Chapter 15

Continuous Time: First-Order Differential Equations

In the Domar growth model, we have solved a simple differential equation by direct integration. For more complicated differential equations, there are various established methods of solution. Even in the latter cases, however, the fundamental idea underlying the methods of solution is still the techniques of integral calculus. For this reason, the solution to a differential equation is often referred to as the *integral* of that equation.

Only *first-order* differential equations will be discussed in the present chapter. In this context, the word *order* refers to the highest order of the derivatives (or differentials) appearing in the differential equation; thus a first-order differential equation can contain only the first derivative, say, dy/dt.

15.1 First-Order Linear Differential Equations with Constant Coefficient and Constant Term

The first derivative dy/dt is the only one that can appear in a first-order differential equation, but it may enter in various powers: dy/dt, $(dy/dt)^2$, or $(dy/dt)^3$. The highest power attained by the derivative in the equation is referred to as the *degree* of the differential equation. In case the derivative dy/dt appears only in the first degree, and so does the dependent variable y, and furthermore, no product of the form $y(dy/dt)$ occurs, then the equation is said to be *linear*. Thus a first-order linear differential equation will generally take the form[†]

$$\frac{dy}{dt} + u(t)y = w(t) \tag{15.1}$$

[†] Note that the derivative term dy/dt in (15.1) has a unit coefficient. This is not to imply that it can never actually have a coefficient other than one, but when such a coefficient appears, we can always "normalize" the equation by dividing each term by the said coefficient. For this reason, the form given in (15.1) may nonetheless be regarded as a *general* representation.

where u and w are two functions of t, as is y. In contrast to dy/dt and y, however, no restriction whatsoever is placed on the independent variable t. Thus the functions u and w may very well represent such expressions as t^2 and e^t or some more complicated functions of t; on the other hand, u and w may also be constants.

This last point leads us to a further classification. When the function u (the coefficient of the dependent variable y) is a constant, and when the function w is a constant additive term, (15.1) reduces to the special case of a first-order linear differential equation with *constant coefficient and constant term*. In this section, we shall deal only with this simple variety of differential equations.

The Homogeneous Case

If u and w are constant functions and if w happens to be identically zero, (15.1) will become

$$\frac{dy}{dt} + ay = 0 \qquad \textbf{(15.2)}$$

where a is some constant. This differential equation is said to be *homogeneous* on account of the zero constant term (compare with homogeneous-equation systems). The defining characteristic of a homogeneous equation is that when all the variables (here, dy/dt and y) are multiplied by a given constant, the equation remains valid. This characteristic holds if the constant term is zero, but will be lost if the constant term is not zero.

Equation (15.2) can be written alternatively as

$$\frac{1}{y}\frac{dy}{dt} = -a \qquad \textbf{(15.2')}$$

But you will recognize that the differential equation (14.16) we met in the Domar model is precisely of this form. Therefore, by analogy, we should be able to write the solution of (15.2) or (15.2′) immediately as follows:

$$y(t) = Ae^{-at} \qquad [\textit{general} \text{ solution}] \qquad \textbf{(15.3)}$$

or

$$y(t) = y(0)e^{-at} \qquad [\textit{definite} \text{ solution}] \qquad \textbf{(15.3')}$$

In (15.3), there appears an arbitrary constant A; therefore it is a *general solution*. When any particular value is substituted for A, the solution becomes a *particular solution* of (15.2). There is an infinite number of particular solutions, one for each possible value of A, including the value $y(0)$. This latter value, however, has a special significance: $y(0)$ is the only value that can make the solution satisfy the initial condition. Since this represents the result of definitizing the arbitrary constant, we shall refer to (15.3′) as the *definite solution* of the differential equation (15.2) or (15.2′).

You should observe two things about the solution of a differential equation: (1) the solution is not a numerical value, but rather a function $y(t)$ — a time path if t symbolizes time; and (2) the solution $y(t)$ is free of any derivative or differential expressions, so that as soon as a specific value of t is substituted into it, a corresponding value of y can be calculated directly.

