - A \sin \varepsilon \sin vt \\ &= A(\cos vt \cos \varepsilon - \sin vt \sin \varepsilon) \\ &= A \cos(vt + \varepsilon) \qquad [\text{by } (16.16)] \end{aligned}$$

This is a modified cosine function of t, with amplitude A and period $2\pi/v$, because every time that t increases by $2\pi/v$, $(vt + \varepsilon)$ will increase by 2π, which will complete a cycle on the cosine curve.

Had y_c consisted only of the expression $(A_5 \cos vt + A_6 \sin vt)$, the implication would have been that the time path of y would be a never-ending, constant-amplitude fluctuation around the equilibrium value of y, as represented by y_p. But there is, in fact, also the multiplicative term e^{ht} to consider. This latter term is of major importance, for, as we shall see, it holds the key to the question of whether the time path will converge.

If $h > 0$, the value of e^{ht} will increase continually as t increases. This will produce a magnifying effect on the amplitude of $(A_5 \cos vt + A_6 \sin vt)$ and cause ever-greater deviations from the equilibrium in each successive cycle. As illustrated in Fig. 16.6*a*, the time path will in this case be characterized by *explosive fluctuation*. If $h = 0$, on the other hand,

FIGURE 16.6



then $e^{ht} = 1$, and the complementary function will simply be $(A_5 \cos vt + A_6 \sin vt)$, which has been shown to have a constant amplitude. In this second case, each cycle will display a uniform pattern of deviation from the equilibrium as illustrated by the time path in Fig. 16.6*b*. This is a time path with *uniform fluctuation*. Last, if $h < 0$, the term e^{ht} will continually decrease as t increases, and each successive cycle will have a smaller amplitude than the preceding one, much as the way a ripple dies down. This case is illustrated in Fig. 16.6*c*, where the time path is characterized by *damped fluctuation*. The solution in (16.25), with $h = -1$, exemplifies this last case. It should be clear that only the case of damped fluctuation can produce a *convergent* time path; in the other two cases, the time path is *nonconvergent* or *divergent*.[†]

In all three diagrams of Fig. 16.6, the intertemporal equilibrium is assumed to be stationary. If it is a moving one, the three types of time path depicted will still fluctuate around it, but since a moving equilibrium generally plots as a curve rather than a horizontal straight

[†] We shall use the two words *nonconvergent* and *divergent* interchangeably, although the latter is more strictly applicable to the explosive than to the uniform variety of nonconvergence.

line, the fluctuation will take on the nature of, say, a series of business cycles around a secular trend.

The Dynamic Stability of Equilibrium

The concept of convergence of the time path of a variable is inextricably tied to the concept of dynamic stability of the intertemporal equilibrium of that variable. Specifically, the equilibrium is dynamically stable if, and only if, the time path is convergent. The condition for convergence of the $y(t)$ path, namely, $h < 0$ (Fig. 16.6c), is therefore also the condition for dynamic stability of the intertemporal equilibrium of y.

You will recall that, for Cases 1 and 2 where the characteristic roots are real, the condition for dynamic stability of equilibrium is that every characteristic root be negative. In the present case (Case 3), with complex roots, the condition seems to be more specialized; it stipulates only that the real part (h) of the complex roots ($h \pm vi$) be negative. However, it is possible to unify all three cases and consolidate the seemingly different conditions into a single, generally applicable one. Just interpret any real root r as a complex root whose imaginary part is zero ($v = 0$). Then the condition "the *real* part of every characteristic root be negative" clearly becomes applicable to all three cases and emerges as the only condition we need.

