14.1 Arithmetic Sequences

OBJECTIVES

1. Write the terms of a sequence
2. Find the general term for an arithmetic sequence
3. Find a specific term for an arithmetic sequence
4. Find the sum of the terms of an arithmetic sequence
5. Determine the sum indicated by summation notation

An **infinite sequence** is a function with a domain that is the set of positive integers. For example, consider the function defined by the equation

$$f(n) = 5n + 1$$

in which the domain is the set of positive integers. If we substitute the numbers of the domain in order, starting with 1, we can list the resulting ordered pairs:

$$(1, 6) \qquad (2, 11) \qquad (3, 16) \qquad (4, 21) \qquad (5, 26)$$

and so on. However, because we know we are using the domain of positive integers in order, starting with 1, there is no need to use ordered pairs. We can simply express the infinite sequence as

$$6, 11, 16, 21, 26, \ldots$$

Often the letter a is used to represent sequential functions, and the functional value of a at n is written a_n (this is read "a sub n") instead of $a(n)$. The sequence is then expressed as

$$a_1, a_2, a_3, a_4, \ldots$$

where a_1 is the *first term*, a_2 is the *second term*, a_3 is the *third term*, and so on. The expression a_n, which defines the sequence, is called the *general term* of the sequence. Knowing the general term of a sequence enables us to find as many terms of the sequence as needed and also to find any specific terms. Consider the following example.

Classroom Example
Find the first five terms of the sequence in which $a_n = 3n^3 + 1$, and then find the 10th term.

EXAMPLE 1

Find the first five terms of the sequence when $a_n = 2n^2 - 3$; find the 20th term.

Solution

The first five terms are generated by replacing n with 1, 2, 3, 4, and 5:

$$a_1 = 2(1)^2 - 3 = -1 \qquad a_2 = 2(2)^2 - 3 = 5$$
$$a_3 = 2(3)^2 - 3 = 15 \qquad a_4 = 2(4)^2 - 3 = 29$$
$$a_5 = 2(5)^2 - 3 = 47$$

The first five terms are thus -1, 5, 15, 29, and 47. The 20th term is

$$a_{20} = 2(20)^2 - 3 = 797$$

Arithmetic Sequences

An **arithmetic sequence** (also called an **arithmetic progression**) is a sequence that has a common difference between successive terms. The following are examples of arithmetic sequences:

$$1, 8, 15, 22, 29, \ldots$$
$$4, 7, 10, 13, 16, \ldots$$

$4, 1, -2, -5, -8, \ldots$

$-1, -6, -11, -16, -21, \ldots$

The common difference in the first sequence is 7. That is, $8 - 1 = 7$, $15 - 8 = 7$, $22 - 15 = 7$, $29 - 22 = 7$, and so on. The common differences for the next three sequences are 3, -3, and -5, respectively.

In a more general setting, we say that the sequence

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$$

is an arithmetic sequence if and only if there is a real number d such that

$$a_{k+1} - a_k = d$$

for every positive integer k. The number d is called the **common difference**.

From the definition, we see that $a_{k+1} = a_k + d$. In other words, we can generate an arithmetic sequence that has a common difference of d by starting with a first term a_1 and then simply adding d to each successive term.

First term:	a_1	
Second term:	$a_1 + d$	
Third term:	$a_1 + 2d$	$(a_1 + d) + d = a_1 + 2d$
Fourth term:	$a_1 + 3d$	
.		
.		
.		
nth term:	$a_1 + (n - 1)d$	

Thus the **general term** of an arithmetic sequence is given by

$$a_n = a_1 + (n - 1)d$$

where a_1 is the first term, and d is the common difference. This formula for the general term can be used to solve a variety of problems involving arithmetic sequences.

Classroom Example
Find the general term of the arithmetic sequence $-4, 3, 10, 17, \ldots$.

EXAMPLE 2 Find the general term of the arithmetic sequence $6, 2, -2, -6, \ldots$.

Solution

The common difference, d, is $2 - 6 = -4$, and the first term, a_1, is 6. Substitute these values into $a_n = a_1 + (n - 1)d$ and simplify to obtain

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ &= 6 + (n - 1)(-4) \\ &= 6 - 4n + 4 \\ &= -4n + 10 \end{aligned}$$

Classroom Example
Find the 15th term of the arithmetic sequence $11, 8, 5, 2, \ldots$.

EXAMPLE 3 Find the 40th term of the arithmetic sequence $1, 5, 9, 13, \ldots$.

Solution

Using $a_n = a_1 + (n - 1)d$, we obtain

$$\begin{aligned} a_{40} &= 1 + (40 - 1)4 \\ &= 1 + (39)(4) \\ &= 157 \end{aligned}$$

Classroom Example
Find the first term of the arithmetic sequence if the fifth term is 28 and the twelfth term is 63.