The Nonhomogeneous Case

When a nonzero constant takes the place of the zero in (15.2), we have a *nonhomogeneous* linear differential equation

$$\frac{dy}{dt} + ay = b \qquad \textbf{(15.4)}$$

The solution of this equation will consist of the sum of two terms, one of which is called the *complementary function* (which we shall denote by y_c), and the other known as the *particular integral* (to be denoted by y_p). As will be shown, each of these has a significant economic interpretation. Here, we shall present only the method of solution; its rationale will become clear later.

Even though our objective is to solve the *non*homogeneous equation (15.4), frequently we shall have to refer to its homogeneous version, as shown in (15.2). For convenient reference, we call the latter the *reduced equation* of (15.4). The nonhomogeneous equation (15.4) itself can accordingly be referred to as the *complete equation.* It turns out that the complementary function y_c is nothing but the general solution of the reduced equation, whereas the particular integral y_p is simply *any* particular solution of the complete equation.

Our discussion of the homogeneous case has already given us the general solution of the reduced equation, and we may therefore write

$$y_c = Ae^{-at} \qquad \text{[by (15.3)]}$$

What about the particular integral? Since the particular integral is *any* particular solution of the complete equation, we can first try the simplest possible type of solution, namely, y being some constant ($y = k$). If y is a constant, then it follows that $dy/dt = 0$, and (15.4) will become $ay = b$, with the solution $y = b/a$. Therefore, the constant solution will work as long as $a \neq 0$. In that case, we have

$$y_p = \frac{b}{a} \qquad (a \neq 0)$$

The sum of the complementary function and the particular integral then constitutes the general solution of the complete equation (15.4):

$$y(t) = y_c + y_p = Ae^{-at} + \frac{b}{a} \qquad \text{[general solution, case of } a \neq 0\text{]} \qquad \textbf{(15.5)}$$

What makes this a general solution is the presence of the arbitrary constant A. We may, of course, definitize this constant by means of an initial condition. Let us say that y takes the value $y(0)$ when $t = 0$. Then, by setting $t = 0$ in (15.5), we find that

$$y(0) = A + \frac{b}{a} \qquad \text{and} \qquad A = y(0) - \frac{b}{a}$$

Thus we can rewrite (15.5) into

$$y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a} \qquad \text{[definite solution, case of } a \neq 0\text{]} \qquad \textbf{(15.5')}$$

It should be noted that the use of the initial condition to definitize the arbitrary constant is—and should be—undertaken as the *final* step, after we have found the general solution to the complete equation. Since the values of both y_c and y_p are related to the value of $y(0)$, both of these must be taken into account in definitizing the constant A.

Example 1 Solve the equation $dy/dt + 2y = 6$, with the initial condition $y(0) = 10$. Here, we have $a = 2$ and $b = 6$; thus, by (15.5'), the solution is

$$y(t) = (10 - 3)e^{-2t} + 3 = 7e^{-2t} + 3$$

Example 2 Solve the equation $dy/dt + 4y = 0$, with the initial condition $y(0) = 1$. Since $a = 4$ and $b = 0$, we have

$$y(t) = (1 - 0)e^{-4t} + 0 = e^{-4t}$$

The same answer could have been obtained from (15.3′), the formula for the homogeneous case. The homogeneous equation (15.2) is merely a special case of the nonhomogeneous equation (15.4) when $b = 0$. Consequently, the formula (15.3′) is also a special case of formula (15.5′) under the circumstance that $b = 0$.

