CHAPTER FOUR

Utility Maximization and Choice

In this chapter we examine the basic model of choice that economists use to explain individuals' behavior. That model assumes that individuals who are constrained by limited incomes will behave as though they are using their purchasing power in such a way as to achieve the highest utility possible. That is, individuals are assumed to behave as though they maximize utility subject to a budget constraint. Although the specific applications of this model are varied, as we will show, all are based on the same fundamental mathematical model, and all arrive at the same general conclusion: To maximize utility, individuals will choose bundles of commodities for which the rate of trade-off between any two goods (the *MRS*) is equal to the ratio of the goods' market prices. Market prices convey information about opportunity costs to individuals, and this information plays an important role in affecting the choices actually made.

Utility maximization and lightning calculations

Before starting a formal study of the theory of choice, it may be appropriate to dispose of two complaints noneconomists often make about the approach we will take. First is the charge that no real person can make the kinds of "lightning calculations" required for utility maximization. According to this complaint, when moving down a supermarket aisle, people just grab what is available with no real pattern or purpose to their actions. Economists are not persuaded by this complaint. They doubt that people behave randomly (everyone, after all, is bound by some sort of budget constraint), and they view the lightning calculation charge as misplaced. Recall, again, Friedman's pool player from Chapter 1. The pool player also cannot make the lightning calculations required to plan a shot according to the laws of physics, but those laws still predict the player's behavior. So too, as we shall see, the utility-maximization model predicts many aspects of behavior even though no one carries around a computer with his or her utility function programmed into it. To be precise, economists assume that people behave *as if* they made such calculations; thus, the complaint that the calculations cannot possibly be made is largely irrelevant. Still, in recent times economists have increasingly tried to model some of the behavioral complications that arise in the actual decisions people make. We look at some of these complications in a variety of problems throughout this book.

Altruism and selfishness

A second complaint against our model of choice is that it appears to be extremely selfish; no one, according to this complaint, has such solely self-centered goals. Although economists are probably more ready to accept self-interest as a motivating force than are other, more Utopian thinkers (Adam Smith observed, "We are not ready to suspect any person of being deficient in selfishness"[1]), this charge is also misplaced. Nothing in the utility-maximization model prevents individuals from deriving satisfaction from philanthropy or generally "doing good." These activities also can be assumed to provide utility. Indeed, economists have used the utility-maximization model extensively to study such issues as donating time and money to charity, leaving bequests to children, or even giving blood. One need not take a position on whether such activities are selfish or selfless because economists doubt people would undertake them if they were against their own best interests, broadly conceived.

AN INITIAL SURVEY

The general results of our examination of utility maximization can be stated succinctly as follows.

OPTIMIZATION PRINCIPLE

Utility maximization. To maximize utility, given a fixed amount of income to spend, an individual will buy those quantities of goods that exhaust his or her total income and for which the psychic rate of trade-off between any two goods (the *MRS*) is equal to the rate at which the goods can be traded one for the other in the marketplace.

That spending all one's income is required for utility maximization is obvious. Because extra goods provide extra utility (there is no satiation) and because there is no other use for income, to leave any unspent would be to fail to maximize utility. Throwing money away is not a utility-maximizing activity.

The condition specifying equality of trade-off rates requires a bit more explanation. Because the rate at which one good can be traded for another in the market is given by the ratio of their prices, this result can be restated to say that the individual will equate the *MRS* (of x for y) to the ratio of the price of x to the price of y (p_x / p_y). This equating of a personal trade-off rate to a market-determined trade-off rate is a result common to all individual utility-maximization problems (and to many other types of maximization problems). It will occur again and again throughout this text.

A numerical illustration

To see the intuitive reasoning behind this result, assume that it were not true that an individual had equated the *MRS* to the ratio of the prices of goods. Specifically, suppose that the individual's *MRS* is equal to 1 and that he or she is willing to trade 1 unit of x for 1 unit of y and remain equally well off. Assume also that the price of x is \$2 per unit and of y is \$1 per unit. It is easy to show that this person can be made better off. Suppose this person reduces x consumption by 1 unit and trades it in the market for 2 units of y. Only 1 extra unit of y was needed to keep this person as happy as before the trade—the second unit of y is a net addition to well-being. Therefore, the individual's spending could not have been allocated optimally in the first place. A similar method of reasoning can be used whenever the *MRS* and the price ratio p_x / p_y differ. The condition for maximum utility must be the equality of these two magnitudes.

[1]Adam Smith, *The Theory of Moral Sentiments* (1759; reprint, New Rochelle, NY: Arlington House, 1969), p. 446.

THE TWO-GOOD CASE: A GRAPHICAL ANALYSIS

This discussion seems eminently reasonable, but it can hardly be called a proof. Rather, we must now show the result in a rigorous manner and, at the same time, illustrate several other important attributes of the maximization process. First we take a graphic analysis; then we take a more mathematical approach.

Budget constraint

Assume that the individual has I dollars to allocate between good x and good y. If p_x is the price of good x and p_y is the price of good y, then the individual is constrained by

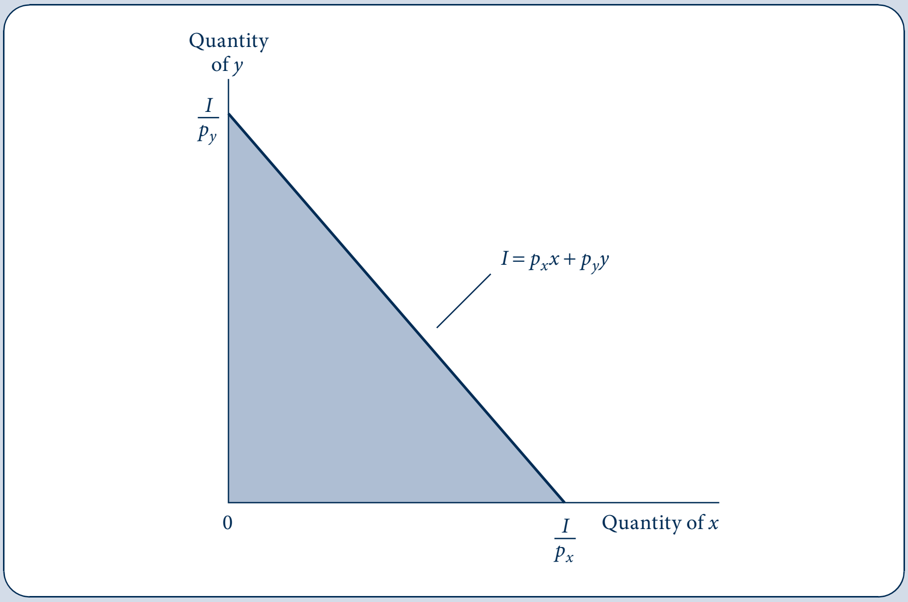
$$p_x x + p_y y \leq I. \quad \textbf{(4.1)}$$

That is, no more than I can be spent on the two goods in question. This budget constraint is shown graphically in Figure 4.1. This person can afford to choose only combinations of x and y in the shaded triangle of the figure. If all of I is spent on good x, it will buy I/p_x units of x. Similarly, if all is spent on y, it will buy I/p_y units of y. The slope of the constraint is easily seen to be $-p_x/p_y$. This slope shows how y can be traded for x in the market. If $p_x = 2$ and $p_y = 1$, then 2 units of y will trade for 1 unit of x.

FIGURE 4.1

The Individual's Budget Constraint for Two Goods

Those combinations of x and y that the individual can afford are shown in the shaded triangle. If, as we usually assume, the individual prefers more rather than less of every good, the outer boundary of this triangle is the relevant constraint where all the available funds are spent either on x or on y. The slope of this straight-line boundary is given by $-p_x/p_y$.



First-order conditions for a maximum

This budget constraint can be imposed on this person's indifference curve map to show the utility-maximization process. Figure 4.2 illustrates this procedure. The individual would be irrational to choose a point such as *A*; he or she can get to a higher utility level just by spending more of his or her income. The assumption of nonsatiation implies that a person should spend all his or her income to receive maximum utility. Similarly, by reallocating expenditures, the individual can do better than point *B*. Point *D* is out of the question because income is not large enough to purchase *D*. It is clear that the position of maximum utility is at point *C*, where the combination x^*, y^* is chosen. This is the only point on indifference curve U_2 that can be bought with *I* dollars; no higher utility level can be bought. *C* is a point of tangency between the budget constraint and the indifference curve. Therefore, at *C* we have

$$\begin{aligned}\text{slope of budget constraint} &= \frac{-p_x}{p_y} = \text{slope of indifference curve}\\ &= \left.\frac{dy}{dx}\right|_{U=\text{constant}}\end{aligned} \tag{4.2}$$

or

$$\frac{p_x}{p_y} = -\left.\frac{dy}{dx}\right|_{U=\text{constant}} = MRS(\text{of } x \text{ for } y). \tag{4.3}$$

FIGURE 4.2

A Graphical Demonstration of Utility Maximization

Point *C* represents the highest utility level that can be reached by the individual, given the budget constraint. Therefore, the combination x^*, y^* is the rational way for the individual to allocate purchasing power. Only for this combination of goods will two conditions hold: All available funds will be spent, and the individual's psychic rate of trade-off (*MRS*) will be equal to the rate at which the goods can be traded in the market (p_x/p_y).