EXERCISE 16.3

Find the y_p and the y_c, the general solution, and the definite solution of each of the following:

1. $y''(t) - 4y'(t) + 8y = 0$; $y(0) = 3$, $y'(0) = 7$
2. $y''(t) + 4y'(t) + 8y = 2$; $y(0) = 2\frac{1}{4}$, $y'(0) = 4$
3. $y''(t) + 3y'(t) - 4y = 12$; $y(0) = 2$, $y'(0) = 2$
4. $y''(t) - 2y'(t) - 10y = 5$; $y(0) = 6$, $y'(0) = 8\frac{1}{2}$
5. $y''(t) + 9y = 3$; $y(0) = 1$, $y'(0) = 3$
6. $2y''(t) - 12y'(t) + 20y = 40$; $y(0) = 4$, $y'(0) = 5$
7. Which of the differential equations in Probs. 1 to 6 yield time paths with (*a*) damped fluctuation; (*b*) uniform fluctuation; (*c*) explosive fluctuation?

16.4 A Market Model with Price Expectations

In the earlier formulation of the dynamic market model, both Q_d and Q_s are taken to be functions of the current price P alone. But sometimes buyers and sellers may base their market behavior not only on the current price but also on the price *trend* prevailing at the time, for the price trend is likely to lead them to certain *expectations* regarding the price level in the future, and these expectations can, in turn, influence their demand and supply decisions.

Price Trend and Price Expectations

In the continuous-time context, the price-trend information is to be found primarily in the two derivatives dP/dt (whether price is rising) and d^2P/dt^2 (whether increasing at an increasing rate). To take the price trend into account, let us now include these derivatives as additional arguments in the demand and supply functions:

$$Q_d = D[P(t), P'(t), P''(t)]$$
$$Q_s = S[P(t), P'(t), P''(t)]$$

If we confine ourselves to the linear version of these functions and simplify the notation for the independent variables to P, P', and P'', we can write

$$\begin{aligned} Q_d &= \alpha - \beta P + mP' + nP'' \qquad (\alpha, \beta > 0) \\ Q_s &= -\gamma + \delta P + uP' + wP'' \qquad (\gamma, \delta > 0) \end{aligned} \tag{16.26}$$

where the parameters α, β, γ, and δ are merely carryovers from the previous market models, but m, n, u, and w are new.

The four new parameters, whose signs have not been restricted, embody the buyers' and sellers' price expectations. If $m > 0$, for instance, a rising price will cause Q_d to increase. This would suggest that buyers expect the rising price to *continue* to rise and, hence, prefer to increase their purchases now, when the price is still relatively low. The opposite sign for m would, on the other hand, signify the expectation of a prompt reversal of the price trend, so the buyers would prefer to cut back current purchases and wait for a lower price to materialize later. The inclusion of the parameter n makes the buyers' behavior depend also on the rate of change of dP/dt. Thus the new parameters m and n inject a substantial element of price speculation into the model. The parameters u and w carry a similar implication on the sellers' side of the picture.

A Simplified Model

For simplicity, we shall assume that only the demand function contains price expectations. Specifically, we let m and n be nonzero, but let $u = w = 0$ in (16.26). Further assume that *the market is cleared at every point of time*. Then we may equate the demand and supply functions to obtain (after normalizing) the differential equation

$$P'' + \frac{m}{n}P' - \frac{\beta + \delta}{n}P = -\frac{\alpha + \gamma}{n} \tag{16.27}$$

This equation is in the form of (16.2) with the following substitutions:

$$y = P \qquad a_1 = \frac{m}{n} \qquad a_2 = -\frac{\beta + \delta}{n} \qquad b = -\frac{\alpha + \gamma}{n}$$

Since this pattern of change of P involves the second derivative P'' as well as the first derivative P', the present model is certainly distinct from the dynamic market model presented in Sec. 15.2.

Note, however, that the present model differs from the previous model in yet another way. In Sec. 15.2, a dynamic adjustment mechanism, $dP/dt = j(Q_d - Q_s)$ is present. Since that equation implies that $dP/dt = 0$ if and only if $Q_d = Q_s$, the intertemporal sense and the market-clearing sense of equilibrium are coincident in that model. In contrast, the present model assumes market clearance at every moment of time. Thus every price attained in the market is an equilibrium price in the market-clearing sense, although it may not qualify as the intertemporal equilibrium price. In other words, the two senses of equilibrium are now disparate. Note, also, that the adjustment mechanism $dP/dt = j(Q_d - Q_s)$, containing a derivative, is what makes the previous market model dynamic.