EXAMPLE 4

Find the first term of the arithmetic sequence if the fourth term is 26 and the ninth term is 61.

Solution

Using $a_n = a_1 + (n - 1)d$ with $a_4 = 26$ (the fourth term is 26) and $a_9 = 61$ (the ninth term is 61), we have

$$26 = a_1 + (4 - 1)d = a_1 + 3d$$

$$61 = a_1 + (9 - 1)d = a_1 + 8d$$

Solving the system of equations

$$\begin{pmatrix} a_1 + 3d = 26 \\ a_1 + 8d = 61 \end{pmatrix}$$

yields $a_1 = 5$ and $d = 7$. Thus the first term is 5.

Sums of Arithmetic Sequences

We often use sequences to solve problems, so we need to be able to find the sum of a certain number of terms of the sequence. Before we develop a general-sum formula for arithmetic sequences, let's consider an approach to a specific problem that we can then use in a general setting.

Classroom Example
Find the sum of the first 25 positive integers.

EXAMPLE 5

Find the sum of the first 100 positive integers.

Solution

We are being asked to find the sum of $1 + 2 + 3 + 4 + \cdots + 100$. Rather than adding in the usual way, we will find the sum in the following manner: Let's simply write the indicated sum forward and backward, and then add in columns:

$$\begin{array}{rcrcrcrcccr} 1 & + & 2 & + & 3 & + & 4 & + & \cdots & + & 100 \\ 100 & + & 99 & + & 98 & + & 97 & + & \cdots & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

We have produced 100 sums of 101. However, this result is double the amount we want because we wrote the sum twice. To find the sum of just the numbers 1 to 100, we need to multiply 100 by 101 and then divide by 2:

$$\frac{100(101)}{2} = \frac{\overset{50}{\cancel{100}}(101)}{\cancel{2}} = 5050$$

Thus the sum of the first 100 positive integers is 5050.

The *forward–backward* approach we used in Example 5 can be used to develop a formula for finding the sum of the first n terms of any arithmetic sequence. Consider an arithmetic sequence $a_1, a_2, a_3, a_4, \ldots, a_n$ with a common difference of d. Use S_n to represent the sum of the first n terms, and proceed as follows:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n$$

Now write this sum in reverse:

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1$$

Add the two equations to produce

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n)$$

That is, we have n sums $a_1 + a_n$, so

$$2S_n = n(a_1 + a_n)$$

from which we obtain a **sum formula**:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Using the nth-term formula and/or the sum formula, we can solve a variety of problems involving arithmetic sequences.

Classroom Example
Find the sum of the first 20 terms of the arithmetic sequence 2, 7, 12, 17, ...

EXAMPLE 6

Find the sum of the first 30 terms of the arithmetic sequence 3, 7, 11, 15, ...

Solution

To use the formula $S_n = \frac{n(a_1 + a_n)}{2}$, we need to know the number of terms (n), the first term (a_1), and the last term (a_n). We are given the number of terms and the first term, so we need to find the last term. Using $a_n = a_1 + (n - 1)d$, we can find the 30th term.

$$a_{30} = 3 + (30 - 1)4 = 3 + 29(4) = 119$$

Now we can use the sum formula.

$$S_{30} = \frac{30(3 + 119)}{2} = 1830$$

Classroom Example
Find the sum $3 + 11 + 19 + \cdots + 283$.

EXAMPLE 7

Find the sum $7 + 10 + 13 + \cdots + 157$.

Solution

To use the sum formula, we need to know the number of terms. Applying the nth-term formula will give us that information:

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ 157 &= 7 + (n - 1)3 \\ 157 &= 7 + 3n - 3 \\ 157 &= 3n + 4 \\ 153 &= 3n \\ 51 &= n \end{aligned}$$

Now we can use the sum formula:

$$S_{51} = \frac{51(7 + 157)}{2} = 4182$$

Keep in mind that we developed the sum formula for an arithmetic sequence by using the forward–backward technique, which we had previously used on a specific problem. Now that we have the sum formula, we have two choices when solving problems. We can either memorize the formula and use it or simply use the forward–backward technique. If you choose to use the formula and some day you forget it, don't panic. Just use the forward–backward technique. In other words, understanding the development of a formula often enables you to do problems even when you forget the formula itself.