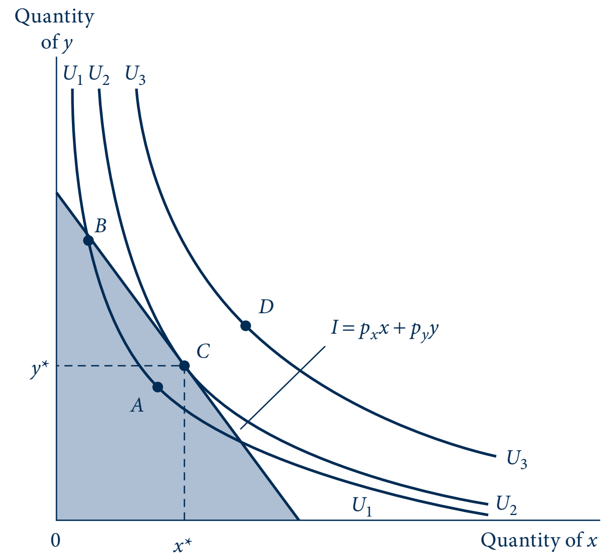
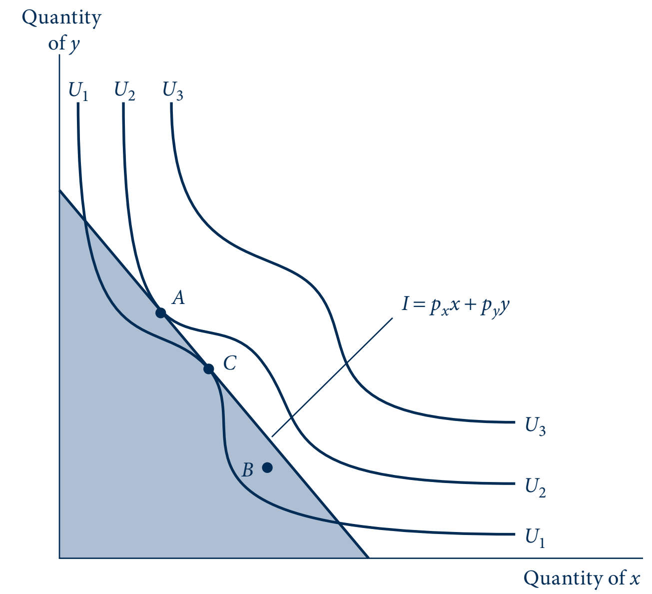


FIGURE 4.3

Example of an Indifference Curve Map for Which the Tangency Condition Does Not Ensure a Maximum

If indifference curves do not obey the assumption of a diminishing *MRS*, not all points of tangency (points for which $MRS - p_x/p_y$) may truly be points of maximum utility. In this example, tangency point *C* is inferior to many other points that can also be purchased with the available funds. In order that the necessary conditions for a maximum (i.e., the tangency conditions) also be sufficient, one usually assumes that the *MRS* is diminishing; that is, the utility function is strictly quasi-concave.



Our intuitive result is proved: For a utility maximum, all income should be spent, and the *MRS* should equal the ratio of the prices of the goods. It is obvious from the diagram that if this condition is not fulfilled, the individual could be made better off by reallocating expenditures.

Second-order conditions for a maximum

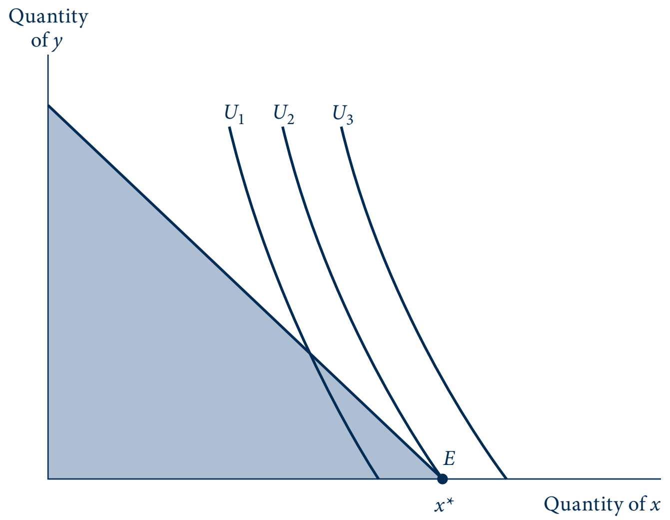
The tangency rule is only a necessary condition for a maximum. To see that it is not a sufficient condition, consider the indifference curve map shown in Figure 4.3. Here, a point of tangency (*C*) is inferior to a point of nontangency (*B*). Indeed, the true maximum is at another point of tangency (*A*). The failure of the tangency condition to produce an unambiguous maximum can be attributed to the shape of the indifference curves in Figure 4.3. If the indifference curves are shaped like those in Figure 4.2, no such problem can arise. But we have already shown that "normally" shaped indifference curves result from the assumption of a diminishing *MRS*. Therefore, if the *MRS* is assumed to be always diminishing, the condition of tangency is both a necessary and sufficient condition for a maximum.[2] Without this assumption, one would have to be careful in applying the tangency rule.

[2]As we saw in Chapters 2 and 3, this is equivalent to assuming that the utility function is quasi-concave. Because we will usually assume quasi-concavity, the necessary conditions for a constrained utility maximum will also be sufficient.

FIGURE 4.4

Corner Solution for Utility Maximization

With the preferences represented by this set of indifference curves, utility maximization occurs at E, where 0 amounts of good y are consumed. The first-order conditions for a maximum must be modified somewhat to accommodate this possibility.



Corner solutions

The utility-maximization problem illustrated in Figure 4.2 resulted in an "interior" maximum, in which positive amounts of both goods were consumed. In some situations individuals' preferences may be such that they can obtain maximum utility by choosing to consume no amount of one of the goods. If someone does not like hamburgers, there is no reason to allocate any income to their purchase. This possibility is reflected in Figure 4.4. There, utility is maximized at E, where $x = x^*$ and $y = 0$; thus, any point on the budget constraint where positive amounts of y are consumed yields a lower utility than does point E. Notice that at E the budget constraint is not precisely tangent to the indifference curve U_2. Instead, at the optimal point the budget constraint is flatter than U_2, indicating that the rate at which x can be traded for y in the market is lower than the individual's psychic trade-off rate (the *MRS*). At prevailing market prices the individual is more than willing to trade away y to get extra x. Because it is impossible in this problem to consume negative amounts of y, however, the physical limit for this process is the X-axis, along which purchases of y are 0. Hence as this discussion makes clear, it is necessary to amend the first-order conditions for a utility maximum a bit to allow for corner solutions of the type shown in Figure 4.4. Following our discussion of the general n-good case, we will use the mathematics from Chapter 2 to show how this can be accomplished.

THE n-GOOD CASE

The results derived graphically in the case of two goods carry over directly to the case of n goods. Again it can be shown that for an interior utility maximum, the *MRS* between any two goods must equal the ratio of the prices of these goods. To study this more general case, however, it is best to use some mathematics.

First-order conditions

With n goods, the individual's objective is to maximize utility from these n goods:

$$\text{utility} = U(x_1, x_2, \ldots, x_n), \tag{4.4}$$

subject to the budget constraint[3]

$$I = p_1x_1 + p_2x_2 + \cdots + p_nx_n \tag{4.5}$$

or

$$I - p_1x_1 - p_2x_2 - \cdots - p_nx_n = 0. \tag{4.6}$$

Following the techniques developed in Chapter 2 for maximizing a function subject to a constraint, we set up the Lagrangian expression

$$\mathscr{L} = U(x_1, x_2, \ldots, x_n) + \lambda(I - p_1x_1 - p_2x_2 - \cdots - p_nx_n). \tag{4.7}$$

Setting the partial derivatives of $\mathscr{L}$ (with respect to $x_1, x_2, \ldots, x_n$ and λ) equal to 0 yields $n + 1$ equations representing the necessary conditions for an interior maximum:

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial x_1} &= \frac{\partial U}{\partial x_1} - \lambda p_1 = 0,\\
\frac{\partial \mathscr{L}}{\partial x_2} &= \frac{\partial U}{\partial x_2} - \lambda p_2 = 0,\\
&\vdots\\
\frac{\partial \mathscr{L}}{\partial x_n} &= \frac{\partial U}{\partial x_n} - \lambda p_n = 0,\\
\frac{\partial \mathscr{L}}{\partial \lambda} &= I - p_1x_1 - p_2x_2 - \cdots - p_nx_n = 0.
\end{aligned} \tag{4.8}$$

These $n + 1$ equations can, in principle, be solved for the optimal $x_1, x_2, \ldots, x_n$ and for λ (see Examples 4.1 and 4.2 to be convinced that such a solution is possible).

Equations 4.8 are necessary but not sufficient for a maximum. The second-order conditions that ensure a maximum are relatively complex and must be stated in matrix terms (see the Extensions to Chapter 2). However, the assumption of strict quasi-concavity (a diminishing *MRS* in the two-good case) is sufficient to ensure that any point obeying Equation 4.8 is in fact a true maximum.

Implications of first-order conditions

The first-order conditions represented by Equation 4.8 can be rewritten in a variety of instructive ways. For example, for any two goods, x_i and x_j, we have

$$\frac{\partial U/\partial x_i}{\partial U/\partial x_j} = \frac{p_i}{p_j}. \tag{4.9}$$

In Chapter 3 we showed that the ratio of the marginal utilities of two goods is equal to the marginal rate of substitution between them. Therefore, the conditions for an optimal allocation of income become

$$MRS(x_i \text{ for } x_j) = \frac{p_i}{p_j}. \tag{4.10}$$

[3]Again, the budget constraint has been written as an equality because, given the assumption of nonsatiation, it is clear that the individual will spend all available income.

This is exactly the result derived graphically earlier in this chapter; to maximize utility, the individual should equate the psychic rate of trade-off to the market trade-off rate.

Interpreting the Lagrange multiplier

Another result can be derived by solving Equations 4.8 for λ:

$$\lambda = \frac{\partial U/\partial x_1}{p_1} = \frac{\partial U/\partial x_2}{p_2} = \cdots = \frac{\partial U/\partial x_n}{p_n} \tag{4.11}$$

These equations state that, at the utility-maximizing point, each good purchased should yield the same marginal utility per dollar spent on that good. Therefore, each good should have an identical (marginal) benefit-to-(marginal)-cost ratio. If this were not true, one good would promise more "marginal enjoyment per dollar" than some other good, and funds would not be optimally allocated.

