Choice and Demand

PART
TWO

In **Part 2** we will investigate the economic theory of *choice*. One goal of this examination is to develop the notion of demand in a formal way so that it can be used in later sections of the text when we turn to the study of markets. A more general goal of this part is to illustrate the approach economists use for explaining how individuals make choices in a wide variety of contexts.

Part 2 begins with a description of the way economists model individual preferences, which are usually referred to by the formal term *utility*. **Chapter 3** shows how economists are able to conceptualize utility in a mathematical way. This permits an examination of the various exchanges that individuals are willing to make voluntarily.

The utility concept is used in **Chapter 4** to illustrate the theory of choice. The fundamental hypothesis of the chapter is that people faced with limited incomes will make economic choices in such a way as to achieve as much utility as possible. Chapter 4 uses mathematical and intuitive analyses to indicate the insights that this hypothesis provides about economic behavior.

Chapters 5 and 6 use the model of utility maximization to investigate how individuals will respond to changes in their circumstances. **Chapter 5** is primarily concerned with responses to changes in the price of a commodity, an analysis that leads directly to the demand curve concept. **Chapter 6** applies this type of analysis to developing an understanding of demand relationships among different goods.

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CHAPTER THREE

Preferences and Utility

In this chapter we look at the way in which economists characterize individuals' preferences. We begin with a fairly abstract discussion of the "preference relation," but quickly turn to the economists' primary tool for studying individual choices—the utility function. We look at some general characteristics of that function and a few simple examples of specific utility functions we will encounter throughout this book.

AXIOMS OF RATIONAL CHOICE

One way to begin an analysis of individuals' choices is to state a basic set of postulates, or axioms, that characterize "rational" behavior. These begin with the concept of "preference": An individual who reports that "A is preferred to B" is taken to mean that all things considered, he or she feels better off under situation A than under situation B. The preference relation is assumed to have three basic properties as follows.

I. *Completeness*. If A and B are *any* two situations, the individual can always specify exactly one of the following three possibilities:
   1. "A is preferred to B,"
   2. "B is preferred to A," or
   3. "A and B are equally attractive."

   Consequently, people are assumed not to be paralyzed by indecision: They completely understand and can always make up their minds about the desirability of any two alternatives. The assumption also rules out the possibility that an individual can report both that A is preferred to B and that B is preferred to A.

II. *Transitivity*. If an individual reports that "A is preferred to B" and "B is preferred to C," then he or she must also report that "A is preferred to C."

   This assumption states that the individual's choices are internally consistent. Such an assumption can be subjected to empirical study. Generally, such studies conclude that a person's choices are indeed transitive, but this conclusion must be modified in cases where the individual may not fully understand the consequences of the choices he or she is making. Because, for the most part, we will assume choices are fully informed (but see the discussion of uncertainty in Chapter 7 and elsewhere), the transitivity property seems to be an appropriate assumption to make about preferences.

III. *Continuity*. If an individual reports "A is preferred to B," then situations suitably "close to" A must also be preferred to B.

   This rather technical assumption is required if we wish to analyze individuals' responses to relatively small changes in income and prices. The purpose of the assumption is to rule out certain kinds of discontinuous, knife-edge preferences that pose problems for a mathematical development of the theory of choice. Assuming continuity does not seem to risk missing types of economic behavior that are important in the real world (but see Problem 3.14 for some counterexamples).

UTILITY

Given the assumptions of completeness, transitivity, and continuity, it is possible to show formally that people are able to rank all possible situations from the least desirable to the most.[1] Following the terminology introduced by the nineteenth-century political theorist Jeremy Bentham, economists call this ranking *utility*.[2] We also will follow Bentham by saying that more desirable situations offer more utility than do less desirable ones. That is, if a person prefers situation A to situation B, we would say that the utility assigned to option A, denoted by $U(A)$, exceeds the utility assigned to B, $U(B)$.

Nonuniqueness of utility measures

We might even attach numbers to these utility rankings; however, these numbers will not be unique. Any set of numbers we arbitrarily assign that accurately reflects the original preference ordering will imply the same set of choices. It makes no difference whether we say that $U(A) = 5$ and $U(B) = 4$, or that $U(A) = 1{,}000{,}000$ and $U(B) = 0.5$. In both cases the numbers imply that A is preferred to B. In technical terms, our notion of utility is defined only up to an order-preserving ("monotonic") transformation.[3] Any set of numbers that accurately reflects a person's preference ordering will do. Consequently, it makes no sense to ask "how much more is A preferred than B?" because that question has no unique answer. Surveys that ask people to rank their "happiness" on a scale of 1 to 10 could just as well use a scale of 7 to 1,000,000. We can only hope that a person who reports he or she is a "6" on the scale one day and a "7" on the next day is indeed happier on the second day. Therefore, utility rankings are like the ordinal rankings of restaurants or movies using one, two, three, or four stars. They simply record the relative desirability of commodity bundles.

This lack of uniqueness in the assignment of utility numbers also implies that it is not possible to compare utilities of different people. If one person reports that a steak dinner provides a utility of "5" and another person reports that the same dinner offers a utility of "100," we cannot say which individual values the dinner more because they could be using different scales. Similarly, we have no way of measuring whether a move from situation A to situation B provides more utility to one person or another. Nonetheless, as we will see, economists can say quite a bit about utility rankings by examining what people voluntarily choose to do.

The ceteris paribus assumption

Because *utility* refers to overall satisfaction, such a measure clearly is affected by a variety of factors. A person's utility is affected not only by his or her consumption of physical commodities but also by psychological attitudes, peer group pressures, personal experiences, and the general cultural environment. Although economists do have a general interest in examining such influences, a narrowing of focus is usually necessary. Consequently, a common

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[1]These properties and their connection to representation of preferences by a utility function are discussed in detail in Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory* (New York: Oxford University Press, 1995).

[2]J. Bentham, *Introduction to the Principles of Morals and Legislation* (London: Hafner, 1848).

[3]We can denote this idea mathematically by saying that any numerical utility ranking (U) can be transformed into another set of numbers by the function F providing that $F(U)$ is order preserving. This can be ensured if $F'(U) > 0$. For example, the transformation $F(U) = U^2$ is order preserving as is the transformation $F(U) = \ln U$. At some places in the text and problems we will find it convenient to make such transformations to make a particular utility ranking easier to analyze.

practice is to devote attention exclusively to choices among quantifiable options (e.g., the relative quantities of food and shelter bought, the number of hours worked per week, or the votes among specific taxing formulas) while holding constant the other things that affect behavior. This *ceteris paribus* ("other things being equal") assumption is invoked in all economic analyses of utility-maximizing choices so as to make the analysis of choices manageable within a simplified setting.

Utility from consumption of goods

As an important example of the ceteris paribus assumption, consider an individual's problem of choosing, at a single point in time, among n consumption goods $x_1, x_2, \ldots, x_n$. We shall assume that the individual's ranking of these goods can be represented by a utility function of the form

$$\text{utility} = U(x_1, x_2, \ldots, x_n; \text{other things}), \tag{3.1}$$

where the x's refer to the quantities of the goods that might be chosen and the "other things" notation is used as a reminder that many aspects of individual welfare are being held constant in the analysis.

Often it is easier to write Equation 3.1 as

$$\text{utility} = U(x_1, x_2, \ldots, x_n) \tag{3.2}$$

Or, if only two goods are being considered, as

$$\text{utility} = U(x, y), \tag{3.2'}$$

where it is clear that everything is being held constant (i.e., outside the frame of analysis) except the goods actually referred to in the utility function. It would be tedious to remind you at each step what is being held constant in the analysis, but it should be remembered that some form of the ceteris paribus assumption will always be in effect.

Arguments of utility functions

The utility function notation is used to indicate how an individual ranks the particular arguments of the function being considered. In the most common case, the utility function (Equation 3.2) will be used to represent how an individual ranks certain bundles of goods that might be purchased at one point in time. On occasion we will use other arguments in the utility function, and it is best to clear up certain conventions at the outset. For example, it may be useful to talk about the utility an individual receives from real wealth (W). Therefore, we shall use the notation

$$\text{utility} = U(W). \tag{3.3}$$

Unless the individual is a rather peculiar, Scrooge-type person, wealth in its own right gives no direct utility. Rather, it is only when wealth is spent on consumption goods that any utility results. For this reason, Equation 3.3 will be taken to mean that the utility from wealth is in fact derived by spending that wealth in such a way as to yield as much utility as possible.