What if $a = 0$, so that the solution in (15.5′) is undefined? In that case, the differential equation is of the extremely simple form

$$\frac{dy}{dt} = b \tag{15.6}$$

By straight integration, its general solution can be readily found to be

$$y(t) = bt + c \tag{15.7}$$

where c is an arbitrary constant. The two component terms in (15.7) can, in fact, again be identified as the complementary function and the particular integral of the given differential equation, respectively. Since $a = 0$, the complementary function can be expressed simply as

$$y_c = Ae^{-at} = Ae^0 = A \qquad (A = \text{an arbitrary constant})$$

As to the particular integral, the fact that the constant solution $y = k$ fails to work in the present case of $a = 0$ suggests that we should try instead a *nonconstant* solution. Let us consider the simplest possible type of the latter, namely, $y = kt$. If $y = kt$, then $dy/dt = k$, and the complete equation (15.6) will reduce to $k = b$, so that we may write

$$y_p = bt \qquad (a = 0)$$

Our new trial solution indeed works! The general solution of (15.6) is therefore

$$y(t) = y_c + y_p = A + bt \qquad [\text{general solution, case of } a = 0] \tag{15.7'}$$

which is identical with the result in (15.7), because c and A are but alternative notations for an arbitrary constant. Note, however, that in the present case, y_c is a constant whereas y_p is a function of time—the exact opposite of the situation in (15.5).

By definitizing the arbitrary constant, we find the definite solution to be

$$y(t) = y(0) + bt \qquad [\text{definite solution, case of } a = 0] \tag{15.7''}$$

Example 3 Solve the equation $dy/dt = 2$, with the initial condition $y(0) = 5$. The solution is, by (15.7″),

$$y(t) = 5 + 2t$$

Verification of the Solution

It is true of all solutions of differential equations that their validity can always be checked by differentiation.

If we try that on the solution (15.5′), we can obtain the derivative

$$\frac{dy}{dt} = -a\left[y(0) - \frac{b}{a}\right]e^{-at}$$

When this expression for dy/dt and the expression for $y(t)$ as shown in (15.5′) are substituted into the left side of the differential equation (15.4), that side should reduce exactly to the value of the constant term b on the right side of (15.4) if the solution is correct. Performing this substitution, we indeed find that

$$-a\left[y(0) - \frac{b}{a}\right]e^{-at} + a\left\{\left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}\right\} = b$$

Thus our solution is correct, provided it also satisfies the initial condition. To check the latter, let us set $t = 0$ in the solution (15.5′). Since the result

$$y(0) = \left[y(0) - \frac{b}{a}\right] + \frac{b}{a} = y(0)$$

is an identity, the initial condition is indeed satisfied.

It is recommended that, as a final step in the process of solving a differential equation, you make it a habit to check the validity of your answer by making sure (1) that the derivative of the time path $y(t)$ is consistent with the given differential equation and (2) that the definite solution satisfies the initial condition.

EXERCISE 15.1

1. Find y_c, y_p, the general solution, and the definite solution, given:

   (a) $\frac{dy}{dt} + 4y = 12;\ y(0) = 2$

   (b) $\frac{dy}{dt} - 2y = 0;\ y(0) = 9$

   (c) $\frac{dy}{dt} + 10y = 15;\ y(0) = 0$

   (d) $2\frac{dy}{dt} + 4y = 6;\ y(0) = 1\frac{1}{2}$

2. Check the validity of your answers to Prob. 1.
3. Find the solution of each of the following by using an appropriate formula developed in the text:

   (a) $\frac{dy}{dt} + y = 4;\ y(0) = 0$

   (b) $\frac{dy}{dt} = 23;\ y(0) = 1$

   (c) $\frac{dy}{dt} - 5y = 0;\ y(0) = 6$

   (d) $\frac{dy}{dt} + 3y = 2;\ y(0) = 4$

   (e) $\frac{dy}{dt} - 7y = 7;\ y(0) = 7$

   (f) $3\frac{dy}{dt} + 6y = 5;\ y(0) = 0$

4. Check the validity of your answers to Prob. 3.

15.2 Dynamics of Market Price

In the (macro) Domar growth model, we found an application of the *homogeneous* case of linear differential equations of the first order. To illustrate the *nonhomogeneous* case, let us present a (micro) dynamic model of the market.