Although the reader is again warned against talking confidently about marginal utility, what Equation 4.11 says is that an extra dollar should yield the same "additional utility" no matter which good it is spent on. The common value for this extra utility is given by the Lagrange multiplier for the consumer's budget constraint (i.e., by λ). Consequently, λ can be regarded as the marginal utility of an extra dollar of consumption expenditure (the marginal utility of "income").

One final way to rewrite the necessary conditions for a maximum is

$$p_i = \frac{\partial U/\partial x_i}{\lambda} \tag{4.12}$$

for every good i that is bought. To interpret this expression, remember (from Equation 4.11) that the Lagrange multiplier, λ, represents the marginal utility value of an extra dollar of income, no matter where it is spent. Therefore, the ratio in Equation 4.12 compares the extra utility value of one more unit of good i to this common value of a marginal dollar in spending. To be purchased, the utility value of an extra unit of a good must be worth, in dollar terms, the price the person must pay for it. For example, a high price for good i can only be justified if it also provides a great deal of extra utility. At the margin, therefore, the price of a good reflects an individual's willingness to pay for one more unit. This is a result of considerable importance in applied welfare economics because willingness to pay can be inferred from market reactions to prices. In Chapter 5 we will see how this insight can be used to evaluate the welfare effects of price changes, and in later chapters we will use this idea to discuss a variety of questions about the efficiency of resource allocation.

Corner solutions

The first-order conditions of Equations 4.8 hold exactly only for interior maxima for which some positive amount of each good is purchased. As discussed in Chapter 2, when corner solutions (such as those illustrated in Figure 4.4) arise, the conditions must be modified slightly.[4] In this case, Equations 4.8 become

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial U}{\partial x_i} - \lambda p_i \le 0 \quad (i = 1, \ldots, n) \tag{4.13}$$

[4]Formally, these conditions are called the *Kuhn–Tucker* conditions for nonlinear programming.

and, if

$$\frac{\partial \mathscr{L}}{\partial x_i} = \frac{\partial U}{\partial x_i} - \lambda p_i < 0, \tag{4.14}$$

then

$$x_i = 0. \tag{4.15}$$

To interpret these conditions, we can rewrite Equation 4.14 as

$$p_i > \frac{\partial U/\partial x_i}{\lambda}. \tag{4.16}$$

Hence the optimal conditions are as before, except that any good whose price (p_i) exceeds its marginal value to the consumer will not be purchased ($x_i = 0$). Thus, the mathematical results conform to the commonsense idea that individuals will not purchase goods that they believe are not worth the money. Although corner solutions do not provide a major focus for our analysis in this book, the reader should keep in mind the possibilities for such solutions arising and the economic interpretation that can be attached to the optimal conditions in such cases.

EXAMPLE 4.1 Cobb–Douglas Demand Functions

As we showed in Chapter 3, the Cobb–Douglas utility function is given by

$$U(x, y) = x^{\alpha} y^{\beta}, \tag{4.17}$$

where, for convenience,[5] we assume $\alpha + \beta = 1$. We can now solve for the utility-maximizing values of x and y for any prices (p_x, p_y) and income (I). Setting up the Lagrangian expression

$$\mathscr{L} = x^{\alpha} y^{\beta} + \lambda(I - p_x x - p_y y) \tag{4.18}$$

yields the first-order conditions

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial x} &= \alpha x^{\alpha-1} y^{\beta} - \lambda p_x = 0,\\
\frac{\partial \mathscr{L}}{\partial y} &= \beta x^{\alpha} y^{\beta-1} - \lambda p_y = 0,\\
\frac{\partial \mathscr{L}}{\partial \lambda} &= I - p_x x - p_y y = 0.
\end{aligned} \tag{4.19}$$

Taking the ratio of the first two terms shows that

$$\frac{\alpha y}{\beta x} = \frac{p_x}{p_y}, \tag{4.20}$$

or

$$p_y y = \frac{\beta}{\alpha} p_x x = \frac{1-\alpha}{\alpha} p_x x, \tag{4.21}$$

where the final equation follows because $\alpha + \beta = 1$. Substitution of this first-order condition in Equation 4.21 into the budget constraint gives

$$I = p_x x + p_y y = p_x x + \frac{1-\alpha}{\alpha} p_x x = p_x x\left(1 + \frac{1-\alpha}{\alpha}\right) = \frac{1}{\alpha} p_x x; \tag{4.22}$$

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[5]As we discussed in Chapter 3, the exponents in the Cobb–Douglas utility function can always be normalized to sum to 1 because $U^{1/(\alpha+\beta)}$ is a monotonic transformation.

solving for x yields

$$x^* = \frac{\alpha I}{p_x}, \tag{4.23}$$

and a similar set of manipulations would give

$$y^* = \frac{\beta I}{p_y}. \tag{4.24}$$

These results show that an individual whose utility function is given by Equation 4.17 will always choose to allocate α proportion of his or her income to buying good x (i.e., $p_x x/I = \alpha$) and β proportion to buying good y ($p_y y/I = \beta$). Although this feature of the Cobb–Douglas function often makes it easy to work out simple problems, it does suggest that the function has limits in its ability to explain actual consumption behavior. Because the share of income devoted to particular goods often changes significantly in response to changing economic conditions, a more general functional form may provide insights not provided by the Cobb–Douglas function. We illustrate a few possibilities in Example 4.2, and the general topic of budget shares is taken up in more detail in the Extensions to this chapter.

Numerical example. First, however, let's look at a specific numerical example for the Cobb–Douglas case. Suppose that x sells for \$1 and y sells for \$4 and that total income is \$8. Succinctly then, assume that $p_x = 1$, $p_y = 4$, $I = 8$. Suppose also that $\alpha = \beta = 0.5$, so that this individual splits his or her income equally between these two goods. Now the demand Equations 4.23 and 4.24 imply

$$\begin{aligned} x^* &= \alpha I/p_x = 0.5I/p_x = 0.5(8)/1 = 4, \\ y^* &= \beta I/p_y = 0.5I/p_y = 0.5(8)/4 = 1, \end{aligned} \tag{4.25}$$

and, at these optimal choices,

$$\text{utility} = x^{0.5}y^{0.5} = (4)^{0.5}(1)^{0.5} = 2. \tag{4.26}$$

Notice also that we can compute the value for the Lagrange multiplier associated with this income allocation by using Equation 4.19:

$$\lambda = \alpha x^{\alpha-1}y^{\beta}/p_x = 0.5(4)^{-0.5}(1)^{0.5}/1 = 0.25. \tag{4.27}$$

This value implies that each small change in income will increase utility by approximately one fourth of that amount. Suppose, for example, that this person had 1 percent more income (\$8.08). In this case he or she would choose $x = 4.04$ and $y = 1.01$, and utility would be $4.04^{0.5} \cdot 1.01^{0.5} = 2.02$. Hence a \$0.08 increase in income increased utility by 0.02, as predicted by the fact that $\lambda = 0.25$.

QUERY: Would a change in p_y affect the quantity of x demanded in Equation 4.23? Explain your answer mathematically. Also develop an intuitive explanation based on the notion that the share of income devoted to good y is given by the parameter of the utility function, β.

EXAMPLE 4.2 CES Demand

To illustrate cases in which budget shares are responsive to economic circumstances, let's look at three specific examples of the CES function.

Case 1: $\delta = 0.5$. In this case, utility is

$$U(x, y) = x^{0.5} + y^{0.5}. \tag{4.28}$$

Setting up the Lagrangian expression

$$\mathscr{L} = x^{0.5} + y^{0.5} + \lambda(I - p_x x - p_y y) \tag{4.29}$$

yields the following first-order conditions for a maximum:

$$\begin{aligned} \partial\mathscr{L}/\partial x &= 0.5x^{-0.5} - \lambda p_x = 0, \\ \partial\mathscr{L}/\partial y &= 0.5y^{-0.5} - \lambda p_y = 0, \\ \partial\mathscr{L}/\partial\lambda &= I - p_x x - p_y y = 0. \end{aligned} \tag{4.30}$$

Division of the first two of these shows that

$$(y/x)^{0.5} = p_x/p_y. \tag{4.31}$$

By substituting this into the budget constraint and doing some messy algebraic manipulation, we can derive the demand functions associated with this utility function:

$$x^* = I/p_x[1 + (p_x/p_y)], \tag{4.32}$$

$$y^* = I/p_y[1 + (p_y/p_x)]. \tag{4.33}$$

Price responsiveness. In these demand functions notice that the share of income spent on, say, good x—that is, $p_x x/I = 1/[1 + (p_x/p_y)]$—is not a constant; it depends on the price ratio p_x/p_y. The higher the relative price of x, the smaller the share of income spent on that good. In other words, the demand for x is so responsive to its own price that an increase in the price reduces total spending on x. That the demand for x is price responsive can also be illustrated by comparing the implied exponent on p_x in the demand function given by Equation 4.32 (-2) to that from Equation 4.23 (-1). In Chapter 5 we will discuss this observation more fully when we examine the elasticity concept in detail.

Case 2: $\delta = -1$. Alternatively, let's look at a demand function with less substitutability[6] than the Cobb–Douglas. If $\delta = -1$, the utility function is given by

$$U(x, y) = -x^{-1} - y^{-1}, \tag{4.34}$$

and it is easy to show that the first-order conditions for a maximum require

$$y/x = (p_x/p_y)^{0.5}. \tag{4.35}$$

Again, substitution of this condition into the budget constraint, together with some messy algebra, yields the demand functions

$$\begin{aligned} x^* &= I/p_x[1 + (p_y/p_x)^{0.5}], \\ y^* &= I/p_y[1 + (p_x/p_y)^{0.5}]. \end{aligned} \tag{4.36}$$

That these demand functions are less price responsive can be seen in two ways. First, now the share of income spent on good x—that is, $p_x x/I = 1/[1 + (p_y/p_x)^{0.5}]$—responds positively to increases in p_x. As the price of x increases, this individual cuts back only modestly on good x; thus, total spending on that good increases. That the demand functions in Equation 4.36 are less price responsive than the Cobb–Douglas is also illustrated by the relatively small implied exponents of each good's own price (-0.5).