In the present model, with no adjustment mechanism, the dynamic nature of the model emanates instead from the expectation terms mP' and nP''.

The Time Path of Price

The intertemporal equilibrium price of this model—the particular integral P_p (formerly y_p)—is easily found by using (16.3). It is

$$P_p = \frac{b}{a_2} = \frac{\alpha + \gamma}{\beta + \delta}$$

Because this is a (positive) constant, it represents a stationary equilibrium.

As for the complementary function P_c (formerly y_c), there are three possible cases.

Case 1 (distinct real roots)

$$\left(\frac{m}{n}\right)^2 > -4\left(\frac{\beta + \delta}{n}\right)$$

The complementary function of this case is, by (16.7),

$$P_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

where

$$r_1, r_2 = \frac{1}{2}\left[-\frac{m}{n} \pm \sqrt{\left(\frac{m}{n}\right)^2 + 4\left(\frac{\beta + \delta}{n}\right)}\right] \tag{16.28}$$

Accordingly, the general solution is

$$P(t) = P_c + P_p = A_1 e^{r_1 t} + A_2 e^{r_2 t} + \frac{\alpha + \gamma}{\beta + \delta} \tag{16.29}$$

Case 2 (double real roots)

$$\left(\frac{m}{n}\right)^2 = -4\left(\frac{\beta + \delta}{n}\right)$$

In this case, the characteristic roots take the single value

$$r = -\frac{m}{2n}$$

thus, by (16.9), the general solution may be written as

$$P(t) = A_3 e^{-mt/2n} + A_4 t e^{-mt/2n} + \frac{\alpha + \gamma}{\beta + \delta} \tag{16.29'}$$

Case 3 (complex roots)

$$\left(\frac{m}{n}\right)^2 < -4\left(\frac{\beta + \delta}{n}\right)$$

In this third and last case, the characteristic roots are the pair of conjugate complex numbers

$$r_1, r_2 = h \pm vi$$

where

$$h = -\frac{m}{2n} \qquad \text{and} \qquad v = \frac{1}{2}\sqrt{-4\left(\frac{\beta+\delta}{n}\right) - \left(\frac{m}{n}\right)^2}$$

Therefore, by (16.24′), we have the general solution

$$P(t) = e^{-mt/2n}(A_5 \cos vt + A_6 \sin vt) + \frac{\alpha+\gamma}{\beta+\delta} \qquad \textbf{(16.29'')}$$

A couple of general conclusions can be deduced from these results. First, if $n > 0$, then $-4(\beta+\delta)/n$ must be negative and hence less than $(m/n)^2$. Hence Cases 2 and 3 can immediately be ruled out. Moreover, with n positive (as are β and δ), the expression under the square-root sign in (16.28) necessarily exceeds $(m/n)^2$, and thus the square root must be greater than $|m/n|$. The $\pm$ sign in (16.28) would then produce one positive root (r_1) and one negative root (r_2). Consequently, the intertemporal equilibrium is dynamically unstable, unless the definitized value of the constant A_1 happens to be zero in (16.29).

Second, if $n < 0$, then all three cases become feasible. Under Case 1, we can be sure that both roots will be negative if m is negative. (Why?) Interestingly, the repeated root of Case 2 will also be negative if m is negative. Moreover, since h, the real part of the complex roots in Case 3, takes the same value as the repeated root r in Case 2, the negativity of m will also guarantee that h is negative. In short, for all three cases, the dynamic stability of equilibrium is ensured when the parameters m and n are both negative.

Example 1

Let the demand and supply functions be

$$Q_d = 42 - 4P - 4P' + P''$$
$$Q_s = -6 + 8P$$

with initial conditions $P(0) = 6$ and $P'(0) = 4$. Assuming market clearance at every point of time, find the time path $P(t)$.