Two other arguments of utility functions will be used in later chapters. In Chapter 16 we will be concerned with the individual's labor–leisure choice and will therefore have to consider the presence of leisure in the utility function. A function of the form

$$\text{utility} = U(c, h) \tag{3.4}$$

will be used. Here, c represents consumption and h represents hours of nonwork time (i.e., leisure) during a particular period.

In Chapter 17 we will be interested in the individual's consumption decisions in different periods. In that chapter we will use a utility function of the form

$$\text{utility} = U(c_1, c_2), \tag{3.5}$$

where c_1 is consumption in this period and c_2 is consumption in the next period. By changing the arguments of the utility function, therefore, we will be able to focus on specific aspects of an individual's choices in a variety of simplified settings.

In summary, we start our examination of individual behavior with the following definition.

DEFINITION

Utility. Individuals' preferences are assumed to be represented by a utility function of the form

$$U(x_1, x_2, \ldots, x_n), \tag{3.6}$$

where $x_1, x_2, \ldots, x_n$ are the quantities of each of n goods that might be consumed in a period. This function is unique only up to an order-preserving transformation.

Economic goods

In this representation the variables are taken to be "goods"; that is, whatever economic quantities they represent, we assume that more of any particular x_i during some period is preferred to less. We assume this is true of every good, be it a simple consumption item such as a hot dog or a complex aggregate such as wealth or leisure. We have pictured this convention for a two-good utility function in Figure 3.1. There, all consumption bundles in the shaded area are preferred to the bundle x^*, y^* because any bundle in the shaded area provides more of at least one of the goods. By our definition of "goods," bundles of goods in the shaded area are ranked higher than x^*, y^*. Similarly, bundles in the area marked "worse" are clearly inferior to x^*, y^* because they contain less of at least one of the goods and no more of the other. Bundles in the two areas indicated by question marks are difficult to compare with x^*, y^* because they contain more of one of the goods and less of the other. Movements into these areas involve trade-offs between the two goods.

TRADES AND SUBSTITUTION

Most economic activity involves voluntary trading between individuals. When someone buys, say, a loaf of bread, he or she is voluntarily giving up one thing (money) for something else (bread) that is of greater value to that individual. To examine this kind of voluntary transaction, we need to develop a formal apparatus for illustrating trades in the utility function context. We first motivate our discussion with a graphical presentation and then turn to some more formal mathematics.

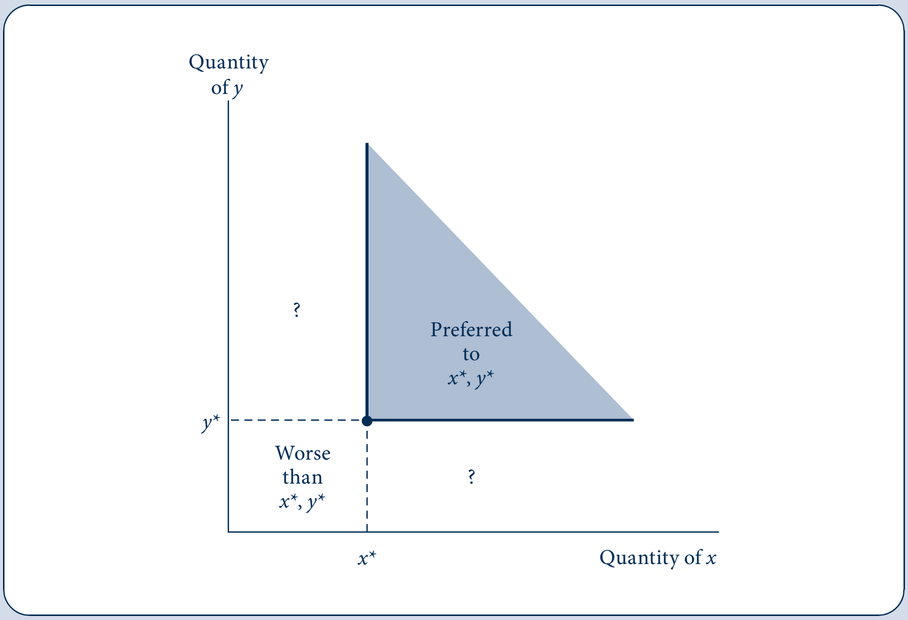
Indifference curves and the marginal rate of substitution

Voluntary trades can best be studied using the graphical device of an *indifference curve.* In Figure 3.2, the curve U_1 represents all the alternative combinations of x and y for which an individual is equally well off (remember again that all other arguments of the utility function are held constant). This person is equally happy consuming, for example, either the combination of goods x_1, y_1 or the combination x_2, y_2. This curve representing all the consumption bundles that the individual ranks equally is called an *indifference curve*.

FIGURE 3.1

More of a Good Is Preferred to Less

The shaded area represents those combinations of x and y that are unambiguously preferred to the combination $x*$, $y*$. Ceteris paribus, individuals prefer more of any good rather than less. Combinations identified by "?" involve ambiguous changes in welfare because they contain more of one good and less of the other.



DEFINITION

Indifference curve. An *indifference curve* (or, in many dimensions, an indifference surface) shows a set of consumption bundles about which the individual is indifferent. That is, the bundles all provide the same level of utility.

The slope of the indifference curve in Figure 3.2 is negative, showing that if the individual is forced to give up some y, he or she must be compensated by an additional amount of x to remain indifferent between the two bundles of goods. The curve is also drawn so that the slope increases as x increases (i.e., the slope starts at negative infinity and increases toward zero). This is a graphical representation of the assumption that people become progressively less willing to trade away y to get more x. In mathematical terms, the absolute value of this slope diminishes as x increases. Hence we have the following definition.

DEFINITION

Marginal rate of substitution. The negative of the slope of an indifference curve (U_1) at some point is termed the *marginal rate of substitution (MRS)* at that point. That is,

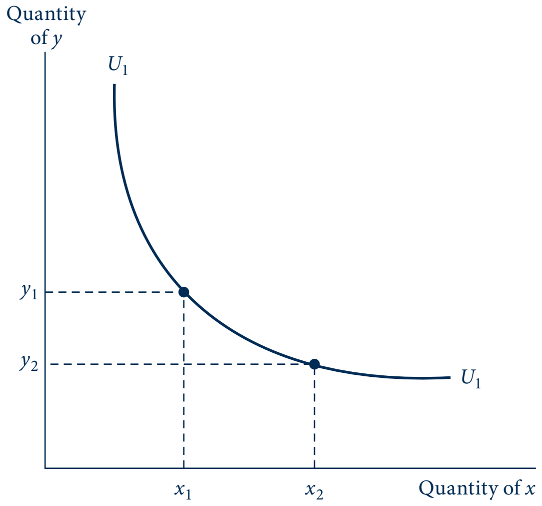
$$MRS = -\left.\frac{dy}{dx}\right|_{U=U_1}, \tag{3.7}$$

where the notation indicates that the slope is to be calculated along the U_1 indifference curve.

FIGURE 3.2

A Single Indifference Curve

The curve U_1 represents those combinations of x and y from which the individual derives the same utility. The slope of this curve represents the rate at which the individual is willing to trade x for y while remaining equally well off. This slope (or, more properly, the negative of the slope) is termed the *marginal rate of substitution*. In the figure, the indifference curve is drawn on the assumption of a diminishing marginal rate of substitution.



Therefore, the slope of U_1 and the *MRS* tell us something about the trades this person will voluntarily make. At a point such as x_1, y_1, the person has a lot of y and is willing to trade away a significant amount to get one more x. Therefore, the indifference curve at x_1, y_1 is rather steep. This is a situation where the person has, say, many hamburgers (y) and little to drink with them (x). This person would gladly give up a few burgers (say, 5) to quench his or her thirst with one more drink.

At x_2, y_2, on the other hand, the indifference curve is flatter. Here, this person has a few drinks and is willing to give up relatively few burgers (say, 1) to get another soft drink. Consequently, the *MRS* diminishes between x_1, y_1 and x_2, y_2. The changing slope of U_1 shows how the particular consumption bundle available influences the trades this person will freely make.