[6]One way to measure substitutability is by the elasticity of substitution, which for the CES function is given by $\sigma = 1/(1 - \delta)$. Here $\delta = 0.5$ implies $\sigma = 2$, $\delta = 0$ (the Cobb–Douglas) implies $\sigma = 1$, and $\delta = -1$ implies $\sigma = 0.5$. See also the discussion of the CES function in connection with the theory of production in Chapter 9.

Case 3: $\delta = -\infty$. This is the important case in which x and y must be consumed in fixed proportions. Suppose, for example, that each unit of y must be consumed together with exactly 4 units of x. The utility function that represents this situation is

$$U(x, y) = \min(x, 4y). \tag{4.37}$$

In this situation, a utility-maximizing person will choose only combinations of the two goods for which $x = 4y$; that is, utility maximization implies that this person will choose to be at a vertex of his or her L-shaped indifference curves. Because of the shape of these indifference curves, calculus cannot be used to solve this problem. Instead, one can adopt the simple procedure of substituting the utility-maximizing condition directly into the budget constraint:

$$I = p_x x + p_y y = p_x x + p_y \frac{x}{4} = (p_x + 0.25p_y)x. \tag{4.38}$$

Hence

$$x^* = \frac{I}{p_x + 0.25p_y}, \tag{4.39}$$

and similar substitutions yield

$$y^* = \frac{I}{4p_x + p_y}. \tag{4.40}$$

In this case, the share of a person's budget devoted to, say, good x rises rapidly as the price of x increases because x and y must be consumed in fixed proportions. For example, if we use the values assumed in Example 4.1 ($p_x = 1$, $p_y = 4$, $I = 8$), Equations 4.39 and 4.40 would predict $x^* = 4$, $y^* = 1$, and, as before, half of the individual's income would be spent on each good. If we instead use $p_x = 2$, $p_y = 4$, and $I = 8$, then $x^* = 8/3$, $y^* = 2/3$, and this person spends two thirds $[p_x x/I = (2 \cdot 8/3)/8 = 2/3]$ of his or her income on good x. Trying a few other numbers suggests that the share of income devoted to good x approaches 1 as the price of x increases.[7]

QUERY: Do changes in income affect expenditure shares in any of the CES functions discussed here? How is the behavior of expenditure shares related to the homothetic nature of this function?

INDIRECT UTILITY FUNCTION

Examples 4.1 and 4.2 illustrate the principle that it is often possible to manipulate the first-order conditions for a constrained utility-maximization problem to solve for the optimal values of $x_1, x_2, \ldots, x_n$. These optimal values in general will depend on the prices of all the goods and on the individual's income. That is,

$$\begin{aligned}
x_1^* &= x_1(p_1, p_2, \ldots, p_n, I), \\
x_2^* &= x_2(p_1, p_2, \ldots, p_n, I), \\
&\vdots \\
x_n^* &= x_n(p_1, p_2, \ldots, p_n, I).
\end{aligned} \tag{4.41}$$

In the next chapter we will analyze in more detail this set of *demand functions*, which show the dependence of the quantity of each x_i demanded on $p_1, p_2, \ldots, p_n$ and I. Here

[7]These relationships for the CES function are pursued in more detail in Problem 4.9 and in Extension E4.3.

we use the optimal values of the x's from Equation 4.42 to substitute in the original utility function to yield

$$\text{maximum utility} = U[x_1^*(p_1, \ldots p_n, I), x_2^*(p_1, \ldots p_n, I), \ldots, x_n^*(p_1, \ldots p_n, I)] \quad \textbf{(4.42)}$$

$$= V(p_1, p_2, \ldots, p_n, I). \quad \textbf{(4.43)}$$

In words, because of the individual's desire to maximize utility given a budget constraint, the optimal level of utility obtainable will depend *indirectly* on the prices of the goods being bought and the individual's income. This dependence is reflected by the indirect utility function V. If either prices or income were to change, the level of utility that could be attained would also be affected. Sometimes, in both consumer theory and many other contexts, it is possible to use this indirect approach to study how changes in economic circumstances affect various kinds of outcomes, such as utility or (later in this book) firms' costs.

THE LUMP SUM PRINCIPLE

Many economic insights stem from the recognition that utility ultimately depends on the income of individuals and on the prices they face. One of the most important of these is the so-called lump sum principle that illustrates the superiority of taxes on a person's general purchasing power to taxes on specific goods. A related insight is that general income grants to low-income people will raise utility more than will a similar amount of money spent subsidizing specific goods. The intuition behind these results derives directly from the utility-maximization hypothesis; an income tax or subsidy leaves the individual free to decide how to allocate whatever final income he or she has. On the other hand, taxes or subsidies on specific goods both change a person's purchasing power and distort his or her choices because of the artificial prices incorporated in such schemes. Hence general income taxes and subsidies are to be preferred if efficiency is an important criterion in social policy.

The lump sum principle as it applies to taxation is illustrated in Figure 4.5. Initially this person has an income of I and is choosing to consume the combination x^*, y^*. A tax on good x would raise its price, and the utility-maximizing choice would shift to combination x_1, y_1. Tax collections would be $t \cdot x_1$ (where t is the tax rate imposed on good x). Alternatively, an income tax that shifted the budget constraint inward to I' would also collect this same amount of revenue.[8] But the utility provided by the income tax (U_2) exceeds that provided by the tax on x alone (U_1). Hence we have shown that the utility burden of the income tax is smaller. A similar argument can be used to illustrate the superiority of income grants to subsidies on specific goods.

EXAMPLE 4.3 Indirect Utility and the Lump Sum Principle

In this example we use the notion of an indirect utility function to illustrate the lump sum principle as it applies to taxation. First we have to derive indirect utility functions for two illustrative cases.

Case 1: Cobb–Douglas. In Example 4.1 we showed that for the Cobb–Douglas utility function with $\alpha = \beta = 0.5$, optimal purchases are

[8] Because $I = (p_x + t)x_1 + p_y y_1$, we have $I' = I - tx_1 = p_x x_1 + p_y y_1$, which shows that the budget constraint with an equal-size income tax also passes through the point x_1, y_1.

$$x^* = \frac{I}{2p_x},$$
$$y^* = \frac{I}{2p_y}. \tag{4.44}$$

Thus, the indirect utility function in this case is

$$V(p_x, p_y, I) = U(x^*, y^*) = (x^*)^{0.5}(y^*)^{0.5} = \frac{I}{2p_x^{0.5}p_y^{0.5}}. \tag{4.45}$$

Notice that when $p_x = 1$, $p_y = 4$, and $I = 8$ we have $V = 8/(2 \cdot 1 \cdot 2) = 2$, which is the utility that we calculated before for this situation.

Case 2: Fixed proportions. In the third case of Example 4.2 we found that

$$x^* = \frac{I}{p_x + 0.25p_y},$$
$$y^* = \frac{I}{4p_x + p_y}. \tag{4.46}$$

Thus, in this case indirect utility is given by

$$\begin{aligned} V(p_x, p_y, I) &= \min(x^*, 4y^*) = x^* = \frac{I}{p_x + 0.25p_y} \\ &= 4y^* = \frac{4}{4p_x + p_y} = \frac{I}{p_x + 0.25p_y}; \end{aligned} \tag{4.47}$$

with $p_x = 1$, $p_y = 4$, and $I = 8$, indirect utility is given by $V = 4$, which is what we calculated before.

The lump sum principle. Consider first using the Cobb–Douglas case to illustrate the lump sum principle. Suppose that a tax of \$1 were imposed on good x. Equation 4.45 shows that indirect utility in this case would fall from 2 to 1.41 [$= 8/(2 \cdot 2^{0.5} \cdot 2)$]. Because this person chooses $x^* = 2$ with the tax, total tax collections will be \$2. Therefore, an equal-revenue income tax would reduce net income to \$6, and indirect utility would be 1.5 [$= 6/(2 \cdot 1 \cdot 2)$]. Thus, the income tax is a clear improvement in utility over the case where x alone is taxed. The tax on good x reduces utility for two reasons: It reduces a person's purchasing power, and it biases his or her choices away from good x. With income taxation, only the first effect is felt and so the tax is more efficient.[9]

The fixed-proportions case supports this intuition. In that case, a \$1 tax on good x would reduce indirect utility from 4 to 8/3 [$= 8/(2 + 1)$]. In this case $x^* = 8/3$ and tax collections would be \$8/3. An income tax that collected \$8/3 would leave this consumer with \$16/3 in net income, and that income would yield an indirect utility of $V = 8/3$ [$= (16/3)/(1 + 1)$]. Hence after-tax utility is the same under both the excise and income taxes. The reason the lump sum principle does not hold in this case is that with fixed-proportions utility, the excise tax does not distort choices because preferences are so rigid.

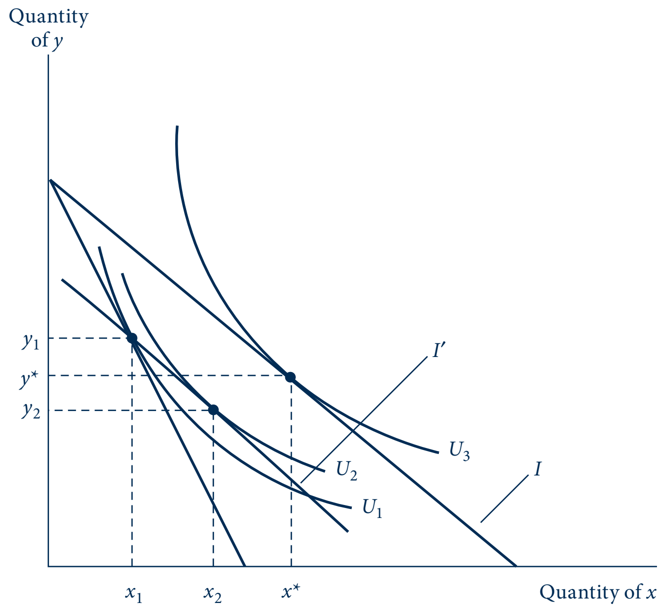
QUERY: Both indirect utility functions illustrated here show that a doubling of income and all prices would leave indirect utility unchanged. Explain why you would expect this to be a property of all indirect utility functions. That is, explain why the indirect utility function is homogeneous of degree zero in all prices and income.