Summation Notation

Sometimes a special notation is used to indicate the sum of a certain number of terms of a sequence. The capital Greek letter *sigma,* Σ, is used as a **summation symbol**. For example,

$$\sum_{i=1}^{5} a_i$$

represents the sum $a_1 + a_2 + a_3 + a_4 + a_5$. The letter i is frequently used as the **index of summation**; the letter i takes on all integer values from the lower limit to the upper limit, inclusive. Thus

$$\sum_{i=1}^{4} b_i = b_1 + b_2 + b_3 + b_4$$

$$\sum_{i=3}^{7} a_i = a_3 + a_4 + a_5 + a_6 + a_7$$

$$\sum_{i=1}^{15} i^2 = 1^2 + 2^2 + 3^2 + \cdots + 15^2$$

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

If $a_1, a_2, a_3, \ldots$ represents an arithmetic sequence, we can now write the sum formula

$$\sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n)$$

Classroom Example

Find the sum $\sum_{i=1}^{28} (5i - 3)$.

EXAMPLE 8 Find the sum $\sum_{i=1}^{50} (3i + 4)$.

Solution

This indicated sum means

$$\sum_{i=1}^{50} (3i + 4) = [3(1) + 4] + [3(2) + 4] + [3(3) + 4] + \cdots + [3(50) + 4]$$

$$= 7 + 10 + 13 + \cdots + 154$$

Because this is an indicated sum of an arithmetic sequence, we can use our sum formula:

$$S_{50} = \frac{50}{2}(7 + 154) = 4025$$

Classroom Example

Find the sum $\sum_{i=4}^{9} 3i^2$.

EXAMPLE 9 Find the sum $\sum_{i=2}^{7} 2i^2$.

Solution

This indicated sum means

$$\sum_{i=2}^{7} 2i^2 = 2(2)^2 + 2(3)^2 + 2(4)^2 + 2(5)^2 + 2(6)^2 + 2(7)^2$$

$$= 8 + 18 + 32 + 50 + 72 + 98$$

This is not the indicated sum of an *arithmetic* sequence; therefore let's simply add the numbers in the usual way. The sum is 278.

Example 9 suggests a word of caution. Be sure to analyze the sequence of numbers that is represented by the summation symbol. You may or may not be able to use a formula for adding the numbers.

Concept Quiz 14.1

For Problems 1–8, answer true or false.

1. An infinite sequence is a function whose domain is the set of all real numbers.

2. An arithmetic sequence is a sequence that has a common difference between successive terms.

3. The sequence 2, 4, 8, 16, . . . is an arithmetic sequence.

4. The odd whole numbers form an arithmetic sequence.

5. The terms of an arithmetic sequence are always positive.

6. The 6th term of an arithmetic sequence is equal to the first term plus 6 times the common difference.

7. The sum formula for n terms of an arithmetic sequence is n times the average of the first and last terms.

8. The indicated sum $\sum_{i=1}^{4}(2i - 7)^2$ is the sum of the first four terms of an arithmetic sequence.

Problem Set 14.1

For Problems 1–10, write the first five terms of the sequence that has the indicated general term. **(Objective 1)**

1. $a_n = 3n - 7$

2. $a_n = 5n - 2$

3. $a_n = -2n + 4$

4. $a_n = -4n + 7$

5. $a_n = 3n^2 - 1$

6. $a_n = 2n^2 - 6$

7. $a_n = n(n - 1)$

8. $a_n = (n + 1)(n + 2)$

9. $a_n = 2^{n+1}$

10. $a_n = 3^{n-1}$

11. Find the 15th and 30th terms of the sequence when $a_n = -5n - 4$.

12. Find the 20th and 50th terms of the sequence when $a_n = -n - 3$.

13. Find the 25th and 50th terms of the sequence when $a_n = (-1)^{n+1}$.

14. Find the 10th and 15th terms of the sequence when $a_n = -n^2 - 10$.

For Problems 15–24, find the general term (the nth term) for each arithmetic sequence. **(Objective 2)**

15. 11, 13, 15, 17, 19, . . .

16. 7, 10, 13, 16, 19, . . .

17. 2, −1, −4, −7, −10, . . .

18. 4, 2, 0, −2, −4, . . .

19. $\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots$

20. $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

21. 2, 6, 10, 14, 18, . . .

22. 2, 7, 12, 17, 22, . . .

23. −3, −6, −9, −12, −15, . . .

24. −4, −8, −12, −16, −20, . . .

For Problems 25–30, find the required term for each arithmetic sequence. **(Objective 3)**

25. The 15th term of 3, 8, 13, 18, . . .

26. The 20th term of 4, 11, 18, 25, . . .

27. The 30th term of 15, 26, 37, 48, . . .

28. The 35th term of 9, 17, 25, 33, . . .

29. The 52nd term of $1, \frac{5}{3}, \frac{7}{3}, 3, \ldots$

30. The 47th term of $\frac{1}{2}, \frac{5}{4}, 2, \frac{11}{4}, \ldots$

For Problems 31–42, solve each problem.