In this example, the parameter values are

$$\alpha = 42 \qquad \beta = 4 \qquad \gamma = 6 \qquad \delta = 8 \qquad m = -4 \qquad n = 1$$

Since n is positive, our previous discussion suggests that only Case 1 can arise, and that the two (real) roots r_1 and r_2 will take opposite signs. Substitution of the parameter values into (16.28) indeed confirms this, for

$$r_1, r_2 = \frac{1}{2}(4 \pm \sqrt{16+48}) = \frac{1}{2}(4 \pm 8) = 6, -2$$

The general solution is, then, by (16.29),

$$P(t) = A_1 e^{6t} + A_2 e^{2t} + 4$$

By taking the initial conditions into account, moreover, we find that $A_1 = A_2 = 1$, so the definite solution is

$$P(t) = e^{6t} + e^{-2t} + 4$$

In view of the positive root $r_1 = 6$, the intertemporal equilibrium ($P_p = 4$) is dynamically unstable.

The preceding solution is found by use of formulas (16.28) and (16.29). Alternatively, we can first equate the given demand and supply functions to obtain the differential equation

$$P'' - 4P' - 12P = -48$$

and then solve this equation as a specific case of (16.2).

Example 2

Given the demand and supply functions

$$Q_d = 40 - 2P - 2P' - P''$$
$$Q_s = -5 + 3P$$

with $P(0) = 12$ and $P'(0) = 1$, find $P(t)$ on the assumption that the market is always cleared.

Here the parameters m and n are both negative. According to our previous general discussion, therefore, the intertemporal equilibrium should be dynamically stable. To find the specific solution, we may first equate Q_d and Q_s to obtain the differential equation (after multiplying through by -1)

$$P'' + 2P' + 5P = 45$$

The intertemporal equilibrium is given by the particular integral

$$P_p = \frac{45}{5} = 9$$

From the characteristic equation of the differential equation,

$$r^2 + 2r + 5 = 0$$

we find that the roots are complex:

$$r_1, r_2 = \frac{1}{2}(-2 \pm \sqrt{4 - 20}) = \frac{1}{2}(-2 \pm 4i) = -1 \pm 2i$$

This means that $h = -1$ and $v = 2$, so the general solution is

$$P(t) = e^{-t}(A_5 \cos 2t + A_6 \sin 2t) + 9$$

To definitize the arbitrary constants A_5 and A_6, we set $t = 0$ in the general solution, to get

$$P(0) = e^0(A_5 \cos 0 + A_6 \sin 0) + 9 = A_5 + 9 \qquad [\cos 0 = 1;\ \sin 0 = 0]$$

Moreover, by differentiating the general solution and then setting $t = 0$, we find that

$$P'(t) = -e^{-t}(A_5 \cos 2t + A_6 \sin 2t) + e^{-t}(-2A_5 \sin 2t + 2A_6 \cos 2t)$$

[product rule and chain rule]

and $$P'(0) = -e^0(A_5 \cos 0 + A_6 \sin 0) + e^0(-2A_5 \sin 0 + 2A_6 \cos 0)$$
$$= -(A_5 + 0) + (0 + 2A_6) = -A_5 + 2A_6$$

Thus, by virtue of the initial conditions $P(0) = 12$ and $P'(0) = 1$, we have $A_5 = 3$ and $A_6 = 2$. Consequently, the definite solution is

$$P(t) = e^{-t}(3 \cos 2t + 2 \sin 2t) + 9$$

This time path is obviously one with periodic fluctuation; the period is $2\pi/v = \pi$. That is, there is a complete cycle every time that t increases by $\pi = 3.14159\ldots$. In view of the multiplicative term e^{-t}, the fluctuation is damped. The time path, which starts from the initial price $P(0) = 12$, converges to the intertemporal equilibrium price $P_p = 9$ in a cyclical fashion.