[9]This discussion assumes that there are no incentive effects of income taxation—probably not a good assumption.

FIGURE 4.5

The Lump Sum Principle of Taxation

A tax on good x would shift the utility-maximizing choice from x^*, y^* to x_1, y_1. An income tax that collected the same amount would shift the budget constraint to I'. Utility would be higher (U_2) with the income tax than with the tax on x alone (U_1).



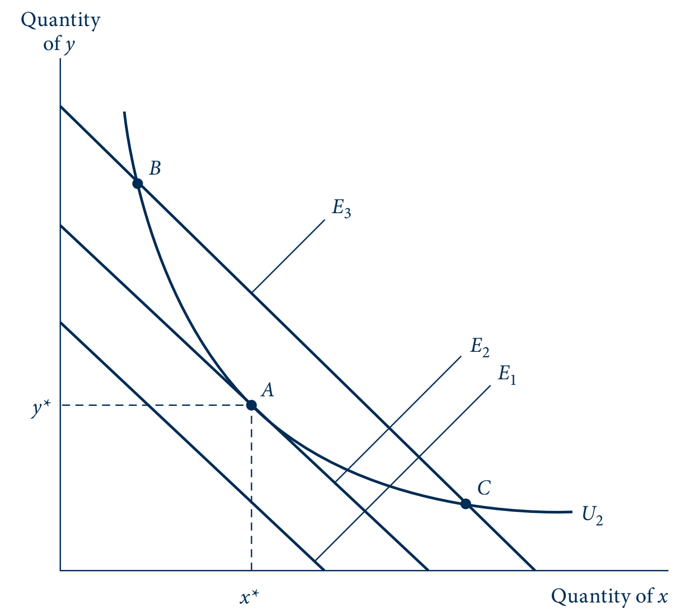
EXPENDITURE MINIMIZATION

In Chapter 2 we pointed out that many constrained maximum problems have associated "dual" constrained minimum problems. For the case of utility maximization, the associated dual minimization problem concerns allocating income in such a way as to achieve a given utility level with the minimal expenditure. This problem is clearly analogous to the primary utility-maximization problem, but the goals and constraints of the problems have been reversed. Figure 4.6 illustrates this dual expenditure-minimization problem. There, the individual must attain utility level U_2; this is now the constraint in the problem. Three possible expenditure amounts (E_1, E_2, and E_3) are shown as three "budget constraint" lines in the figure. Expenditure level E_1 is clearly too small to achieve U_2; hence it cannot solve the dual problem. With expenditures given by E_3, the individual can reach U_2 (at either of the two points B or C), but this is not the minimal expenditure level required. Rather, E_2 clearly provides just enough total expenditures to reach U_2 (at point A), and this is in fact the solution to the dual problem. By comparing Figures 4.2 and 4.6, it is obvious that both the primary utility-maximization approach and the dual expenditure-minimization approach yield the same solution (x^*, y^*); they are simply alternative ways of viewing the same process. Often the expenditure-minimization approach is more useful, however, because expenditures are directly observable, whereas utility is not.

FIGURE 4.6

The Dual Expenditure-Minimization Problem

The dual of the utility-maximization problem is to attain a given utility level (U_2) with minimal expenditures. An expenditure level of E_1 does not permit U_2 to be reached, whereas E_3 provides more spending power than is strictly necessary. With expenditure E_2, this person can just reach U_2 by consuming x^* and y^*.



A mathematical statement

More formally, the individual's dual expenditure-minimization problem is to choose $x_1, x_2, \ldots, x_n$ to minimize

$$\text{total expenditures} = E = p_1x_1 + p_2x_2 + \cdots + p_nx_n, \tag{4.48}$$

subject to the constraint

$$\text{utility} = \bar{U} = U(x_1, x_2, \ldots, x_n). \tag{4.49}$$

The optimal amounts of $x_1, x_2, \ldots, x_n$ chosen in this problem will depend on the prices of the various goods ($p_1, p_2, \ldots, p_n$) and on the required utility level $\bar{U}$. If any of the prices were to change or if the individual had a different utility "target," then another commodity bundle would be optimal. This dependence can be summarized by an *expenditure function*.

DEFINITION

Expenditure function. The individual's expenditure function shows the minimal expenditures necessary to achieve a given utility level for a particular set of prices. That is,

$$\text{minimal expenditures} = E(p_1, p_2, \ldots, p_n, U). \tag{4.50}$$

This definition shows that the expenditure function and the indirect utility function are inverse functions of one another (compare Equations 4.43 and 4.50). Both depend on market prices but involve different constraints (income or utility). In the next chapter we will see how this relationship is useful in allowing us to examine the theory of how individuals respond to price changes. First, however, let's look at two expenditure functions.

EXAMPLE 4.4 Two Expenditure Functions

There are two ways one might compute an expenditure function. The first, most straightforward method would be to state the expenditure-minimization problem directly and apply the Lagrangian technique. Some of the problems at the end of this chapter ask you to do precisely that. Here, however, we will adopt a more streamlined procedure by taking advantage of the relationship between expenditure functions and indirect utility functions. Because these two functions are inverses of each other, calculation of one greatly facilitates the calculation of the other. We have already calculated indirect utility functions for two important cases in Example 4.3. Retrieving the related expenditure functions is simple algebra.

Case 1: Cobb–Douglas utility. Equation 4.45 shows that the indirect utility function in the two-good, Cobb–Douglas case is

$$V(p_x, p_y, I) = \frac{I}{2p_x^{0.5}p_y^{0.5}}. \tag{4.51}$$

If we now interchange the role of utility (which we will now treat as the utility "target" denoted by U) and income (which we will now term "expenditures," E, and treat as a function of the parameters of this problem), then we have the expenditure function

$$E(p_x, p_y, U) = 2p_x^{0.5}p_y^{0.5}U. \tag{4.52}$$

Checking this against our former results, now we use a utility target of $U = 2$ with, again, $p_x = 1$ and $p_y = 4$. With these parameters, Equation 4.52 shows that the required minimal expenditures are \$8 $(= 2 \cdot 1^{0.5} \cdot 4^{0.5} \cdot 2)$. Not surprisingly, both the primal utility-maximization problem and the dual expenditure-minimization problem are formally identical.

Case 2: Fixed proportions. For the fixed-proportions case, Equation 4.47 gave the indirect utility function as

$$V(p_x, p_y, I) = \frac{I}{p_x + 0.25p_y}. \tag{4.53}$$

If we again switch the role of utility and expenditures, we quickly derive the expenditure function:

$$E(p_x, p_y, U) = (p_x + 0.25p_y)U. \tag{4.54}$$

A check of the hypothetical values used in Example 4.3 ($p_x = 1$, $p_y = 4$, $U = 4$) again shows that it would cost \$8 $[= (1 + 0.25 \cdot 4) \cdot 4]$ to reach the utility target of 4.

Compensating for a price change. These expenditure functions allow us to investigate how a person might be compensated for a price change. Specifically, suppose that the price of good y were to increase from \$4 to \$5. This would clearly reduce a person's utility, so we might ask what amount of monetary compensation would mitigate the harm. Because the expenditure function allows utility to be held constant, it provides a direct estimate of this amount. Specifically, in the Cobb–Douglas case, expenditures would have to be increased from \$8 to \$8.94 $(= 2 \cdot 1 \cdot 5^{0.5} \cdot 2)$ to provide enough extra purchasing power to precisely compensate for this price increase. With fixed proportions, expenditures would have to be increased from \$8 to \$9 to compensate for the price increase. Hence the compensations are about the same in these simple cases.

There is one important difference between the two examples, however. In the fixed-proportions case, the \$1 of extra compensation simply permits this person to return to his or her previous consumption bundle ($x = 4, y = 1$). That is the only way to restore utility to $U = 4$ for this rigid person. In the Cobb–Douglas case, however, this person will not use the extra compensation to revert to his or her previous consumption bundle. Instead, utility maximization will require that the \$8.94 be allocated so that $x = 4.47$ and $y = 0.894$. This will still provide a utility level of $U = 2$, but this person will economize on the now more expensive good y. In the next chapter we will pursue this analysis of the welfare effects of price changes in much greater detail.

QUERY: How should a person be compensated for a price decrease? What sort of compensation would be required if the price of good y fell from \$4 to \$3?

PROPERTIES OF EXPENDITURE FUNCTIONS

Because expenditure functions are widely used in applied economics, it is important to understand a few of the properties shared by all such functions. Here we look at three properties. All these follow directly from the fact that expenditure functions are based on individual utility maximization.