The Framework

Suppose that, for a particular commodity, the demand and supply functions are as follows:

$$\begin{aligned} Q_d &= \alpha - \beta P \qquad (\alpha, \beta > 0) \\ Q_s &= -\gamma + \delta P \qquad (\gamma, \delta > 0) \end{aligned} \tag{15.8}$$

Then, according to (3.4), the equilibrium price should be†

$$P^* = \frac{\alpha + \gamma}{\beta + \delta} \qquad (= \text{some positive constant}) \tag{15.9}$$

If it happens that the initial price $P(0)$ is precisely at the level of P^*, the market will clearly be in equilibrium already, and no dynamic analysis will be needed. In the more interesting case of $P(0) \neq P^*$, however, P^* is attainable (if ever) only after a due process of adjustment, during which not only will price change over time but Q_d and Q_s, being functions of P, must change over time as well. In this light, then, the price and quantity variables can *all* be taken to be *functions of time.*

Our dynamic question is this: Given sufficient time for the adjustment process to work itself out, does it tend to bring price to the equilibrium level P^*? That is, does the time path $P(t)$ tend to converge to P^*, as $t \to \infty$?

The Time Path

To answer this question, we must first find the time path $P(t)$. But that, in turn, requires a specific pattern of price change to be prescribed first. In general, price changes are governed by the relative strength of the demand and supply forces in the market. Let us assume, for the sake of simplicity, that the rate of price change (with respect to time) at any moment is always directly proportional to the *excess demand* $(Q_d - Q_s)$ prevailing at that moment. Such a pattern of change can be expressed symbolically as

$$\frac{dP}{dt} = j(Q_d - Q_s) \qquad (j > 0) \tag{15.10}$$

where j represents a (constant) *adjustment coefficient.* With this pattern of change, we can have $dP/dt = 0$ if and only if $Q_d = Q_s$. In this connection, it may be instructive to note two senses of the term *equilibrium price:* the intertemporal sense (P being constant over time) and the market-clearing sense (the equilibrium price being one that equates Q_d and Q_s). In the present model, the two senses happen to coincide with each other, but this may not be true of all models.

By virtue of the demand and supply functions in (15.8), we can express (15.10) specifically in the form

$$\frac{dP}{dt} = j(\alpha - \beta P + \gamma - \delta P) = j(\alpha + \gamma) - j(\beta + \delta)P$$

or

$$\frac{dP}{dt} + j(\beta + \delta)P = j(\alpha + \gamma) \tag{15.10'}$$

† We have switched from the symbols (a, b, c, d) of (3.4) to $(\alpha, \beta, \gamma, \delta)$ here to avoid any possible confusion with the use of a and b as parameters in the differential equation (15.4) which we shall presently apply to the market model.

Since this is precisely in the form of the differential equation (15.4), and since the coefficient of P is nonzero, we can apply the solution formula (15.5′) and write the solution—the time path of price—as

$$P(t) = \left[P(0) - \frac{\alpha+\gamma}{\beta+\delta}\right]e^{-j(\beta+\delta)t} + \frac{\alpha+\gamma}{\beta+\delta}$$
$$= [P(0) - P^*]e^{-kt} + P^* \qquad [\text{by (15.9)};\ k \equiv j(\beta+\delta)] \qquad \textbf{(15.11)}$$