EXERCISE 16.4

1. Let the parameters m, n, u, and w in (16.26) be all nonzero.
   (a) Assuming market clearance at every point of time, write the new differential equation of the model.
   (b) Find the intertemporal equilibrium price.
   (c) Under what circumstances can periodic fluctuation be ruled out?
2. Let the demand and supply functions be as in (16.26), but with $u = w = 0$ as in the text discussion.
   (a) If the market is not always cleared, but adjusts according to

   $$\frac{dP}{dt} = j(Q_d - Q_s) \qquad (j > 0)$$

   write the appropriate new differential equation.
   (b) Find the intertemporal equilibrium price $\overline{P}$ and the market-clearing equilibrium price P^*.
   (c) State the condition for having a fluctuating price path. Can fluctuation occur if $n > 0$?
3. Let the demand and supply be

   $$Q_d = 9 - P + P' + 3P'' \qquad Q_s = -1 + 4P - P' + 5P''$$

   with $P(0) = 4$ and $P'(0) = 4$.
   (a) Find the price path, assuming market clearance at every point of time.
   (b) Is the time path convergent? With fluctuation?

16.5 The Interaction of Inflation and Unemployment

In this section, we illustrate the use of a second-order differential equation with a macro model dealing with the problem of inflation and unemployment.

The Phillips Relation

One of the most widely used concepts in the modern analysis of the problem of inflation and unemployment is the Phillips relation.† In its original formulation, this relation depicts an empirically based negative relation between the rate of growth of money wage and the rate of unemployment:

$$w = f(U) \qquad [f'(U) < 0] \qquad \textbf{(16.30)}$$

† A. W. Phillips, "The Relationship Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861–1957," *Economica*, November 1958, pp. 283–299.

where the lowercase letter w denotes the rate of growth of money wage W (i.e., $w \equiv \dot{W}/W$) and U is the rate of unemployment. It thus pertains only to the labor market. Later usage, however, has adapted the Phillips relation into a function that links the *rate of inflation* (instead of w) to the rate of unemployment. This adaptation may be justified by arguing that mark-up pricing is in wide use, so that a positive w, reflecting growing money-wage cost, would necessarily carry inflationary implications. And this makes the rate of inflation, like w, a function of U. The inflationary pressure of a positive w can, however, be offset by an increase in labor productivity, assumed to be exogeneous, and denoted here by T. Specifically, the inflationary effect can materialize only to the extent that money wage grows faster than productivity. Denoting the rate of inflation—that is, the rate of growth of the price level P—by the lowercase letter p, ($p \equiv \dot{P}/P$), we may thus write

$$p = w - T \tag{16.31}$$

Combining (16.30) and (16.31), and adopting the linear version of the function $f(U)$, we then get an adapted Phillips relation

$$p = \alpha - T - \beta U \qquad (\alpha, \beta > 0) \tag{16.32}$$

The Expectations-Augmented Phillips Relation

More recently, economists have preferred to use the *expectations-augmented* version of the Phillips relation

$$w = f(U) + g\pi \qquad (0 < g \le 1) \tag{16.30'}$$

where π denotes the expected rate of inflation. The underlying idea of (16.30′), as propounded by the Nobel laureate Professor Friedman,[†] is that if an inflationary trend has been in effect long enough, people are apt to form certain inflation expectations which they then attempt to incorporate into their money-wage demands. Thus w should be an increasing function of π. Carried over to (16.32), this idea results in the equation

$$p = \alpha - T - \beta U + g\pi \qquad (0 < g \le 1) \tag{16.33}$$

With the introduction of a new variable to denote the expected rate of inflation, it becomes necessary to hypothesize how inflation expectations are specifically formed.[‡] Here we adopt the *adaptive expectations* hypothesis

$$\frac{d\pi}{dt} = j(p - \pi) \qquad (0 < j \le 1) \tag{16.34}$$

Note that, rather than explain the absolute magnitude of π, this equation describes instead its pattern of change over time. If the actual rate of inflation p turns out to exceed the expected rate π, the latter, having now been proven to be too low, is revised upward ($d\pi/dt > 0$). Conversely, if p falls short of π, then π is revised in the downward direction. In format, (16.34) closely resembles the adjustment mechanism $dP/dt = j(Q_d - Q_s)$ of

[†] Milton Friedman, "The Role of Monetary Policy," *American Economic Review*, March 1968, pp. 1–17.