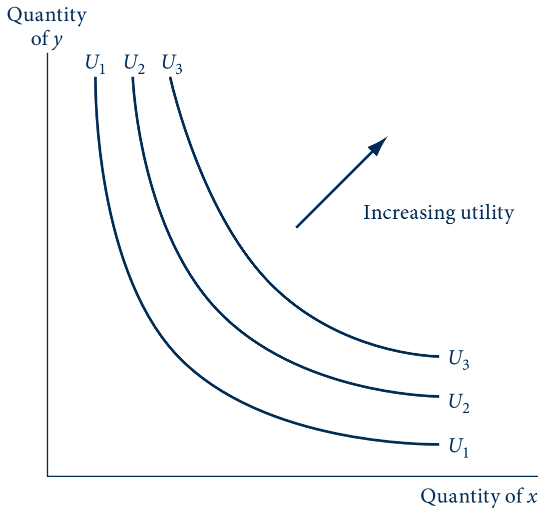
Indifference curve map

In Figure 3.2 only one indifference curve was drawn. The x, y quadrant, however, is densely packed with such curves, each corresponding to a different level of utility. Because every bundle of goods can be ranked and yields some level of utility, each point in Figure 3.2 must have an indifference curve passing through it. Indifference curves are similar to contour lines on a map in that they represent lines of equal "altitude" of utility. In Figure 3.3 several indifference curves are shown to indicate that there are infinitely many in the plane. The level of utility represented by these curves increases as we move in a northeast direction; the utility of curve U_1 is less than that of U_2, which is less than that of U_3. This is because of the assumption made in Figure 3.1: More of a good is preferred to less. As was discussed earlier, there is no unique way to assign numbers to these

FIGURE 3.3

There Are Infinitely Many Indifference Curves in the x–y Plane

There is an indifference curve passing through each point in the x–y plane. Each of these curves records combinations of x and y from which the individual receives a certain level of satisfaction. Movements in a northeast direction represent movements to higher levels of satisfaction.



utility levels. The curves only show that the combinations of goods on U_3 are preferred to those on U_2, which are preferred to those on U_1.

Indifference curves and transitivity

As an exercise in examining the relationship between consistent preferences and the representation of preferences by utility functions, consider the following question: Can any two of an individual's indifference curves intersect? Two such intersecting curves are shown in Figure 3.4. We wish to know if they violate our basic axioms of rationality. Using our map analogy, there would seem to be something wrong at point E, where "altitude" is equal to two different numbers, U_1 and U_2. But no point can be both 100 and 200 feet above sea level.

To proceed formally, let us analyze the bundles of goods represented by points A, B, C, and D. By the assumption of nonsatiation (i.e., more of a good always increases utility), "A is preferred to B" and "C is preferred to D." But this person is equally satisfied with B and C (they lie on the same indifference curve), so the axiom of transitivity implies that A must be preferred to D. But that cannot be true because A and D are on the same indifference curve and are by definition regarded as equally desirable. This contradiction shows that indifference curves cannot intersect. Therefore, we should always draw indifference curve maps as they appear in Figure 3.3.

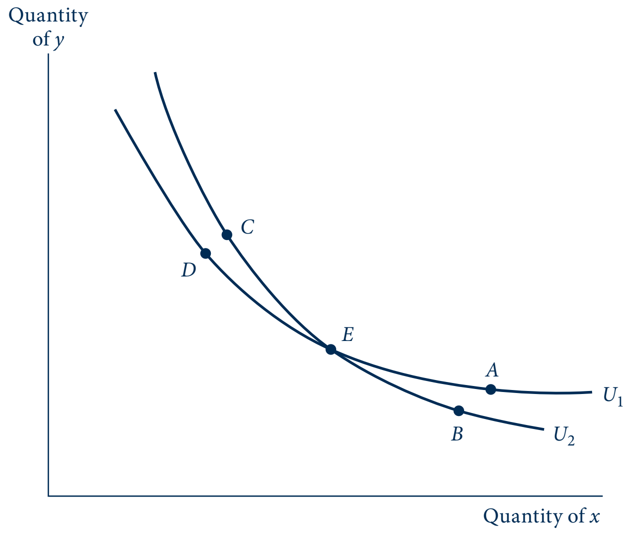
Convexity of indifference curves

An alternative way of stating the principle of a diminishing marginal rate of substitution uses the mathematical notion of a convex set. A set of points is said to be *convex* if any two points within the set can be joined by a straight line that is contained completely

FIGURE 3.4

Intersecting Indifference Curves Imply Inconsistent Preferences

Combinations A and D lie on the same indifference curve and therefore are equally desirable. But the axiom of transitivity can be used to show that A is preferred to D. Hence intersecting indifference curves are not consistent with rational preferences.



within the set. The assumption of a diminishing *MRS* is equivalent to the assumption that all combinations of x and y that are preferred or indifferent to a particular combination x^*, y^* form a convex set.[4] This is illustrated in Figure 3.5a, where all combinations preferred or indifferent to x^*, y^* are in the shaded area. Any two of these combinations—say, x_1, y_1 and x_2, y_2—can be joined by a straight line also contained in the shaded area. In Figure 3.5b this is not true. A line joining x_1, y_1 and x_2, y_2 passes outside the shaded area. Therefore, the indifference curve through x^*, y^* in Figure 3.5b does not obey the assumption of a diminishing *MRS* because the set of points preferred or indifferent to x^*, y^* is not convex.

Convexity and balance in consumption

By using the notion of convexity, we can show that individuals prefer some balance in their consumption. Suppose that an individual is indifferent between the combinations x_1, y_1 and x_2, y_2. If the indifference curve is strictly convex, then the combination $(x_1 + x_2)/2$, $(y_1 + y_2)/2$ will be preferred to either of the initial combinations.[5] Intuitively, "well-balanced" bundles of commodities are preferred to bundles that are heavily weighted toward one commodity. This is illustrated in Figure 3.6. Because the indifference curve is assumed to be convex, all points on the straight line joining (x_1, y_1) and (x_2, y_2) are preferred to these initial points. Therefore, this will be true of the point $(x_1 + x_2)/2$, $(y_1 + y_2)/2$,

[4]This definition is equivalent to assuming that the utility function is quasi-concave. Such functions were discussed in Chapter 2, and we shall return to examine them in the next section. Sometimes the term *strict quasi-concavity* is used to rule out the possibility of indifference curves having linear segments. We generally will assume strict quasi-concavity, but in a few places we will illustrate the complications posed by linear portions of indifference curves.

[5]In the case in which the indifference curve has a linear segment, the individual will be indifferent among all three combinations.

FIGURE 3.5

The Notion of Convexity as an Alternative Definition of a Diminishing *MRS*

In (a) the indifference curve is *convex* (any line joining two points above U_1 is also above U_1). In (b) this is not the case, and the curve shown here does not everywhere have a diminishing *MRS*.

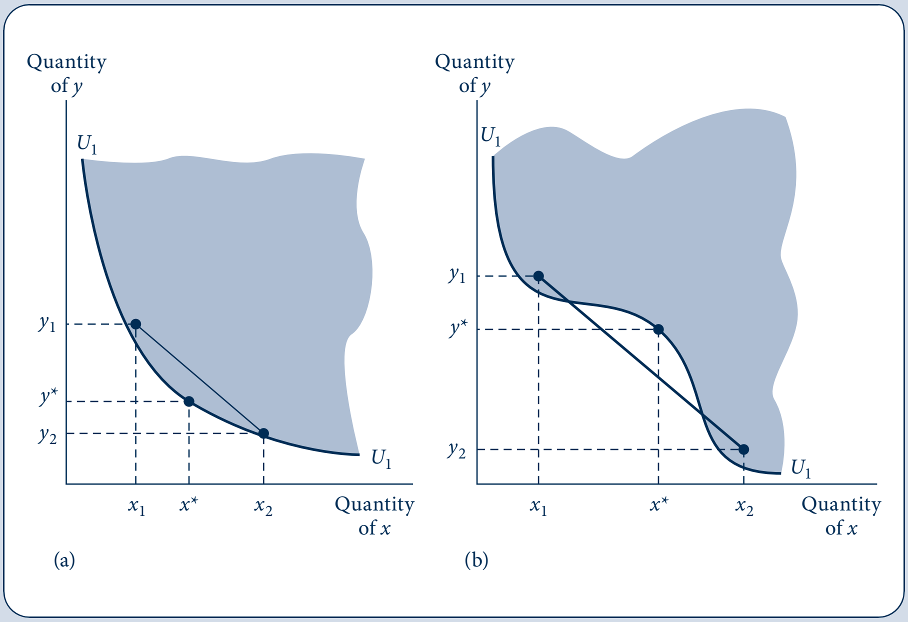
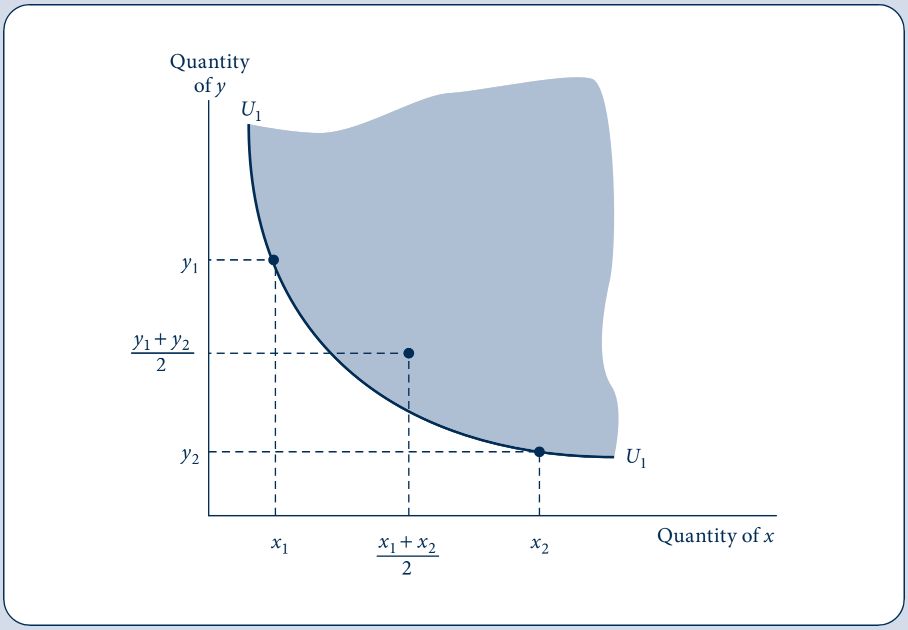