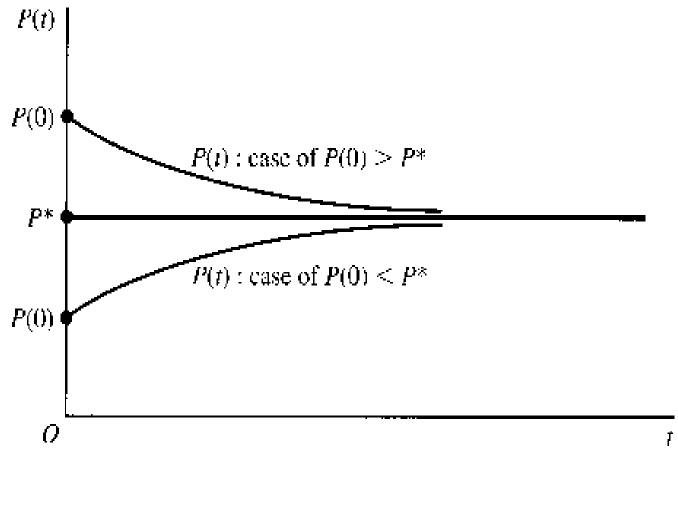
The Dynamic Stability of Equilibrium

In the end, the question originally posed, namely, whether $P(t) \to P^*$ as $t \to \infty$, amounts to the question of whether the first term on the right of (15.11) will tend to zero as $t \to \infty$. Since $P(0)$ and P^* are both constant, the key factor will be the exponential expression e^{-kt}. In view of the fact that $k > 0$, that expression does tend to zero as $t \to \infty$. Consequently, on the assumptions of our model, the time path will indeed lead the price toward the equilibrium position. In a situation of this sort, where the time path of the relevant variable $P(t)$ *converges* to the level P^* interpreted here in its role as the intertemporal (rather than market-clearing) equilibrium—the equilibrium is said to be *dynamically stable*.

The concept of dynamic stability is an important one. Let us examine it further by a more detailed analysis of (15.11). Depending on the relative magnitudes of $P(0)$ and P^*, the solution (15.11) really encompasses three possible cases. The first is $P(0) = P^*$, which implies $P(t) = P^*$. In that event, the time path of price can be drawn as the horizontal straight line in Fig. 15.1. As mentioned earlier, the attainment of equilibrium is in this case a fait accompli. Second, we may have $P(0) > P^*$. In this case, the first term on the right of (15.11) is positive, but it will decrease as the increase in t lowers the value of e^{-kt}. Thus the time path will approach the equilibrium level P^* from above, as illustrated by the top curve in Fig. 15.1. Third, in the opposite case of $P(0) < P^*$, the equilibrium level P^* will be approached from below, as illustrated by the bottom curve in the same figure. In general, to have dynamic stability, the *deviation* of the time path from equilibrium must either be identically zero (as in case 1) or steadily decrease with time (as in cases 2 and 3).

A comparison of (15.11) with (15.5′) tells us that the P^* term, the counterpart of b/a, is nothing but the particular integral y_p, whereas the exponential term is the (definitized) complementary function y_c. Thus, we now have an economic interpretation for y_c and y_p; y_p represents the *intertemporal equilibrium level* of the relevant variable, and y_c is the *deviation from equilibrium*. Dynamic stability requires the asymptotic vanishing of the complementary function as t becomes infinite.

FIGURE 15.1



In this model, the particular integral is a constant, so we have a *stationary equilibrium* in the intertemporal sense, represented by P^*. If the particular integral is nonconstant, as in (15.7′), on the other hand, we may interpret it as a *moving equilibrium*.

An Alternative Use of the Model

What we have done in the preceding is to analyze the dynamic stability of equilibrium (the convergence of the time path), given certain sign specifications for the parameters. An alternative type of inquiry is: In order to ensure dynamic stability, what specific restrictions must be imposed upon the parameters?

The answer to that is contained in the solution (15.11). If we allow $P(0) \neq P^*$, we see that the first (y_c) term in (15.11) will tend to zero as $t \to \infty$ if and only if $k > 0$—that is, if and only if