[‡] This is in contrast to Sec. 16.4, where price expectations were discussed without introducing a new variable to represent the expected price. As a result, the assumptions regarding the formation of expectations were only implicitly embedded in the parameters m, n, u, and w in (16.26).

the market model. But here the driving force behind the adjustment is the discrepancy between the *actual* and *expected* rates of inflation, rather than Q_d and Q_s.

The Feedback from Inflation to Unemployment

It is possible to consider (16.33) and (16.34) as constituting a complete model. Since there are three variables in a two-equation system, however, one of the variables has to be taken as exogenous. If π and p are considered endogenous, for instance, then U must be treated as exogenous. A more satisfying alternative is to introduce a third equation to explain the variable U, so that the model will be richer in behavioral characteristics. More significantly, this will provide us with an opportunity to take into account the feedback effect of inflation on unemployment. Equation (16.33) tells us how U affects p—largely from the supply side of the economy. But p surely can affect U in return. For example, the rate of inflation may influence the consumption-saving decisions of the public, hence also the aggregate demand for domestic production, and the latter will, in turn, affect the rate of unemployment. Even in the conduct of government policies of demand management, the rate of inflation can make a difference in their effectiveness. Depending on the rate of inflation, a given level of money expenditure (fiscal policy) could translate into varying levels of real expenditure, and similarly, a given rate of nominal-money expansion (monetary policy) could mean varying rates of real-money expansion. And these, in turn, would imply differing effects on output and unemployment.

For simplicity, we shall only take into consideration the feedback through the conduct of monetary policy. Denoting the nominal money balance by M and its rate of growth by $m \equiv \dot{M}/M$, let us postulate that[†]

$$\frac{dU}{dt} = -k(m - p) \qquad (k > 0) \tag{16.35}$$

Recalling (10.25), and applying it backward, we see that the expression $(m - p)$ represents the rate of growth of *real* money:

$$m - p = \frac{\dot{M}}{M} - \frac{\dot{P}}{P} = r_M - r_P = r_{(M/P)}$$

Thus (16.35) stipulates that dU/dt is negatively related to the rate of growth of real-money balance. Inasmuch as the variable p now enters into the determination of dU/dt, the model now contains a feedback from inflation to unemployment.

The Time Path of π

Together, (16.33) through (16.35) constitute a closed model in the three variables π, p, and U. By eliminating two of the three variables, however, we can condense the model into a single differential equation in a single variable. Suppose that we let that single variable be π. Then we may first substitute (16.33) into (16.34) to get

$$\frac{d\pi}{dt} = j(\alpha - T - \beta U) - j(1 - g)\pi \tag{16.36}$$

[†] In an earlier discussion, we denoted the money supply by M_s, to distinguish it from the demand for money M_d. Here, we can simply use the unsubscripted letter M, since there is no fear of confusion.

Had this equation contained the expression dU/dt instead of U, we could have substituted (16.35) into (16.36) directly. But as (16.36) stands, we must first deliberately create a dU/dt term by differentiating (16.36) with respect to t, with the result

$$\frac{d^2\pi}{dt^2} = -j\beta\frac{dU}{dt} - j(1-g)\frac{d\pi}{dt} \tag{16.37}$$

Substitution of (16.35) into this then yields

$$\frac{d^2\pi}{dt^2} = j\beta km - j\beta kp - j(1-g)\frac{d\pi}{dt} \tag{16.37'}$$

There is still a p variable to be eliminated. To achieve that, we note that (16.34) implies

$$p = \frac{1}{j}\frac{d\pi}{dt} + \pi \tag{16.38}$$

Using this result in (16.37'), and simplifying, we finally obtain the desired differential equation in the variable π alone:

$$\frac{d^2\pi}{dt^2} + \underbrace{[\beta k + j(1-g)]}_{a_1}\frac{d\pi}{dt} + \underbrace{(j\beta k)}_{a_2}\pi = \underbrace{j\beta km}_{b} \tag{16.37''}$$

The particular integral of this equation is simply

$$\pi_p = \frac{b}{a_2} = m$$

Thus, in this model, the intertemporal equilibrium value of the expected rate of inflation hinges exclusively on the rate of growth of nominal money.