1. *Homogeneity*. For both of the functions illustrated in Example 4.4, a doubling of all prices will precisely double the value of required expenditures. Technically, these expenditure functions are "homogeneous of degree one" in all prices.[10] This is a general property of expenditure functions. Because the individual's budget constraint is linear in prices, any proportional increase in both prices and purchasing power will permit the person to buy the same utility-maximizing commodity bundle that was chosen before the price increase. In Chapter 5 we will see that, for this reason, demand functions are homogeneous of degree zero in all prices and income.
2. *Expenditure functions are nondecreasing in prices*. This property can be succinctly summarized by the mathematical statement

$$\frac{\partial E}{\partial p_i} \geq 0 \quad \text{for every good } i. \tag{4.55}$$

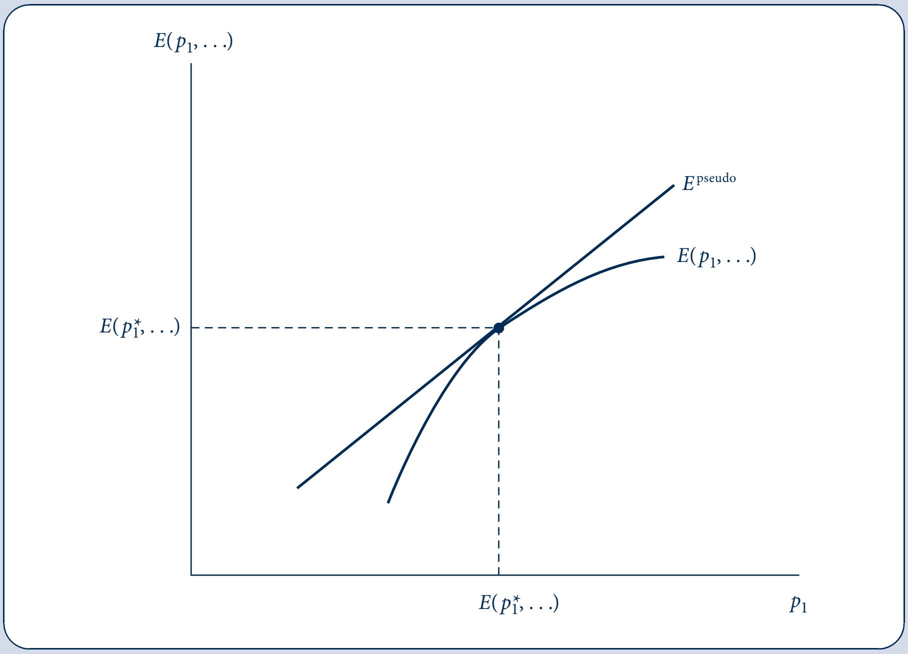
This seems intuitively obvious. Because the expenditure function reports the minimum expenditure necessary to reach a given utility level, an increase in any price must increase this minimum. More formally, suppose the price of one good increases and that all other prices stay the same. Let A represent the bundle of goods purchased before the price increase and B the bundle purchased after the price increase. Clearly bundle B costs more after the price increase than it did previously. The only change between the two situations is an increase in one of the prices; therefore, spending on that good increases and all other spending stays the same. However, we also know that, before the price increase, bundle A cost less than bundle B because A was the expenditure-minimizing bundle. Hence actual expenditures when B is chosen after

[10] As described in Chapter 2, the function $f(x_1, x_2, \ldots, x_n)$ is said to be homogeneous of degree k if $f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n)$. In this case, $k = 1$.

FIGURE 4.7

Expenditure Functions Are Concave in Prices

At p_1^* this person spends $E(p_1^*, \ldots)$. If he or she continues to buy the same set of goods as p_1 changes, then expenditures would be given by E^{pseudo}. Because his or her consumption patterns will likely change as p_1 changes, actual expenditures will be less than this.



the price increase must exceed those on *A* before the price increase. A similar chain of logic could be used to show that a decrease in price should cause expenditures to decrease (or possibly stay the same).

3. *Expenditure functions are concave in prices*. In Chapter 2 we discussed concave functions, which are defined as functions that always lie below tangents to them. Although the technical mathematical conditions that describe such functions are complicated, it is relatively simple to show how the concept applies to expenditure functions by considering the variation in a single price. Figure 4.7 shows an individual's expenditures as a function of the single price, p_1. At the initial price, p_1^*, this person's expenditures are given by $E(p_1^*, \ldots)$. Now consider prices higher or lower than p_1^*. If this person continued to buy the same bundle of goods, expenditures would increase or decrease linearly as this price changed. This would give rise to the pseudo-expenditure function E^{pseudo} in the figure. This line shows a level of expenditures that would allow this person to buy the original bundle of goods despite the changing value of p_1. If, as seems more likely, this person adjusted his or her purchases as p_1 changed, we know (because of expenditure minimization) that actual expenditures would be less than these pseudo-amounts. Hence the actual expenditure function, *E*, will lie everywhere below E^{pseudo} and the function will be concave.[11] The concavity of the expenditure function is a useful property for a number of applications, especially those related to the substitution effect from price changes (see Chapter 5).

[11]One result of concavity is that $f_{ii} = \partial^2 E/\partial p_i^2 \leq 0$. This is precisely what Figure 4.7 shows.

Summary

In this chapter we explored the basic economic model of utility maximization subject to a budget constraint. Although we approached this problem in a variety of ways, all these approaches led to the same basic result.

- To reach a constrained maximum, an individual should spend all available income and should choose a commodity bundle such that the *MRS* between any two goods is equal to the ratio of those goods' market prices. This basic tangency will result in the individual equating the ratios of the marginal utility to market price for every good that is actually consumed. Such a result is common to most constrained optimization problems.
- The tangency conditions are only the first-order conditions for a unique constrained maximum, however. To ensure that these conditions are also sufficient, the individual's indifference curve map must exhibit a diminishing *MRS*. In formal terms, the utility function must be strictly quasi-concave.
- The tangency conditions must also be modified to allow for corner solutions in which the optimal level of consumption of some goods is zero. In this case, the ratio of marginal utility to price for such a good will be below the common marginal benefit–marginal cost ratio for goods actually bought.
- A consequence of the assumption of constrained utility maximization is that the individual's optimal choices will depend implicitly on the parameters of his or her budget constraint. That is, the choices observed will be implicit functions of all prices and income. Therefore, utility will also be an indirect function of these parameters.
- The dual to the constrained utility-maximization problem is to minimize the expenditure required to reach a given utility target. Although this dual approach yields the same optimal solution as the primal constrained maximum problem, it also yields additional insight into the theory of choice. Specifically, this approach leads to expenditure functions in which the spending required to reach a given utility target depends on goods' market prices. Therefore, expenditure functions are, in principle, measurable.

Problems

4.1

Each day Paul, who is in third grade, eats lunch at school. He likes only Twinkies (t) and soda (s), and these provide him a utility of

$$\text{utility} = U(t, s) = \sqrt{ts}.$$

a. If Twinkies cost \$0.10 each and soda costs \$0.25 per cup, how should Paul spend the \$1 his mother gives him to maximize his utility?

b. If the school tries to discourage Twinkie consumption by increasing the price to \$0.40, by how much will Paul's mother have to increase his lunch allowance to provide him with the same level of utility he received in part (a)?

4.2

a. A young connoisseur has \$600 to spend to build a small wine cellar. She enjoys two vintages in particular: a 2001 French Bordeaux (w_F) at \$40 per bottle and a less expensive 2005 California varietal wine (w_C) priced at \$8. If her utility is

$$U(w_F, w_C) = w_F^{2/3} w_C^{1/3},$$

then how much of each wine should she purchase?

b. When she arrived at the wine store, our young oenologist discovered that the price of the French Bordeaux had fallen to \$20 a bottle because of a decrease in the value of the euro. If the price of the California wine remains stable at \$8 per bottle, how much of each wine should our friend purchase to maximize utility under these altered conditions?

c. Explain why this wine fancier is better off in part (b) than in part (a). How would you put a monetary value on this utility increase?

4.3

a. On a given evening, J. P. enjoys the consumption of cigars (c) and brandy (b) according to the function

$$U(c, b) = 20c - c^2 + 18b - 3b^2.$$

How many cigars and glasses of brandy does he consume during an evening? (Cost is no object to J. P.)

b. Lately, however, J. P. has been advised by his doctors that he should limit the sum of glasses of brandy and cigars consumed to 5. How many glasses of brandy and cigars will he consume under these circumstances?

4.4

a. Mr. Odde Ball enjoys commodities x and y according to the utility function

$$U(x, y) = \sqrt{x^2 + y^2}.$$

Maximize Mr. Ball's utility if $p_x = \$3$, $p_y = \$4$, and he has \$50 to spend. *Hint:* It may be easier here to maximize U^2 rather than U. Why will this not alter your results?

b. Graph Mr. Ball's indifference curve and its point of tangency with his budget constraint. What does the graph say about Mr. Ball's behavior? Have you found a true maximum?

4.5

Mr. A derives utility from martinis (m) in proportion to the number he drinks:

$$U(m) = m.$$

Mr. A is particular about his martinis, however: He only enjoys them made in the exact proportion of two parts gin (g) to one part vermouth (v). Hence we can rewrite Mr. A's utility function as

$$U(m) = U(g, v) = \min\left(\frac{g}{2}, v\right).$$

a. Graph Mr. A's indifference curve in terms of g and v for various levels of utility. Show that, regardless of the prices of the two ingredients, Mr. A will never alter the way he mixes martinis.

b. Calculate the demand functions for g and v.

c. Using the results from part (b), what is Mr. A's indirect utility function?

d. Calculate Mr. A's expenditure function; for each level of utility, show spending as a function of p_g and p_v. *Hint:* Because this problem involves a fixed-proportions utility function, you cannot solve for utility-maximizing decisions by using calculus.

4.6

Suppose that a fast-food junkie derives utility from three goods—soft drinks (x), hamburgers (y), and ice cream sundaes (z)—according to the Cobb–Douglas utility function

$$U(x, y, z) = x^{0.5}y^{0.5}(1 + z)^{0.5}.$$

Suppose also that the prices for these goods are given by $p_x = 1$, $p_y = 4$, and $p_z = 8$ and that this consumer's income is given by $I = 8$.

a. Show that, for $z = 0$, maximization of utility results in the same optimal choices as in Example 4.1. Show also that any choice that results in $z > 0$ (even for a fractional z) reduces utility from this optimum.

b. How do you explain the fact that $z = 0$ is optimal here?

c. How high would this individual's income have to be for any z to be purchased?