$$j(\beta + \delta) > 0$$

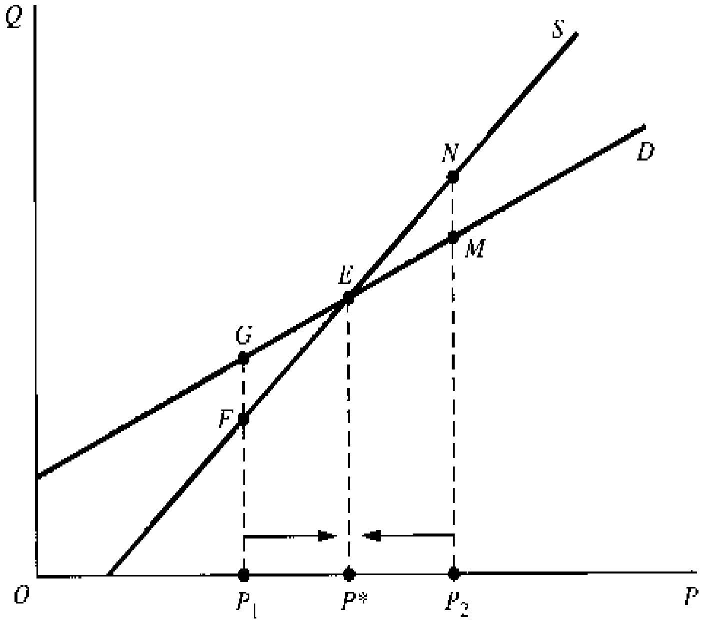
Thus, we can take this last inequality as the required restriction on the parameters j (the adjustment coefficient of price), β (the negative of the slope of the demand curve, plotted with Q on the *vertical* axis), and δ (the slope of the supply curve, plotted similarly).

In case the price adjustment is of the "normal" type, with $j > 0$, so that excess demand drives price up rather than down, then this restriction becomes merely $(\beta + \delta) > 0$ or, equivalently,

$$\delta > -\beta$$

To have dynamic stability in that event, the slope of the supply must exceed the slope of the demand. When both demand and supply are normally sloped ($-\beta < 0$, $\delta > 0$), as in (15.8), this requirement is obviously met. But even if one of the curves is sloped "perversely," the condition may still be fulfilled, such as when $\delta = 1$ and $-\beta = 1/2$ (positively sloped demand). The latter situation is illustrated in Fig. 15.2, where the equilibrium price P^* is, as usual, determined by the point of intersection of the two curves. If the initial price happens to be at P_1, then Q_d (distance P_1G) will exceed Q_s (distance P_1F), and the excess demand (FG) will drive price up. On the other hand, if price is initially at P_2, then

FIGURE 15.2



there will be a *negative* excess demand MN, which will drive the price down. As the two arrows in the figure show, therefore, the price adjustment in this case will be *toward* the equilibrium, no matter which side of P^* we start from. We should emphasize, however, that while these arrows can display the direction, they are incapable of indicating the magnitude of change. Thus Fig. 15.2 is basically static, not dynamic, in nature, and can serve only to illustrate, not to replace, the dynamic analysis presented.

EXERCISE 15.2

1. If both the demand and supply in Fig. 15.2 are negatively sloped instead, which curve should be steeper in order to have dynamic stability? Does your answer conform to the criterion $\delta > -\beta$?
2. Show that (15.10′) can be rewritten as $dP/dt + k(P - P^*) = 0$. If we let $P - P^* \equiv \Delta$ (signifying deviation), so that $d\Delta/dt = dP/dt$, the differential equation can be further rewritten as
$$\frac{d\Delta}{dt} + k\Delta = 0$$
Find the time path $\Delta(t)$, and discuss the condition for dynamic stability.
3. The dynamic market model discussed in this section is closely patterned after the static one in Sec. 3.2. What specific new feature is responsible for transforming the static model into a dynamic one?
4. Let the demand and supply be
$$Q_d = \alpha - \beta P + \sigma \frac{dP}{dt} \qquad Q_s = -\gamma + \delta P \qquad (\alpha, \beta, \gamma, \delta > 0)$$
(*a*) Assuming that the rate of change of price over time is directly proportional to the excess demand, find the time path $P(t)$ (general solution).
(*b*) What is the intertemporal equilibrium price? What is the market-clearing equilibrium price?
(*c*) What restriction on the parameter σ would ensure dynamic stability?
5. Let the demand and supply be
$$Q_d = \alpha - \beta P - \eta \frac{dP}{dt} \qquad Q_s = \delta P \qquad (\alpha, \beta, \eta, \delta > 0)$$
(*a*) Assuming that the market is cleared at every point of time, find the time path $P(t)$ (general solution).
(*b*) Does this market have a dynamically stable intertemporal equilibrium price?
(*c*) The assumption of the present model that $Q_d = Q_s$ for all t is identical with that of the static market model in Sec. 3.2. Nevertheless, we still have a dynamic model here. How come?