For the complementary function, the two roots are, as before,

$$r_1, r_2 = \frac{1}{2}\left(-a_1 \pm \sqrt{a_1^2 - 4a_2}\right) \tag{16.39}$$

where, as may be noted from (16.37''), both a_1 and a_2 are positive. On a priori grounds, it is not possible to determine whether a_1^2 would exceed, equal, or be less than $4a_2$. Thus all three cases of characteristic roots—distinct real roots, repeated real roots, or complex roots—can conceivably arise. Whichever case presents itself, however, the intertemporal equilibrium will prove dynamically stable in the present model. This can be explained as follows: Suppose, first, that Case 1 prevails, with $a_1^2 > 4a_2$. Then the square root in (16.39) yields a real number. Since a_2 is positive, $\sqrt{a_1^2 - 4a_2}$ is necessarily less than $\sqrt{a_1^2} = a_1$. It follows that r_1 is negative, as is r_2, implying a dynamically stable equilibrium. What if $a_1^2 = 4a_2$ (Case 2)? In that event, the square root is zero, so that $r_1 = r_2 = -a_1/2 < 0$. And the negativity of the repeated roots again implies dynamic stability. Finally, for Case 3, the real part of the complex roots is $h = -a_1/2$. Since this has the same value as the repeated roots under Case 2, the identical conclusion regarding dynamic stability applies.

Although we have only studied the time path of π, the model can certainly yield information on the other variables, too. To find the time path of, say, the U variable, we can *either* start off by condensing the model into a differential equation in U rather than π (see Exercise 16.5-2) *or* deduce the U path from the π path already found (see Example 1).

Example 1

Let the three equations of the model take the specific forms

$$p = \frac{1}{6} - 3U + \pi \tag{16.40}$$

$$\frac{d\pi}{dt} = \frac{3}{4}(p - \pi) \tag{16.41}$$

$$\frac{dU}{dt} = -\frac{1}{2}(m - p) \tag{16.42}$$

Then we have the parameter values $\beta = 3$, $h = 1$, $j = \frac{3}{4}$, and $k = \frac{1}{2}$; thus, with reference to (16.37″), we find

$$a_1 = \beta k + j(1 - g) = \frac{3}{2} \qquad a_2 = j\beta k = \frac{9}{8} \qquad \text{and} \qquad b = j\beta k m = \frac{9}{8}m$$

The particular integral is $b/a_2 = m$. With $a_1^2 < 4a_2$, the characteristic roots are complex:

$$r_1, r_2 = \frac{1}{2}\left(-\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{9}{2}}\right) = \frac{1}{2}\left(-\frac{3}{2} \pm \frac{3}{2}i\right) = -\frac{3}{4} \pm \frac{3}{4}i$$

That is, $h = -\frac{3}{4}$ and $v = \frac{3}{4}$. Consequently, the general solution for the expected rate of inflation is

$$\pi(t) = e^{-3t/4}\left(A_5 \cos\frac{3}{4}t + A_6 \sin\frac{3}{4}t\right) + m \tag{16.43}$$

which depicts a time path with damped fluctuation around the equilibrium value m.

From this, we can also deduce the time paths for the p and U variables. According to (16.41), p can be expressed in terms of π and $d\pi/dt$ by the equation

$$p = \frac{4}{3}\frac{d\pi}{dt} + \pi$$

The π path in the general solution (16.43) implies the derivative

$$\begin{aligned}\frac{d\pi}{dt} &= -\frac{3}{4}e^{-3t/4}\left(A_5 \cos\frac{3}{4}t + A_6 \sin\frac{3}{4}t\right)\\ &\quad + e^{-3t/4}\left(-\frac{3}{4}A_5 \sin\frac{3}{4}t + \frac{3}{4}A_6 \cos\frac{3}{4}t\right) \qquad \text{[product rule and chain rule]}\end{aligned}$$

Using the solution (16.43) and its derivative, we thus have

$$p(t) = e^{-3t/4}\left(A_6 \cos\frac{3}{4}t - A_5 \sin\frac{3}{4}t\right) + m \tag{16.44}$$

Like the *expected* rate of inflation π, the *actual* rate of inflation p also has a fluctuating time path converging to the equilibrium value m.