FIGURE 3.6

Balanced Bundles of Goods Are Preferred to Extreme Bundles

If indifference curves are convex (if they obey the assumption of a diminishing *MRS*), then the line joining any two points that are indifferent will contain points preferred to either of the initial combinations. Intuitively, balanced bundles are preferred to unbalanced ones.



which lies at the midpoint of such a line. Indeed, any proportional combination of the two indifferent bundles of goods will be preferred to the initial bundles because it will represent a more balanced combination. Thus, strict convexity is equivalent to the assumption of a diminishing *MRS*. Both assumptions rule out the possibility of an indifference curve being straight over any portion of its length.

EXAMPLE 3.1 Utility and the *MRS*

Suppose a person's ranking of hamburgers (y) and soft drinks (x) could be represented by the utility function

$$\text{utility} = \sqrt{x \cdot y}. \tag{3.8}$$

An indifference curve for this function is found by identifying that set of combinations of x and y for which utility has the same value. Suppose we arbitrarily set utility equal to 10. Then the equation for this indifference curve is

$$\text{utility} = 10 = \sqrt{x \cdot y}. \tag{3.9}$$

Because squaring this function is order preserving, the indifference curve is also represented by

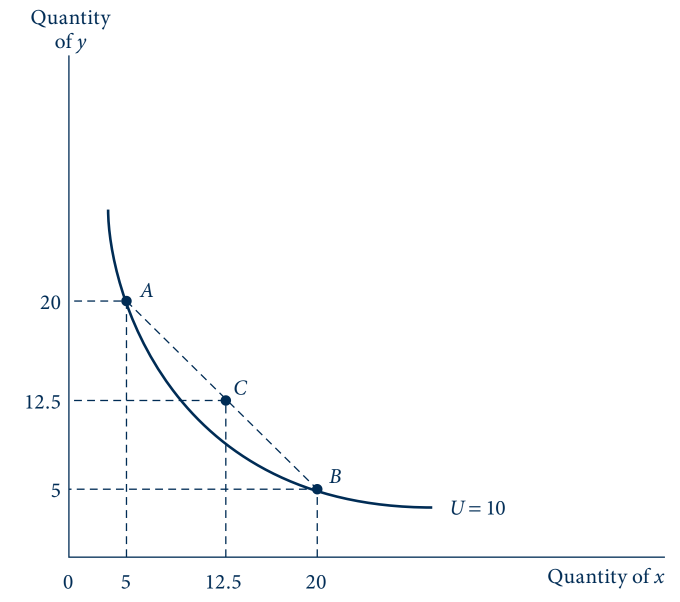
$$100 = x \cdot y, \tag{3.10}$$

which is easier to graph. In Figure 3.7 we show this indifference curve; it is a familiar rectangular hyperbola. One way to calculate the *MRS* is to solve Equation 3.10 for y,

$$y = 100/x, \tag{3.11}$$

FIGURE 3.7 Indifference Curve for Utility $= \sqrt{x \cdot y}$

This indifference curve illustrates the function $10 = U = \sqrt{x \cdot y}$. At point A (5, 20), the *MRS* is 4, implying that this person is willing to trade $4y$ for an additional x. At point B (20, 5), however, the *MRS* is 0.25, implying a greatly reduced willingness to trade.



And then use the definition (Equation 3.7):

$$MRS = -dy/dx \text{ (along } U_1) = 100/x^2. \quad \textbf{(3.12)}$$

Clearly this *MRS* decreases as x increases. At a point such as A on the indifference curve with a lot of hamburgers (say, $x = 5$, $y = 20$), the slope is steep so the *MRS* is high:

$$MRS \text{ at } (5, 20) = 100/x^2 = 100/25 = 4. \quad \textbf{(3.13)}$$

Here the person is willing to give up 4 hamburgers to get 1 more soft drink. On the other hand, at B where there are relatively few hamburgers (here $x = 20$, $y = 5$), the slope is flat and the *MRS* is low:

$$MRS \text{ at } (20, 5) = 100/x^2 = 100/400 = 0.25. \quad \textbf{(3.14)}$$

Now he or she will only give up one quarter of a hamburger for another soft drink. Notice also how convexity of the indifference curve U_1 is illustrated by this numerical example. Point C is midway between points A and B; at C this person has 12.5 hamburgers and 12.5 soft drinks. Here utility is given by

$$\text{utility} = \sqrt{x \cdot y} = \sqrt{(12.5)^2} = 12.5, \quad \textbf{(3.15)}$$

which clearly exceeds the utility along U_1 (which was assumed to be 10).

QUERY: From our derivation here, it appears that the *MRS* depends only on the quantity of x consumed. Why is this misleading? How does the quantity of y implicitly enter into Equations 3.13 and 3.14?

THE MATHEMATICS OF INDIFFERENCE CURVES

A mathematical derivation of the indifference curve concept provides additional insights about the nature of preferences. In this section we look at a two-good example that ties directly to the graphical treatment provided previously. Later in the chapter we look at the many-good case, but conclude that this more complicated case adds only a few additional insights.

The marginal rate of substitution

Suppose an individual receives utility from consuming two goods whose quantities are given by x and y. This person's ranking of bundles of these goods can be represented by a utility function of the form $U(x, y)$. Those combinations of the two goods that yield a specific level of utility, say k, are represented by solutions to the implicit equation $U(x, y) = k$. In Chapter 2 (see Equation 2.23) we showed that the trade-offs implied by such an equation are given by:

$$\frac{dy}{dx}\bigg|_{U(x,\, y) = k} = -\frac{U_x}{U_y}. \quad \textbf{(3.16)}$$

That is, the rate at which x can be traded for y is given by the negative of the ratio of the "marginal utility" of good x to that of good y. Assuming additional amounts of both goods provide added utility, this trade-off rate will be negative, implying that increases in the quantity of good x must be met by decreases in the quantity of good y to keep utility constant. Earlier we defined the *marginal rate of substitution* as the negative (or absolute value) of this trade-off, so now we have:

$$MRS = -\frac{dy}{dx}\Big|_{U(x,\ y)=k} = \frac{U_x}{U_y}. \tag{3.17}$$

This derivation helps in understanding why the *MRS* does not depend specifically on how utility is measured. Because the *MRS* is a ratio of two utility measures, the units "drop out" in the computation. For example, suppose good x represents food and that we have chosen a utility function for which an extra unit of food yields 6 extra units of utility (sometimes these units are called *utils*). Suppose also that y represents clothing and with this utility function each extra unit of clothing provides 2 extra units of utility. In this case it is clear that this person is willing to give up 3 units of clothing (thereby losing 6 utils) in exchange for 1 extra unit of food (thereby gaining 6 utils):

$$MRS = -\frac{dy}{dx} = \frac{U_x}{U_y} = \frac{6\ utils\ per\ unit\ x}{2\ utils\ per\ unit\ y} = 3\ units\ y\ per\ unit\ x. \tag{3.18}$$

Notice that the utility measure used here (utils) drops out in making this computation and what remains is purely in terms of the units of the two goods. This shows that the *MRS* will be unchanged no matter what specific utility ranking is used.[6]

Convexity of Indifference Curves

In Chapter 1 we described how economists were able to resolve the water–diamond paradox by proposing that the price of water is low because one more gallon provides relatively little in terms of increased utility. Water is (for the most part) plentiful; therefore, its marginal utility is low. Of course, in a desert, water would be scarce and its marginal utility (and price) could be high. Thus, one might conclude that the marginal utility associated with water consumption decreases as more water is consumed—in formal terms, the second (partial) derivative of the utility function (i.e., $U_{xx} = \partial^2 U/\partial x^2$) should be negative.