15.3 Variable Coefficient and Variable Term

In the more general case of a first-order linear differential equation
$$\frac{dy}{dt} + u(t)y = w(t) \tag{15.12}$$

$u(t)$ and $w(t)$ represent a variable coefficient and a variable term, respectively. How do we find the time path $y(t)$ in this case?

The Homogeneous Case

For the homogeneous case, where $w(t) = 0$, the solution is still easy to obtain. Since the differential equation is in the form

$$\frac{dy}{dt} + u(t)y = 0 \qquad \text{or} \qquad \frac{1}{y}\frac{dy}{dt} = -u(t) \tag{15.13}$$

we have, by integrating both sides in turn with respect to t,

$$\text{Left side} = \int \frac{1}{y}\frac{dy}{dt}\,dt = \int \frac{dy}{y} = \ln y + c \qquad (\text{assuming } y > 0)$$

$$\text{Right side} = \int -u(t)\,dt = -\int u(t)\,dt$$

In the latter, the integration process cannot be carried further because $u(t)$ has not been given a specific form; thus we have to settle for just a general integral expression. When the two sides are equated, the result is

$$\ln y = -c - \int u(t)\,dt$$

Then the desired y path can be obtained by taking the antilog of $\ln y$:

$$y(t) = e^{\ln y} = e^{-c}e^{-\int u(t)\,dt} = Ae^{-\int u(t)\,dt} \qquad \text{where } A \equiv e^{-c} \tag{15.14}$$

This is the general solution of the differential equation (15.13).

To highlight the variable nature of the coefficient $u(t)$, we have so far explicitly written out the argument t. For notational simplicity, however, we shall from here on omit the argument and shorten $u(t)$ to u.

As compared with the general solution (15.3) for the constant-coefficient case, the only modification in (15.14) is the replacement of the e^{-at} expression by the more complicated expression $e^{-\int u\,dt}$. The rationale behind this change can be better understood if we interpret the at term in e^{-at} as an integral: $\int a\,dt = at$ (plus a constant which can be absorbed into the A term, since e raised to a constant power is again a constant). In this light, the difference between the two general solutions in fact turns into a similarity. For in both cases we are taking the coefficient of the y term in the differential equation—a constant term a in one case, and a variable term u in the other—and integrating that with respect to t, and then taking the negative of the resulting integral as the exponent of e.

Once the general solution is obtained, it is a relatively simple matter to get the definite solution with the help of an appropriate initial condition.

Example 1

Find the general solution of the equation $\frac{dy}{dt} + 3t^2 y = 0$. Here we have $u = 3t^2$, and $\int u\,dt = \int 3t^2\,dt = t^3 + c$. Therefore, by (15.14), we may write the solution as

$$y(t) = Ae^{-(t^3+c)} = Ae^{-t^3}e^{-c} = Be^{-t^3} \qquad \text{where } B \equiv Ae^{-c}$$

Observe that if we had omitted the constant of integration c, we would have lost no information, because then we would have obtained $y(t) = Ae^{-t^3}$, which is really the identical solution since A and B both represent arbitrary constants. In other words, the expression e^{-c}, where the constant c makes its only appearance, can always be subsumed under the other constant A.

The Nonhomogeneous Case

For the nonhomogeneous case, where $w(t) \neq 0$, the solution is not as easy to obtain. We shall try to find that solution via the concept of exact differential equations, to be discussed in Sec. 15.4. It does no harm, however, to state the result here first: Given the differential equation (15.12), the general solution is

$$y(t) = e^{-\int u\,dt}\left(A + \int we^{\int u\,dt}\,dt\right) \qquad (15.15)$$

where A is an arbitrary constant that can be definitized if we have an appropriate initial condition.

It is of interest that this general solution, like the solution in the constant-coefficient constant-term case, again consists of two additive components. Furthermore, one of these two, $Ae^{-\int u\,dt}$, is nothing but the general solution of the reduced (homogeneous) equation, derived earlier in (15.14), and is therefore in the nature of a complementary function.