As for the U variable, (16.40) tells us that it can be expressed in terms of π and p as follows:

$$U = \frac{1}{3}(\pi - p) + \frac{1}{18}$$

By virtue of the solutions (16.43) and (16.44), therefore, we can write the time path of the rate of unemployment as

$$U(t) = \frac{1}{3}e^{-3t/4}\left[(A_5 - A_6)\cos\frac{3}{4}t + (A_5 + A_6)\sin\frac{3}{4}t\right] + \frac{1}{18} \tag{16.45}$$

This path is, again, one with damped fluctuation, with $\frac{1}{18}$ as $\overline{U}$, the dynamically stable intertemporal equilibrium value of U.

Because the intertemporal equilibrium values of π and p are both equal to the monetary-policy parameter m, the value of m—the rate of growth of nominal money—provides the axis around which the time paths of π and p fluctuate. If a change occurs in m, a new equilibrium value of π and p will immediately replace the old one, and whatever values the π and p variables happen to take at the moment of the monetary-policy change will become the initial values from which the new π and p paths emanate.

In contrast, the intertemporal equilibrium value $\overline{U}$ does not depend on m. According to (16.45), U converges to the constant $\frac{1}{18}$ regardless of the rate of growth of nominal money, and hence regardless of the equilibrium rate of inflation. This constant equilibrium value of U is referred to as the *natural rate of unemployment*. The fact that the natural rate of unemployment is consistent with any equilibrium rate of inflation can be represented in the Up space by a vertical straight line parallel to the p axis. That vertical line relating the equilibrium values of U and p to each other, is known as the *long-run Phillips curve*. The vertical shape of this curve, however, is contingent upon a special parameter value assumed in this example. When that value is altered, as in Exercise 16.5-4, the long-run Phillips curve may no longer be vertical.

EXERCISE 16.5

1. In the inflation-unemployment model, retain (16.33) and (16.34) but delete (16.35) and let U be exogenous instead.
   (*a*) What kind of differential equation will now arise?
   (*b*) How many characteristic roots can you obtain? Is it possible now to have periodic fluctuation in the complementary function?
2. In the text discussion, we condensed the inflation-unemployment model into a differential equation in the variable π. Show that the model can alternatively be condensed into a second-order differential equation in the variable U, with the same a_1 and a_2 coefficients as in (16.37″), but a different constant term $b = kj[\alpha - T - (1 - g)m]$.
3. Let the adaptive expectations hypothesis (16.34) be replaced by the so-called perfect foresight hypothesis $\pi = p$, but retain (16.33) and (16.35).
   (*a*) Derive a differential equation in the variable p.
   (*b*) Derive a differential equation in the variable U.
   (*c*) How do these equations differ fundamentally from the one we obtained under the adaptive expectations hypothesis?
   (*d*) What change in parameter restriction is now necessary to make the new differential equations meaningful?
4. In Example 1, retain (16.41) and (16.42) but replace (16.40) by
   $$p = \frac{1}{6} - 3U + \frac{1}{3}\pi$$
   (*a*) Find $p(t)$, $\pi(t)$, and $U(t)$.
   (*b*) Are the time paths still fluctuating? Still convergent?
   (*c*) What are $\overline{p}$ and $\overline{U}$, the intertemporal equilibrium values of p and U?
   (*d*) Is it still true that $\overline{U}$ is functionally unrelated to $\overline{p}$? If we now link these two equilibrium values to each other in a long-run Phillips curve, can we still get a vertical curve? What assumption in Example 1 is thus crucial for deriving a vertical long-run Phillips curve?