Intuitively it seems that this commonsense idea should also explain why indifference curves are convex. The fact that people are increasingly less willing to part with good y to get more x (while holding utility constant) seems to refer to the same phenomenon—that people do not want too much of any one good. Unfortunately, the precise connection between diminishing marginal utility and a diminishing *MRS* is complex, even in the two-good case. As we showed in Chapter 2, a function will (by definition) have convex indifference curves, providing it is quasi-concave. But the conditions required for quasi-concavity are messy, and the assumption of diminishing marginal utility (i.e., negative second-order partial derivatives) will not ensure that they hold.[7] Still, as we shall see, there are good reasons for assuming that utility functions (and many other functions used in microeconomics) are quasi-concave; thus, we will not be too concerned with situations in which they are not.

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[6]More formally, let $F[U(x, y)]$ be any monotonic transformation of the utility function with $F'(U) > 0$. With this new utility ranking the *MRS* is given by:

$$MRS = \frac{\partial F/\partial x}{\partial F/\partial y} = \frac{F'(U).U_x}{F'(U).U_y} = \frac{U_x}{U_y},$$

which is the same as the *MRS* for the original utility function.

[7]Specifically, for the function $U(x, y)$ to be quasi-concave the following condition must hold (see Equation 2.114):

$$U_{xx}U_x^2 - 2U_{xy}U_xU_y + U_{yy}U_y^2 < 0.$$

The assumptions that $U_{xx}, U_{yy} < 0$ will not ensure this. One must also be concerned with the sign of the cross partial derivative U_{xy}.

EXAMPLE 3.2 Showing Convexity of Indifference Curves

Calculation of the *MRS* for specific utility functions is frequently a good shortcut for showing convexity of indifference curves. In particular, the process can be much simpler than applying the definition of quasi-concavity, although it is more difficult to generalize to more than two goods. Here we look at how Equation 3.17 can be used for three different utility functions (for more practice, see Problem 3.1).

1. $U(x, y) = \sqrt{x \cdot y}$.
   This example just repeats the case illustrated in Example 3.1. One shortcut to applying Equation 3.17 that can simplify the algebra is to take the logarithm of this utility function. Because taking logs is order preserving, this will not alter the *MRS* to be calculated. Thus, let
   $$U^*(x, y) = \ln[U(x, y)] = 0.5 \ln x + 0.5 \ln y. \qquad \textbf{(3.19)}$$
   Applying Equation 3.17 yields
   $$MRS = \frac{\partial U^*/\partial x}{\partial U^*/\partial y} = \frac{0.5/x}{0.5/y} = \frac{y}{x}, \qquad \textbf{(3.20)}$$
   which seems to be a much simpler approach than we used previously.[8] Clearly this *MRS* is diminishing as x increases and y decreases. Therefore, the indifference curves are convex.
2. $U(x, y) = x + xy + y$.
   In this case there is no advantage to transforming this utility function. Applying Equation 3.17 yields
   $$MRS = \frac{\partial U/\partial x}{\partial U/\partial y} = \frac{1 + y}{1 + x}. \qquad \textbf{(3.21)}$$
   Again, this ratio clearly decreases as x increases and y decreases; thus, the indifference curves for this function are convex.
3. $U(x, y) = \sqrt{x^2 + y^2}$
   For this example it is easier to use the transformation
   $$U^*(x, y) = [U(x, y)]^2 = x^2 + y^2. \qquad \textbf{(3.22)}$$
   Because this is the equation for a quarter-circle, we should begin to suspect that there might be some problems with the indifference curves for this utility function. These suspicions are confirmed by again applying the definition of the *MRS* to yield
   $$MRS = \frac{\partial U^*/\partial x}{\partial U^* \partial y} = \frac{2x}{2y} = \frac{x}{y}. \qquad \textbf{(3.23)}$$
   For this function, it is clear that, as x increases and y decreases, the *MRS increases*! Hence the indifference curves are concave, not convex, and this is clearly not a quasi-concave function.

QUERY: Does a doubling of x and y change the *MRS* in each of these three examples? That is, does the *MRS* depend only on the ratio of x to y, not on the absolute scale of purchases? (See also Example 3.3.)

[8] In Example 3.1 we looked at the $U = 10$ indifference curve. Thus, for that curve, $y = 100/x$, and the *MRS* in Equation 3.20 would be $MRS = 100/x^2$ as calculated before.

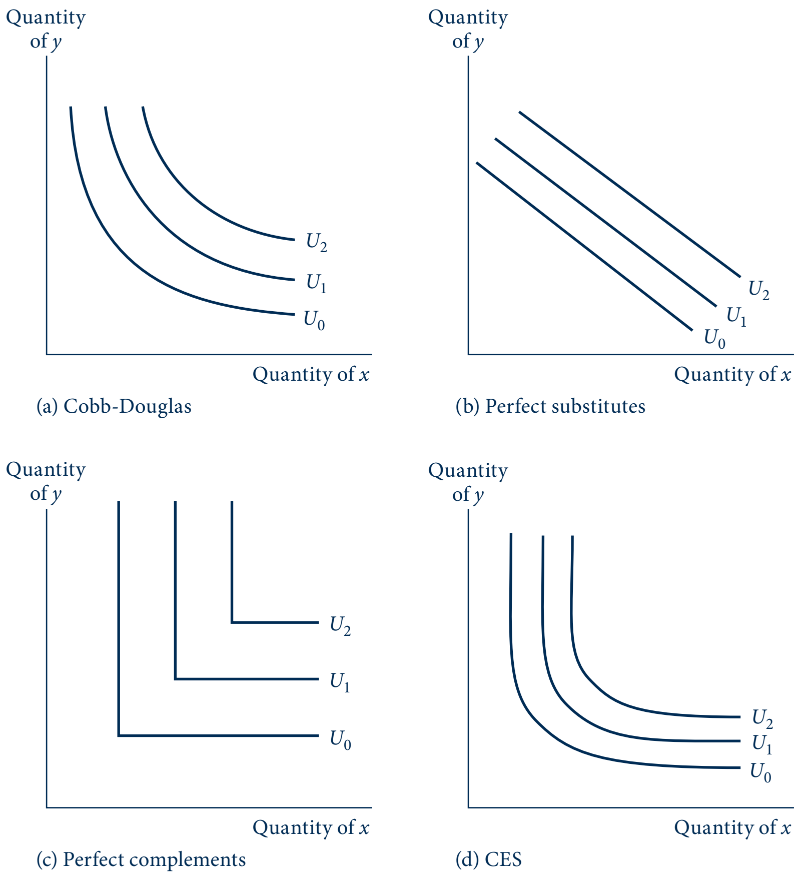
UTILITY FUNCTIONS FOR SPECIFIC PREFERENCES

Individuals' rankings of commodity bundles and the utility functions implied by these rankings are unobservable. All we can learn about people's preferences must come from the behavior we observe when they respond to changes in income, prices, and other factors. Nevertheless, it is useful to examine a few of the forms particular utility functions might take. Such an examination may offer insights into observed behavior, and (more to the point) understanding the properties of such functions can be of some help in solving problems. Here we will examine four specific examples of utility functions for two goods. Indifference curve maps for these functions are illustrated in the four panels of Figure 3.8. As should be visually apparent, these cover a few possible shapes. Even greater variety is possible once we move to functions for three or more goods, and some of these possibilities are mentioned in later chapters.

FIGURE 3.8

Examples of Utility Functions

The four indifference curve maps illustrate alternative degrees of substitutability of x for y. The Cobb–Douglas and constant elasticity of substitution (CES) functions (drawn here for relatively low substitutability) fall between the extremes of perfect substitution (b) and no substitution (c).



Cobb–Douglas utility

Figure 3.8a shows the familiar shape of an indifference curve. One commonly used utility function that generates such curves has the form

$$\text{utility} = U(x, y) = x^{\alpha} y^{\beta}, \tag{3.24}$$

where α and β are positive constants.