4.7

The lump sum principle illustrated in Figure 4.5 applies to transfer policy and taxation. This problem examines this application of the principle.

a. Use a graph similar to Figure 4.5 to show that an income grant to a person provides more utility than does a subsidy on good x that costs the same amount to the government.
b. Use the Cobb–Douglas expenditure function presented in Equation 4.52 to calculate the extra purchasing power needed to increase this person's utility from $U = 2$ to $U = 3$.
c. Use Equation 4.52 again to estimate the degree to which good x must be subsidized to increase this person's utility from $U = 2$ to $U = 3$. How much would this subsidy cost the government? How would this cost compare with the cost calculated in part (b)?
d. Problem 4.10 asks you to compute an expenditure function for a more general Cobb–Douglas utility function than the one used in Example 4.4. Use that expenditure function to re-solve parts (b) and (c) here for the case $\alpha = 0.3$, a figure close to the fraction of income that low-income people spend on food.
e. How would your calculations in this problem have changed if we had used the expenditure function for the fixed-proportions case (Equation 4.54) instead?

4.8

Two of the simplest utility functions are:

1. Fixed proportions: $U(x, y) = \min[x, y]$.
2. Perfect substitutes: $U(x, y) = x + y$

a. For each of these utility functions, compute the following:
   - Demand functions for x and y
   - Indirect utility function
   - Expenditure function

b. Discuss the particular forms of these functions you calculated—why do they take the specific forms they do?

4.9

Suppose that we have a utility function involving two goods that is linear of the form $U(x, y) = ax + by$. Calculate the expenditure function for this utility function. *Hint:* The expenditure function will have kinks at various price ratios.

Analytical Problems

4.10 Cobb–Douglas utility

In Example 4.1 we looked at the Cobb–Douglas utility function $U(x, y) = x^{\alpha}y^{1-\alpha}$, where $0 \le \alpha \le 1$. This problem illustrates a few more attributes of that function.

a. Calculate the indirect utility function for this Cobb–Douglas case.
b. Calculate the expenditure function for this case.
c. Show explicitly how the compensation required to offset the effect of an increase in the price of x is related to the size of the exponent α.

4.11 CES utility

The CES utility function we have used in this chapter is given by

$$U(x, y) = \frac{x^{\delta}}{\delta} + \frac{y^{\delta}}{\delta}.$$

a. Show that the first-order conditions for a constrained utility maximum with this function require individuals to choose goods in the proportion

$$\frac{x}{y} = \left(\frac{p_x}{p_y}\right)^{1/(\delta-1)}.$$

b. Show that the result in part (a) implies that individuals will allocate their funds equally between x and y for the Cobb–Douglas case ($\delta = 0$), as we have shown before in several problems.

c. How does the ratio $p_x x/p_y y$ depend on the value of δ? Explain your results intuitively. (For further details on this function, see Extension E4.3.)
d. Derive the indirect utility and expenditure functions for this case and check your results by describing the homogeneity properties of the functions you calculated.

4.12 Stone–Geary utility

Suppose individuals require a certain level of food (x) to remain alive. Let this amount be given by x_0. Once x_0 is purchased, individuals obtain utility from food and other goods (y) of the form

$$U(x, y) = (x - x_0)^{\alpha} y^{\beta},$$

where $\alpha + \beta = 1$.

a. Show that if $I > p_x x_0$ then the individual will maximize utility by spending $\alpha(I - p_x x_0) + p_x x_0$ on good x and $\beta(I - p_x x_0)$ on good y. Interpret this result.
b. How do the ratios $p_x x/I$ and $p_y y/I$ change as income increases in this problem? (See also Extension E4.2 for more on this utility function.)

4.13 CES indirect utility and expenditure functions

In this problem, we will use a more standard form of the CES utility function to derive indirect utility and expenditure functions. Suppose utility is given by

$$U(x, y) = (x^{\delta} + y^{\delta})^{1/\delta}$$

[in this function the elasticity of substitution $\sigma = 1/(1 - \delta)$].

a. Show that the indirect utility function for the utility function just given is

$$V = I(p_x^r + p_y^r)^{-1/r},$$

where $r = \delta/(\delta - 1) = 1 - \sigma$.
b. Show that the function derived in part (a) is homogeneous of degree zero in prices and income.
c. Show that this function is strictly increasing in income.
d. Show that this function is strictly decreasing in any price.
e. Show that the expenditure function for this case of CES utility is given by

$$E = V(p_x^r + p_y^r)^{1/r}.$$

f. Show that the function derived in part (e) is homogeneous of degree one in the goods' prices.
g. Show that this expenditure function is increasing in each of the prices.
h. Show that the function is concave in each price.

4.14 Altruism

Michele, who has a relatively high income I, has altruistic feelings toward Sofia, who lives in such poverty that she essentially has no income. Suppose Michele's preferences are represented by the utility function

$$U_1(c_1, c_2) = c_1^{1-a} c_2^{a},$$

where c_1 and c_2 are Michele and Sofia's consumption levels, appearing as goods in a standard Cobb–Douglas utility function. Assume that Michele can spend her income either on her own or Sofia's consumption (through charitable donations) and that \$1 buys a unit of consumption for either (thus, the "prices" of consumption are $p_1 = p_2 = 1$).

a. Argue that the exponent a can be taken as a measure of the degree of Michele's altruism by providing an interpretation of extremes values $a = 0$ and $a = 1$. What value would make her a perfect altruist (regarding others the same as oneself)?
b. Solve for Michele's optimal choices and demonstrate how they change with a.
c. Solve for Michele's optimal choices under an income tax at rate t. How do her choices change if there is a charitable deduction (so income spent on charitable deductions is not taxed)? Does the charitable deduction have a bigger incentive effect on more or less altruistic people?

d. Return to the case without taxes for simplicity. Now suppose that Michele's altruism is represented by the utility function

$$U_1(c_1, U_2) = c_1^{1-a} U_2^a,$$

which is similar to the representation of altruism in Extension E3.4 to the previous chapter. According to this specification, Michele cares directly about Sofia's utility level and only indirectly about Sofia's consumption level.

1. Solve for Michele's optimal choices if Sofia's utility function is symmetric to Michele's: $U_2(c_2, U_1) = c_2^{1-a} U_1^a$. Compare your answer with part (b). Is Michele more or less charitable under the new specification? Explain.
2. Repeat the previous analysis assuming Sofia's utility function is $U_2(c_2) = c_2$.

Suggestions for Further Reading

Barten, A. P., and Volker Böhm. "Consumer Theory." In K. J. Arrow and M. D. Intriligator, Eds., *Handbook of Mathematical Economics*, vol. II. Amsterdam: North-Holland, 1982.

Sections 10 and 11 have compact summaries of many of the concepts covered in this chapter.

Deaton, A., and J. Muelbauer. *Economics and Consumer Behavior*. Cambridge, UK: Cambridge University Press, 1980.

Section 2.5 provides a nice geometric treatment of duality concepts.

Dixit, A. K. *Optimization in Economic Theory*. Oxford, UK: Oxford University Press, 1990.

Chapter 2 provides several Lagrangian analyses focusing on the Cobb–Douglas utility function.

Hicks, J. R. *Value and Capital*. Oxford, UK: Clarendon Press, 1946.

Chapter II and the Mathematical Appendix provide some early suggestions of the importance of the expenditure function.

Luenberger, D. G. *Microeconomic Theory*. New York: McGraw Hill, 1992.

In Chapter 4 the author shows several interesting relationships between his "Benefit Function" (see Problem 3.15) and the more standard expenditure function. This chapter also offers insights on a number of unusual preference structures.

Mas-Colell, A., M. D. Whinston, and J. R. Green. *Microeconomic Theory*. Oxford, UK: Oxford University Press, 1995.

Chapter 3 contains a thorough analysis of utility and expenditure functions.

Samuelson, Paul A. *Foundations of Economic Analysis*. Cambridge, MA: Harvard University Press, 1947.

Chapter V and Appendix A provide a succinct analysis of the first-order conditions for a utility maximum. The appendix provides good coverage of second-order conditions.

Silberberg, E., and W. Suen. *The Structure of Economics: A Mathematical Analysis*, 3rd ed. Boston: Irwin/McGraw-Hill, 2001.

A useful, although fairly difficult, treatment of duality in consumer theory.

Theil, H. *Theory and Measurement of Consumer Demand*. Amsterdam: North-Holland, 1975.

Good summary of basic theory of demand together with implications for empirical estimation.

Budget Shares

EXTENSIONS

The nineteenth-century economist Ernst Engel was one of the first social scientists to intensively study people's actual spending patterns. He focused specifically on food consumption. His finding that the fraction of income spent on food decreases as income increases has come to be known as *Engel's law* and has been confirmed in many studies. Engel's law is such an empirical regularity that some economists have suggested measuring poverty by the fraction of income spent on food. Two other interesting applications are: (1) the study by Hayashi (1995) showing that the share of income devoted to foods favored by the elderly is much higher in two-generation households than in one-generation households; and (2) findings by Behrman (1989) from less-developed countries showing that people's desires for a more varied diet as their incomes increase may in fact result in reducing the fraction of income spent on particular nutrients. In the remainder of this extension we look at some evidence on budget shares (denoted by $s_i = p_i x_i / I$) together with a bit more theory on the topic.

E4.1 The variability of budget shares

Table E4.1 shows some recent budget share data from the United States. Engel's law is clearly visible in the table: As income increases families spend a smaller proportion of their funds on food. Other important variations in the table include the declining share of income spent on health-care needs and the much larger share of income devoted to retirement plans by higher-income people. Interestingly, the shares of income devoted to shelter and transportation are relatively constant over the range of income shown in the table; apparently, high-income people buy bigger houses and cars.

The variable income shares in Table E4.1 illustrate why the Cobb–Douglas utility function is not useful for detailed empirical studies of household behavior. When utility is given by $U(x, y) = x^{\alpha} y^{\beta}$ (where $\alpha + \beta = 1$), the implied demand equations are $x = \alpha I/p_x$ and $y = \beta I/p_y$. Therefore,

$$s_x = p_x x/I = \alpha \quad \text{and} \\ s_y = p_y y/I = \beta, \tag{i}$$

and budget shares are constant for all observed income levels and relative prices. Because of this shortcoming, economists have investigated a number of other possible forms for the utility function that permit more flexibility.