31. If the 6th term of an arithmetic sequence is 12 and the 10th term is 16, find the first term.

32. If the 5th term of an arithmetic sequence is 14 and the 12th term is 42, find the first term.

33. If the 3rd term of an arithmetic sequence is 20 and the 7th term is 32, find the 25th term.

34. If the 5th term of an arithmetic sequence is −5 and the 15th term is −25, find the 50th term.

35. Find the sum of the first 50 terms of the arithmetic sequence 5, 7, 9, 11, 13, ...

36. Find the sum of the first 30 terms of the arithmetic sequence 0, 2, 4, 6, 8, ...

37. Find the sum of the first 40 terms of the arithmetic sequence 2, 6, 10, 14, 18, ...

38. Find the sum of the first 60 terms of the arithmetic sequence −2, 3, 8, 13, 18, ...

39. Find the sum of the first 75 terms of the arithmetic sequence 5, 2, −1, −4, −7, ...

40. Find the sum of the first 80 terms of the arithmetic sequence 7, 3, −1, −5, −9, ...

41. Find the sum of the first 50 terms of the arithmetic sequence $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

42. Find the sum of the first 100 terms of the arithmetic sequence $-\frac{1}{3}, \frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, \ldots$

For Problems 43–50, find the indicated sum. **(Objective 4)**

43. $1 + 5 + 9 + 13 + \cdots + 197$

44. $3 + 8 + 13 + 18 + \cdots + 398$

45. $2 + 8 + 14 + 20 + \cdots + 146$

46. $6 + 9 + 12 + 15 + \cdots + 93$

47. $(-7) + (-10) + (-13) + (-16) + \cdots + (-109)$

48. $(-5) + (-9) + (-13) + (-17) + \cdots + (-169)$

49. $(-5) + (-3) + (-1) + 1 + \cdots + 119$

50. $(-7) + (-4) + (-1) + 2 + \cdots + 131$

For Problems 51–58, solve each problem.

51. Find the sum of the first 200 odd whole numbers.

52. Find the sum of the first 175 positive even whole numbers.

53. Find the sum of all even numbers between 18 and 482, inclusive.

54. Find the sum of all odd numbers between 17 and 379, inclusive.

55. Find the sum of the first 30 terms of the arithmetic sequence with the general term $a_n = 5n - 4$.

56. Find the sum of the first 40 terms of the arithmetic sequence with the general term $a_n = 4n - 7$.

57. Find the sum of the first 25 terms of the arithmetic sequence with the general term $a_n = -4n - 1$.

58. Find the sum of the first 35 terms of the arithmetic sequence with the general term $a_n = -5n - 3$.

For Problems 59–70, find each sum. **(Objective 5)**

59. $\sum_{i=1}^{45} (5i + 2)$

60. $\sum_{i=1}^{38} (3i + 6)$

61. $\sum_{i=1}^{30} (-2i + 4)$

62. $\sum_{i=1}^{40} (-3i + 3)$

63. $\sum_{i=4}^{32} (3i - 10)$

64. $\sum_{i=6}^{47} (4i - 9)$

65. $\sum_{i=10}^{20} 4i$

66. $\sum_{i=15}^{30} (-5i)$

67. $\sum_{i=1}^{5} i^2$

68. $\sum_{i=1}^{6} (i^2 + 1)$

69. $\sum_{i=3}^{8} (2i^2 + i)$

70. $\sum_{i=4}^{7} (3i^2 - 2)$

Thoughts Into Words

71. Before developing the formula $a_n = a_1 + (n - 1)d$, we stated the equation $a_{k+1} - a_k = d$. In your own words, explain what this equation says.

72. Explain how to find the sum $1 + 2 + 3 + 4 + \cdots + 175$ without using the sum formula.

73. Explain in words how to find the sum of the first n terms of an arithmetic sequence.

74. Explain how one can tell that a particular sequence is an arithmetic sequence.

Further Investigations

The general term of a sequence can consist of one expression for certain values of n and another expression (or expressions) for other values of n. That is, a *multiple description* of the sequence can be given. For example,

$$a_n = \begin{cases} 2n + 3 & \text{for } n \text{ odd} \\ 3n - 2 & \text{for } n \text{ even} \end{cases}$$

means that we use $a_n = 2n + 3$ for $n = 1, 3, 5, 7, \ldots$, and we use $a_n = 3n - 2$ for $n = 2, 4, 6, 8, \ldots$. The first six terms of this sequence are 5, 4, 9, 10, 13, and 16.

For Problems 75–78, write the first six terms of each sequence.

75. $a_n = \begin{cases} 2n + 1 & \text{for } n \text{ odd} \\ 2n - 1 & \text{for } n \text{ even} \end{cases}$

76. $a_n = \begin{cases} \frac{1}{n} & \text{for } n \text{ odd} \\ n^2 & \text{for } n \text{ even} \end{cases}$