In Examples 3.1 and 3.2, we studied a particular case of this function for which $\alpha = \beta = 0.5$. The more general case presented in Equation 3.24 is termed a *Cobb–Douglas utility function*, after two researchers who used such a function for their detailed study of production relationships in the U.S. economy (see Chapter 9). In general, the relative sizes of α and β indicate the relative importance of the two goods to this individual. Because utility is unique only up to a monotonic transformation, it is often convenient to normalize these parameters so that $\alpha + \beta = 1$. In this case, utility would be given by

$$U(x, y) = x^{\delta} y^{1-\delta} \tag{3.25}$$

where $\delta = \alpha/(\alpha + \beta)$, $1 - \delta = \beta/(\alpha + \beta)$.

Perfect substitutes

The linear indifference curves in Figure 3.8b are generated by a utility function of the form

$$\text{utility} = U(x, y) = \alpha x + \beta y, \tag{3.26}$$

where, again, α and β are positive constants. That the indifference curves for this function are straight lines should be readily apparent: Any particular level curve can be calculated by setting $U(x, y)$ equal to a constant that specifies a straight line. The linear nature of these indifference curves gave rise to the term *perfect substitutes* to describe the implied relationship between x and y. Because the *MRS* is constant (and equal to α/β) along the entire indifference curve, our previous notions of a diminishing *MRS* do not apply in this case. A person with these preferences would be willing to give up the same amount of y to get one more x no matter how much x was being consumed. Such a situation might describe the relationship between different brands of what is essentially the same product. For example, many people (including the author) do not care where they buy gasoline. A gallon of gas is a gallon of gas despite the best efforts of the Exxon and Shell advertising departments to convince me otherwise. Given this fact, I am always willing to give up 10 gallons of Exxon in exchange for 10 gallons of Shell because it does not matter to me which I use or where I got my last tankful. Indeed, as we will see in the next chapter, one implication of such a relationship is that I will buy all my gas from the least expensive seller. Because I do not experience a diminishing *MRS* of Exxon for Shell, I have no reason to seek a balance among the gasoline types I use.

Perfect complements

A situation directly opposite to the case of perfect substitutes is illustrated by the L-shaped indifference curves in Figure 3.8c. These preferences would apply to goods that "go together"—coffee and cream, peanut butter and jelly, and cream cheese and lox are familiar examples. The indifference curves shown in Figure 3.8c imply that these pairs of goods will be used in the fixed proportional relationship represented by the vertices of the curves. A person who prefers 1 ounce of cream with 8 ounces of coffee will want 2 ounces of cream with 16 ounces of coffee. Extra coffee without cream is of no value to this person, just as extra cream would be of no value without coffee. Only by choosing the goods together can utility be increased.

These concepts can be formalized by examining the mathematical form of the utility function that generates these L-shaped indifference curves:

$$\text{utility} = U(x, y) = \min(\alpha x, \beta y). \tag{3.27}$$

Here α and β are positive parameters, and the operator "min" means that utility is given by the smaller of the two terms in the parentheses. In the coffee–cream example, if we let ounces of coffee be represented by x and ounces of cream by y, utility would be given by

$$\text{utility} = U(x, y) = \min(x, 8y). \tag{3.28}$$

Now 8 ounces of coffee and 1 ounce of cream provide 8 units of utility. But 16 ounces of coffee and 1 ounce of cream still provide only 8 units of utility because $\min(16, 8) = 8$. The extra coffee without cream is of no value, as shown by the horizontal section of the indifference curves for movement away from a vertex; utility does not increase when only x increases (with y constant). Only if coffee and cream are both doubled (to 16 and 2, respectively) will utility increase to 16.

More generally, neither of the two goods specified in the utility function given by Equation 3.27 will be consumed in superfluous amounts if $\alpha x = \beta y$. In this case, the ratio of the quantity of good x consumed to that of good y will be a constant given by

$$\frac{y}{x} = \frac{\alpha}{\beta}. \tag{3.29}$$

Consumption will occur at the vertices of the indifference curves shown in Figure 3.8c.

CES utility

The three specific utility functions illustrated thus far are special cases of the more general CES function, which takes the form

$$\text{utility} = U(x, y) = \frac{x^\delta}{\delta} + \frac{y^\delta}{\delta}, \tag{3.30}$$

where $\delta \leq 1$, $\delta \neq 0$, and

$$\text{utility} = U(x, y) = \ln x + \ln y \tag{3.31}$$

when $\delta = 0$. It is obvious that the case of perfect substitutes corresponds to the limiting case, $\delta = 1$, in Equation 3.30 and that the Cobb–Douglas[9] case corresponds to $\delta = 0$ in Equation 3.31. Less obvious is that the case of fixed proportions corresponds to $\delta = -\infty$ in Equation 3.30, but that result can also be shown using a limits argument.

The use of the term *elasticity of substitution* for this function derives from the notion that the possibilities illustrated in Figure 3.8 correspond to various values for the substitution parameter, σ, which for this function is given by $\sigma = 1/(1 - \delta)$. For perfect substitutes, then $\sigma = \infty$, and the fixed proportions case has $\sigma = 0$.[10] Because the CES function allows us to explore all these cases, and many cases in between, it will prove useful for illustrating the degree of substitutability present in various economic relationships.

The specific shape of the CES function illustrated in Figure 3.8a is for the case $\delta = -1$. That is,

$$\text{utility} = -x^{-1} - y^{-1} = -\frac{1}{x} - \frac{1}{y}. \tag{3.32}$$

[9]The CES function could easily be generalized to allow for differing weights to be attached to the two goods. Because the main use of the function is to examine substitution questions, we usually will not make that generalization. In some of the applications of the CES function, we will also omit the denominators of the function because these constitute only a scale factor when δ is positive. For negative values of δ, however, the denominator is needed to ensure that marginal utility is positive.

[10]The elasticity of substitution concept is discussed in more detail in connection with production functions in Chapter 9.

For this situation, $\sigma = 1/(1 - \delta) = 1/2$, and, as the graph shows, these sharply curved indifference curves apparently fall between the Cobb–Douglas and fixed proportion cases. The negative signs in this utility function may seem strange, but the marginal utilities of both x and y are positive and diminishing, as would be expected. This explains why δ must appear in the denominators in Equation 3.30. In the particular case of Equation 3.32, utility increases from $-\infty$ (when $x = y = 0$) toward 0 as x and y increase. This is an odd utility scale, perhaps, but perfectly acceptable and often useful.

EXAMPLE 3.3 Homothetic Preferences

All the utility functions described in Figure 3.8 are homothetic (see Chapter 2). That is, the marginal rate of substitution for these functions depends only on the *ratio* of the amounts of the two goods, not on the total quantities of the goods. This fact is obvious for the case of the perfect substitutes (when the *MRS* is the same at every point) and the case of perfect complements (where the *MRS* is infinite for $y/x > \alpha/\beta$, undefined when $y/x = \alpha/\beta$, and zero when $y/x < \alpha/\beta$). For the general Cobb–Douglas function, the *MRS* can be found as

$$MRS = \frac{\partial U/\partial x}{\partial U/\partial y} = \frac{\alpha x^{\alpha-1}y^{\beta}}{\beta x^{\alpha}y^{\beta-1}} = \frac{\alpha}{\beta}\cdot\frac{y}{x}, \tag{3.33}$$

which clearly depends only on the ratio y/x. Showing that the CES function is also homothetic is left as an exercise (see Problem 3.12).

The importance of homothetic functions is that one indifference curve is much like another. Slopes of the curves depend only on the ratio y/x, not on how far the curve is from the origin. Indifference curves for higher utility are simple copies of those for lower utility. Hence we can study the behavior of an individual who has homothetic preferences by looking only at one indifference curve or at a few nearby curves without fearing that our results would change dramatically at different levels of utility.

QUERY: How might you define homothetic functions geometrically? What would the locus of all points with a particular *MRS* look like on an individual's indifference curve map?

EXAMPLE 3.4 Nonhomothetic Preferences

Although all the indifference curve maps in Figure 3.8 exhibit homothetic preferences, this need not always be true. Consider the quasi-linear utility function

$$\text{utility} = U(x, y) = x + \ln y. \tag{3.34}$$

For this function, good y exhibits diminishing marginal utility, but good x does not. The *MRS* can be computed as

$$MRS = \frac{\partial U/\partial x}{\partial U/\partial y} = \frac{1}{1/y} = y. \tag{3.35}$$

The *MRS* diminishes as the chosen quantity of y decreases, but it is independent of the quantity of x consumed. Because x has a constant marginal utility, a person's willingness to give up y to get one more unit of x depends only on how much y he or she has. Contrary to the homothetic case, a doubling of both x and y doubles the *MRS* rather than leaving it unchanged.

QUERY: What does the indifference curve map for the utility function in Equation 3.34 look like? Why might this approximate a situation where y is a specific good and x represents everything else?