E4.2 Linear expenditure system

A generalization of the Cobb–Douglas function that incorporates the idea that certain minimal amounts of each good

TABLE E4.1 BUDGET SHARES OF U.S. HOUSEHOLDS, 2008

	Annual Income		
	$10,000–$14,999	$40,000–$49,999	Over $70,000
Expenditure Item			
Food	15.7	13.4	11.8
Shelter	23.1	21.2	19.3
Utilities, fuel, and public services	11.2	8.6	5.8
Transportation	14.1	17.8	16.8
Health insurance	5.3	4.0	2.6
Other health-care expenses	2.6	2.8	2.3
Entertainment (including alcohol)	4.6	5.2	5.8
Education	2.3	1.2	2.6
Insurance and pensions	2.2	8.5	14.6
Other (apparel, personal care, other housing expenses, and misc.)	18.9	17.3	18.4

Consumer Expenditure Report, 2008, Bureau of Labor Statistics website: http://www.bls.gov.

must be bought by an individual (x_0, y_0) is the utility function

$$U(x, y) = (x - x_0)^{\alpha}(y - y_0)^{\beta} \tag{ii}$$

for $x \geq x_0$ and $y \geq y_0$, where again $\alpha + \beta = 1$.

Demand functions can be derived from this utility function in a way analogous to the Cobb–Douglas case by introducing the concept of supernumerary income (I^*), which represents the amount of purchasing power remaining after purchasing the minimum bundle

$$I^* = I - p_x x_0 - p_y y_0. \tag{iii}$$

Using this notation, the demand functions are

$$\begin{aligned} x &= (p_x x_0 + \alpha I^*)/p_x, \\ y &= (p_y y_0 + \beta I^*)/p_y. \end{aligned} \tag{iv}$$

In this case, the individual then spends a constant fraction of supernumerary income on each good once the minimum bundle has been purchased. Manipulation of Equation iv yields the share equations

$$\begin{aligned} s_x &= \alpha + (\beta p_x x_0 - \alpha p_y y_0)/I, \\ s_y &= \beta + (\alpha p_y y_0 - \beta p_x x_0)/I, \end{aligned} \tag{v}$$

which show that this demand system is not homothetic. Inspection of Equation v shows the unsurprising result that the budget share of a good is positively related to the minimal amount of that good needed and negatively related to the minimal amount of the other good required. Because the notion of necessary purchases seems to accord well with real-world observation, this linear expenditure system (LES), which was first developed by Stone (1954), is widely used in empirical studies. The utility function in Equation ii is also called a Stone–Geary utility function.

Traditional purchases

One of the most interesting uses of the LES is to examine how its notion of necessary purchases changes as conditions change. For example, Oczkowski and Philip (1994) study how access to modern consumer goods may affect the share of income that individuals in transitional economies devote to traditional local items. They show that villagers of Papua, New Guinea reduce such shares significantly as outside goods become increasingly accessible. Hence such improvements as better roads for moving goods provide one of the primary routes by which traditional cultural practices are undermined.

E4.3 CES utility

In Chapter 3 we introduced the CES utility function

$$U(x, y) = \frac{x^{\delta}}{\delta} + \frac{y^{\delta}}{\delta} \tag{vi}$$

for $\delta \leq 1$, $\delta \neq 0$. The primary use of this function is to illustrate alternative substitution possibilities (as reflected in the value of the parameter δ). Budget shares implied by this utility function provide a number of such insights. Manipulation of the first-order conditions for a constrained utility maximum with the CES function yields the share equations

$$\begin{aligned} s_x &= 1/[1 + (p_y/p_x)^K], \\ s_y &= 1/[1 + (p_x/p_y)^K], \end{aligned} \tag{vii}$$

where $K = \delta/(\delta - 1)$.

The homothetic nature of the CES function is shown by the fact that these share expressions depend only on the price ratio, p_x/p_y. Behavior of the shares in response to changes in relative prices depends on the value of the parameter K. For the Cobb–Douglas case, $\delta = 0$ and so $K = 0$ and $s_x = s_y = 1/2$. When $\delta > 0$, substitution possibilities are great and $K < 0$. In this case, Equation vii shows that s_x and p_x/p_y move in opposite directions. If p_x/p_y increases, the individual substitutes y for x to such an extent that s_x decreases. Alternatively, if $\delta < 0$, then substitution possibilities are limited, $K > 0$, and s_x and p_x/p_y move in the same direction. In this case, an increase in p_x/p_y causes only minor substitution of y for x, and s_x actually increases because of the relatively higher price of good x.

North American free trade

CES demand functions are most often used in large-scale computer models of general equilibrium (see Chapter 13) that economists use to evaluate the impact of major economic changes. Because the CES model stresses that shares respond to changes in relative prices, it is particularly appropriate for looking at innovations such as changes in tax policy or in international trade restrictions, where changes in relative prices are likely. One important area of such research has been on the impact of the North American Free Trade Agreement for Canada, Mexico, and the United States. In general, these models find that all the countries involved might be expected to gain from the agreement, but that Mexico's gains may be the greatest because it is experiencing the greatest change in relative prices. Kehoe and Kehoe (1995) present a number of computable equilibrium models that economists have used in these examinations.[1]

E4.4 The almost ideal demand system

An alternative way to study budget shares is to start from a specific expenditure function. This approach is especially convenient because the envelope theorem shows that budget shares can be derived directly from expenditure functions through logarithmic differentiation (for more details, see Chapter 5):

$$\begin{aligned} \frac{\partial \ln E(p_x, p_y, V)}{\partial \ln p_x} &= \frac{1}{E(p_x, p_y, V)} \cdot \frac{\partial E}{\partial p_x} \cdot \frac{\partial p_x}{\partial \ln p_x} \\ &= \frac{x p_x}{E} = s_x. \end{aligned} \tag{viii}$$

[1]The research on the North American Free Trade Agreement is discussed in more detail in the Extensions to Chapter 13.

Deaton and Muellbauer (1980) make extensive use of this relationship to study the characteristics of a particular class of expenditure functions that they term an *almost ideal demand system* (AIDS). Their expenditure function takes the form

$$\begin{aligned}\ln E(p_x, p_y, V) = a_0 &+ a_1 \ln p_x + a_2 \ln p_y \\ &+ 0.5b_1(\ln p_x)^2 + b_2 \ln p_x \ln p_y \\ &+ 0.5b_3(\ln p_y)^2 + Vc_0 p_x^{c_1} p_y^{c_2}.\end{aligned} \quad \textbf{(ix)}$$

This form approximates any expenditure function. For the function to be homogeneous of degree one in the prices, the parameters of the function must obey the constraints $a_1 + a_2 = 1$, $b_1 + b_2 = 0$, $b_2 + b_3 = 0$, and $c_1 + c_2 = 0$. Using the results of Equation viii shows that, for this function,

$$\begin{aligned}s_x &= a_1 + b_1 \ln p_x + b_2 \ln p_y + c_1 V c_0 p_x^{c_1} p_y^{c_2}, \\ s_y &= a_2 + b_2 \ln p_x + b_3 \ln p_y + c_2 V c_0 p_x^{c_1} p_y^{c_2}.\end{aligned} \quad \textbf{(x)}$$

Notice that, given the parameter restrictions, $s_x + s_y = 1$. Making use of the inverse relationship between indirect utility and expenditure functions and some additional algebraic manipulation will put these budget share equations into a simple form suitable for econometric estimation:

$$\begin{aligned}s_x &= a_1 + b_1 \ln p_x + b_2 \ln p_y + c_1(E/p), \\ s_y &= a_2 + b_2 \ln p_x + b_3 \ln p_y + c_2(E/p),\end{aligned} \quad \textbf{(xi)}$$

where p is an index of prices defined by

$$\begin{aligned}\ln p = a_0 &+ a_1 \ln p_x + a_2 \ln p_y + 0.5b_1(\ln p_x)^2 \\ &+ b_2 \ln p_x \ln p_y + 0.5b_3(\ln p_y)^2.\end{aligned} \quad \textbf{(xii)}$$

In other words, the AIDS share equations state that budget shares are linear in the logarithms of prices and in total real expenditures. In practice, simpler price indices are often substituted for the rather complex index given by Equation xii, although there is some controversy about this practice (see the Extensions to Chapter 5).

British expenditure patterns

Deaton and Muellbauer apply this demand system to the study of British expenditure patterns between 1954 and 1974. They find that food and housing have negative coefficients of real expenditures, implying that the share of income devoted to these items decreases (at least in Britain) as people get richer. The authors also find significant relative price effects in many of their share equations, and prices have especially large effects in explaining the share of expenditures devoted to transportation and communication. In applying the AIDS model to real-world data, the authors also encounter a variety of econometric difficulties, the most important of which is that many of the equations do not appear to obey the restrictions necessary for homogeneity. Addressing such issues has been a major topic for further research on this demand system.

References

Behrman, Jere R. "Is Variety the Spice of Life? Implications for Caloric Intake." *Review of Economics and Statistics* (November 1989): 666–72.

Deaton, Angus, and John Muellbauer. "An Almost Ideal Demand System." *American Economic Review* (June 1980): 312–26.

Hyashi, Fumio. "Is the Japanese Extended Family Altruistically Linked? A Test Based on Engel Curves." *Journal of Political Economy* (June 1995): 661–74.

Kehoe, Patrick J., and Timothy J. Kehoe. *Modeling North American Economic Integration*. London: Kluwer Academic Publishers, 1995.

Oczkowski, E., and N. E. Philip. "Household Expenditure Patterns and Access to Consumer Goods in a Transitional Economy." *Journal of Economic Development* (June 1994): 165–83.

Stone, R. "Linear Expenditure Systems and Demand Analysis." *Economic Journal* (September 1954): 511–27.