Example 2

Find the general solution of the equation $\frac{dy}{dt} + 2ty = t$. Here we have

$$u = 2t \qquad w = t \qquad \text{and} \qquad \int u\,dt = t^2 + k \qquad (k \text{ arbitrary})$$

Thus, by (15.15), we have

$$\begin{aligned} y(t) &= e^{-(t^2+k)}\left(A + \int te^{t^2+k}\,dt\right) \\ &= e^{-t^2}e^{-k}\left(A + e^k \int te^{t^2}\,dt\right) \\ &= Ae^{-k}e^{-t^2} + e^{-t^2}\left(\frac{1}{2}e^{t^2} + c\right) \qquad [e^{-k}e^k = 1] \\ &= (Ae^{-k} + c)e^{-t^2} + \frac{1}{2} \\ &= Be^{-t^2} + \frac{1}{2} \qquad \text{where } B \equiv Ae^{-k} + c \text{ is arbitrary} \end{aligned}$$

The validity of this solution can again be checked by differentiation.

It is interesting to note that, in this example, we could again have omitted the constant of integration k, as well as the constant of integration c, without affecting the final outcome. This is because both k and c may be subsumed under the arbitrary constant B in the final solution. You are urged to try out the simpler process of applying (15.15) without using the constants k and c, and verify that the same solution will emerge.

Example 3

Solve the equation $\frac{dy}{dt} + 4ty = 4t$. This time we shall omit the constants of integration. Since

$$u = 4t \qquad w = 4t \qquad \text{and} \qquad \int u\,dt = 2t^2 \qquad [\text{constant omitted}]$$

the general solution is, by (15.15),

$$\begin{aligned} y(t) &= e^{-2t^2}\left(A + \int 4te^{2t^2}\,dt\right) = e^{-2t^2}\left(A + e^{2t^2}\right) \qquad [\text{constant omitted}] \\ &= Ae^{-2t^2} + 1 \end{aligned}$$

As may be expected, the omission of the constants of integration serves to simplify the procedure substantially.

The differential equation $\frac{dy}{dt} + uy = w$ in (15.12) is more general than the equation $\frac{dy}{dt} + ay = b$ in (15.4), since u and w are not necessarily constant, as are a and b. Accordingly, solution formula (15.15) is also more general than solution formula (15.5). In fact, when we set $u = a$ and $w = b$, (15.15) should reduce to (15.5). This is indeed the case. For when we have

$$u = a \qquad w = b \qquad \text{and} \qquad \int u\,dt = at \qquad \text{[constant omitted]}$$

then (15.15) becomes

$$y(t) = e^{-at}\left(A + \int be^{at}\,dt\right) = e^{-at}\left(A + \frac{b}{a}e^{at}\right) \qquad \text{[constant omitted]}$$

$$= Ae^{-at} + \frac{b}{a}$$

which is identical with (15.5).

EXERCISE 15.3

Solve the following first-order linear differential equations; if an initial condition is given, definitize the arbitrary constant:

1. $\frac{dy}{dt} + 5y = 15$
2. $\frac{dy}{dt} + 2ty = 0$
3. $\frac{dy}{dt} + 2ty = t$; $y(0) = \frac{3}{2}$
4. $\frac{dy}{dt} + t^2y = 5t^2$; $y(0) = 6$
5. $2\frac{dy}{dt} + 12y + 2e^t = 0$; $y(0) = \frac{6}{7}$
6. $\frac{dy}{dt} + y = t$

15.4 Exact Differential Equations

We shall now introduce the concept of exact differential equations and use the solution method pertaining thereto to obtain the solution formula (15.15) previously cited for the differential equation (15.12). Even though our immediate purpose is to use it to solve a *linear* differential equation, an exact differential equation can be either linear or nonlinear by itself.

Exact Differential Equations

Given a function of two variables $F(y, t)$, its total differential is

$$dF(y,t) = \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial t}\,dt$$