THE MANY-GOOD CASE

All the concepts we have studied thus far for the case of two goods can be generalized to situations where utility is a function of arbitrarily many goods. In this section, we will briefly explore those generalizations. Although this examination will not add much to what we have already shown, considering peoples' preferences for many goods can be important in applied economics, as we will see in later chapters.

If utility is a function of n goods of the form $U(x_1, x_2, \ldots, x_n)$, then the equation

$$U(x_1, x_2, \ldots, x_n) = k \tag{3.36}$$

defines an indifference surface in n dimensions. This surface shows all those combinations of the n goods that yield the same level of utility. Although it is probably impossible to picture what such a surface would look like, we will continue to assume that it is convex. That is, balanced bundles of goods will be preferred to unbalanced ones. Hence the utility function, even in many dimensions, will be assumed to be quasi-concave.

The *MRS* with many goods

We can study the trades that a person might voluntarily make between any two of these goods (say, x_1 and x_2) by again using the implicit function theorem:

$$MRS = -\left.\frac{dx_2}{dx_1}\right|_{U(x_1,\, x_2,\, \ldots,\, x_n)=k} = \frac{U_{x_1}(x_1, x_2, \ldots, x_n)}{U_{x_2}(x_1, x_2, \ldots, x_n)}. \tag{3.37}$$

The notation here makes the important point that an individual's willingness to trade x_1 for x_2 will depend not only on the quantities of these two goods but also on the quantities of all the other goods. An individual's willingness to trade food for clothing will depend not only on the quantities of food and clothing he or she has but also on how much "shelter" he or she has. In general it would be expected that changes in the quantities of any of these other goods would affect the trade-off represented by Equation 3.37. It is this possibility that can sometimes make it difficult to generalize the findings of simple two-good models to the many-good case. One must be careful to specify what is being assumed about the quantities of the other goods. In later chapters we will occasionally look at such complexities. However, for the most part, the two-good model will be good enough for developing intuition about economic relationships.

SUMMARY

In this chapter we have described the way in which economists formalize individuals' preferences about the goods they choose. We drew several conclusions about such preferences that will play a central role in our analysis of the theory of choice in the following chapters:

- If individuals obey certain basic behavioral postulates in their preferences among goods, they will be able to rank all commodity bundles, and that ranking can be represented by a utility function. In making choices, individuals will behave as though they were maximizing this function.
- Utility functions for two goods can be illustrated by an indifference curve map. Each indifference curve contour on this map shows all the commodity bundles that yield a given level of utility.
- The negative of the slope of an indifference curve is defined as the marginal rate of substitution (*MRS*). This shows the rate at which an individual would willingly give up an amount of one good (y) if he or she were compensated by receiving one more unit of another good (x).
- The assumption that the *MRS* decreases as x is substituted for y in consumption is consistent with the notion that individuals prefer some balance in their consumption choices. If the *MRS* is always decreasing, individuals will have strictly convex indifference curves. That is, their utility function will be strictly quasi-concave.
- A few simple functional forms can capture important differences in individuals' preferences for two (or more) goods. Here we examined the Cobb–Douglas function, the linear function (perfect substitutes), the fixed proportions

function (perfect complements), and the CES function (which includes the other three as special cases).

- It is a simple matter mathematically to generalize from two-good examples to many goods. And, as we shall see, studying peoples' choices among many goods can yield many insights. But the mathematics of many goods is not especially intuitive; therefore, we will primarily rely on two-good cases to build such intuition.

Problems

3.1

Graph a typical indifference curve for the following utility functions, and determine whether they have convex indifference curves (i.e., whether the *MRS* declines as x increases).

a. $U(x, y) = 3x + y$.

b. $U(x, y) = \sqrt{x \cdot y}$.

c. $U(x, y) = \sqrt{x} + y$.

d. $U(x, y) = \sqrt{x^2 - y^2}$.

e. $U(x, y) = \dfrac{xy}{x + y}$.

3.2

In footnote 7 we showed that for a utility function for two goods to have a strictly diminishing *MRS* (i.e., to be strictly quasi-concave), the following condition must hold:

$$U_{xx}U_x^2 - 2U_{xy}U_xU_y + U_{yy}U_y^2 < 0$$

Use this condition to check the convexity of the indifference curves for each of the utility functions in Problem 3.1. Describe the precise relationship between diminishing marginal utility and quasi-concavity for each case.

3.3

Consider the following utility functions:

a. $U(x, y) = xy$.

b. $U(x, y) = x^2y^2$.

c. $U(x, y) = \ln x + \ln y$.

Show that each of these has a diminishing *MRS* but that they exhibit constant, increasing, and decreasing marginal utility, respectively. What do you conclude?

3.4

As we saw in Figure 3.5, one way to show convexity of indifference curves is to show that, for any two points (x_1, y_1) and (x_2, y_2) on an indifference curve that promises $U = k$, the utility associated with the point $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ is at least as great as k. Use this approach to discuss the convexity of the indifference curves for the following three functions. Be sure to graph your results.

a. $U(x, y) = \min(x, y)$.

b. $U(x, y) = \max(x, y)$.

c. $U(x, y) = x + y$.

3.5

The Phillie Phanatic (PP) always eats his ballpark franks in a special way; he uses a foot-long hot dog together with precisely half a bun, 1 ounce of mustard, and 2 ounces of pickle relish. His utility is a function only of these four items, and any extra amount of a single item without the other constituents is worthless.

a. What form does PP's utility function for these four goods have?
b. How might we simplify matters by considering PP's utility to be a function of only one good? What is that good?
c. Suppose foot-long hot dogs cost $1.00 each, buns cost $0.50 each, mustard costs $0.05 per ounce, and pickle relish costs $0.15 per ounce. How much does the good defined in part (b) cost?
d. If the price of foot-long hot dogs increases by 50 percent (to $1.50 each), what is the percentage increase in the price of the good?
e. How would a 50 percent increase in the price of a bun affect the price of the good? Why is your answer different from part (d)?
f. If the government wanted to raise $1.00 by taxing the goods that PP buys, how should it spread this tax over the four goods so as to minimize the utility cost to PP?

3.6

Many advertising slogans seem to be asserting something about people's preferences. How would you capture the following slogans with a mathematical utility function?

a. Promise margarine is just as good as butter.
b. Things go better with Coke.
c. You can't eat just one Pringle's potato chip.
d. Krispy Kreme glazed doughnuts are just better than Dunkin' Donuts.
e. Miller Brewing advises us to drink (beer) "responsibly." [What would "irresponsible" drinking be?]

3.7

a. A consumer is willing to trade 3 units of x for 1 unit of y when she has 6 units of x and 5 units of y. She is also willing to trade in 6 units of x for 2 units of y when she has 12 units of x and 3 units of y. She is indifferent between bundle (6, 5) and bundle (12, 3). What is the utility function for goods x and y? *Hint:* What is the shape of the indifference curve?
b. A consumer is willing to trade 4 units of x for 1 unit of y when she is consuming bundle (8, 1). She is also willing to trade in 1 unit of x for 2 units of y when she is consuming bundle (4, 4). She is indifferent between these two bundles. Assuming that the utility function is Cobb–Douglas of the form $U(x, y) = x^{\alpha}y^{\beta}$, where α and β are positive constants, what is the utility function for this consumer?
c. Was there a redundancy of information in part (b)? If yes, how much is the minimum amount of information required in that question to derive the utility function?

3.8

Find utility functions given each of the following indifference curves [defined by $U(\cdot) = k$]:

a. $z = \dfrac{k^{1/\delta}}{x^{\alpha/\delta}y^{\beta/\delta}}$.
b. $y = 0.5\sqrt{x^2 - 4(x^2 - k)} - 0.5x$.
c. $z = \dfrac{\sqrt{y^4 - 4x(x^2y - k)}}{2x} - \dfrac{y^2}{2x}$.

Analytical Problems

3.9 Initial endowments

Suppose that a person has initial amounts of the two goods that provide utility to him or her. These initial amounts are given by $\bar{x}$ and $\bar{y}$.

a. Graph these initial amounts on this person's indifference curve map.
b. If this person can trade x for y (or vice versa) with other people, what kinds of trades would he or she voluntarily make? What kinds would not be made? How do these trades relate to this person's *MRS* at the point $(\bar{x}, \bar{y})$?
c. Suppose this person is relatively happy with the initial amounts in his or her possession and will only consider trades that increase utility by at least amount k. How would you illustrate this on the indifference curve map?

3.10 Cobb–Douglas utility

Example 3.3 shows that the *MRS* for the Cobb–Douglas function

$$U(x, y) = x^{\alpha} y^{\beta}$$

is given by

$$MRS = \frac{\alpha}{\beta}\left(\frac{y}{x}\right).$$

a. Does this result depend on whether $\alpha + \beta = 1$? Does this sum have any relevance to the theory of choice?
b. For commodity bundles for which $y = x$, how does the *MRS* depend on the values of α and β? Develop an intuitive explanation of why, if $\alpha > \beta$, $MRS > 1$. Illustrate your argument with a graph.
c. Suppose an individual obtains utility only from amounts of x and y that exceed minimal subsistence levels given by x_0, y_0. In this case,

$$U(x, y) = (x - x_0)^{\alpha}(y - y_0)^{\beta}$$

Is this function homothetic? (For a further discussion, see the Extensions to Chapter 4.)

3.11 Independent marginal utilities

Two goods have independent marginal utilities if

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y} = 0.$$

Show that if we assume diminishing marginal utility for each good, then any utility function with independent marginal utilities will have a diminishing *MRS*. Provide an example to show that the converse of this statement is not true.

3.12 CES utility

a. Show that the CES function

$$\alpha\frac{x^{\delta}}{\delta} + \beta\frac{y^{\delta}}{\delta}$$

is homothetic. How does the *MRS* depend on the ratio y/x?
b. Show that your results from part (a) agree with our discussion of the cases $\delta = 1$ (perfect substitutes) and $\delta = 0$ (Cobb–Douglas).
c. Show that the *MRS* is strictly diminishing for all values of $\delta < 1$.
d. Show that if $x = y$, the *MRS* for this function depends only on the relative sizes of α and β.
e. Calculate the *MRS* for this function when $y/x = 0.9$ and $y/x = 1.1$ for the two cases $\delta = 0.5$ and $\delta = -1$. What do you conclude about the extent to which the *MRS* changes in the vicinity of $x = y$? How would you interpret this geometrically?

3.13 The quasi-linear function

Consider the function $U(x, y) = x + \ln y$. This is a function that is used relatively frequently in economic modeling as it has some useful properties.

a. Find the *MRS* of the function. Now, interpret the result.
b. Confirm that the function is quasi-concave.
c. Find the equation for an indifference curve for this function.
d. Compare the marginal utility of x and y. How do you interpret these functions? How might consumers choose between x and y as they try to increase their utility by, for example, consuming more when their income increases? (We will look at this "income effect" in detail in the Chapter 5 problems.)
e. Considering how the utility changes as the quantities of the two goods increase, describe some situations where this function might be useful.

3.14 Preference relations

The formal study of preferences uses a general vector notation. A bundle of n commodities is denoted by the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and a preference relation ($\succ$) is defined over all potential bundles. The statement $\mathbf{x}^1 \succ \mathbf{x}^2$ means that bundle $\mathbf{x}^1$ is preferred to bundle $\mathbf{x}^2$. Indifference between two such bundles is denoted by $\mathbf{x}^1 \approx \mathbf{x}^2$.

The preference relation is "complete" if for any two bundles the individual is able to state either $\mathbf{x}^1 \succ \mathbf{x}^2$, $\mathbf{x}^2 \succ \mathbf{x}^1$, or $\mathbf{x}^1 \approx \mathbf{x}^2$. The relation is "transitive" if $\mathbf{x}^1 \succ \mathbf{x}^2$ and $\mathbf{x}^2 \succ \mathbf{x}^3$ implies that $\mathbf{x}^1 \succ \mathbf{x}^3$. Finally, a preference relation is "continuous" if for any bundle $\mathbf{y}$ such that $\mathbf{y} \succ \mathbf{x}$, any bundle suitably close to $\mathbf{y}$ will also be preferred to $\mathbf{x}$. Using these definitions, discuss whether each of the following preference relations is complete, transitive, and continuous.

a. Summation preferences: This preference relation assumes one can indeed add apples and oranges. Specifically, $\mathbf{x}^1 \succ \mathbf{x}^2$ if and only if $\sum_{i=1}^{n} x_i^1 > \sum_{i=1}^{n} x_i^2$. If $\sum_{i=1}^{n} x_i^1 = \sum_{i=1}^{n} x_i^2$, $\mathbf{x}^1 \approx \mathbf{x}^2$.

b. Lexicographic preferences: In this case the preference relation is organized as a dictionary: If $x_1^1 > x_1^2$, $\mathbf{x^1} \succ \mathbf{x^2}$ (regardless of the amounts of the other $n-1$ goods). If $x_1^1 = x_1^2$ and $x_2^1 > x_2^2$, $\mathbf{x}^1 \succ \mathbf{x}^2$ (regardless of the amounts of the other $n-2$ goods). The lexicographic preference relation then continues in this way throughout the entire list of goods.

c. Preferences with satiation: For this preference relation there is assumed to be a consumption bundle ($\mathbf{x}^*$) that provides complete "bliss." The ranking of all other bundles is determined by how close they are to $\mathbf{x}^*$. That is, $\mathbf{x}^1 \succ \mathbf{x}^2$ if and only if $|\,\mathbf{x}^1 - \mathbf{x}^*| < |\,\mathbf{x}^2 - \mathbf{x}^*|$ where $|\mathbf{x^i} - \mathbf{x}^*| = \sqrt{(x_1^i - x_1^*)^2 + (x_2^i - x_x^*)^2 + \ldots + (x_n^i - x_n^*)^2}$.

3.15 The benefit function

In a 1992 article David G. Luenberger introduced what he termed the *benefit function* as a way of incorporating some degree of cardinal measurement into utility theory.[11] The author asks us to specify a certain elementary consumption bundle and then measure how many replications of this bundle would need to be provided to an individual to raise his or her utility level to a particular target. Suppose there are only two goods and that the utility target is given by $U^*(x, y)$. Suppose also that the elementary consumption bundle is given by (x_0, y_0). Then the value of the benefit function, $b(U^*)$, is that value of α for which $U(\alpha x_0, \alpha y_0) = U^*$.

a. Suppose utility is given by $U(x, y) = x^{\beta} y^{1-\beta}$. Calculate the benefit function for $x_0 = y_0 = 1$.

b. Using the utility function from part (a), calculate the benefit function for $x_0 = 1$, $y_0 = 0$. Explain why your results differ from those in part (a).

c. The benefit function can also be defined when an individual has initial endowments of the two goods. If these initial endowments are given by $\bar{x}, \bar{y}$, then $b(U^*, \bar{x}, \bar{y})$ is given by that value of α which satisfies the equation $U(\bar{x} + \alpha x_0, \bar{y} + \alpha y_0) = U^*$. In this situation the "benefit" can be either positive (when $U(\bar{x}, \bar{y}) < U^*$) or negative (when $U(\bar{x}, \bar{y}) > U^*$). Develop a graphical description of these two possibilities, and explain how the nature of the elementary bundle may affect the benefit calculation.

d. Consider two possible initial endowments, $\bar{x}_1, \bar{y}_1$ and $\bar{x}_2, \bar{y}_2$. Explain both graphically and intuitively why $b\left(U^*, \frac{\bar{x}_1 + \bar{x}_2}{2}, \frac{\bar{y}_1 + \bar{y}_2}{2}\right) < 0.5b\left(U^*, \bar{x}_1, \bar{y}_1\right) + 0.5b\left(U^*, \bar{x}_2, \bar{y}_2\right)$. (*Note*: This shows that the benefit function is concave in the initial endowments.)

Suggestions for Further Reading

Aleskerov, Fuad, and Bernard Monjardet. *Utility Maximization, Choice, and Preference.* Berlin: Springer-Verlag, 2002.

A complete study of preference theory. Covers a variety of threshold models and models of "context-dependent" decision making.

Jehle, G. R., and P. J. Reny. *Advanced Microeconomic Theory,* 2nd ed. Boston: Addison Wesley/Longman, 2001.

Chapter 2 has a good proof of the existence of utility functions when basic axioms of rationality hold.

Kreps, David M. *A Course in Microeconomic Theory.* Princeton, NJ: Princeton University Press, 1990.

Chapter 1 covers preference theory in some detail. Good discussion of quasi-concavity.

Kreps, David M. *Notes on the Theory of Choice.* London: Westview Press, 1988.

Good discussion of the foundations of preference theory. Most of the focus of the book is on utility in uncertain situations.

[11]Luenberger, David G. "Benefit Functions and Duality." *Journal of Mathematical Economics* 21: 461–81. The presentation here has been simplified considerably from that originally presented by the author, mainly by changing the direction in which "benefits" are measured.

Mas-Colell, Andrea, Michael D. Whinston, and Jerry R. Green. *Microeconomic Theory*. New York: Oxford University Press, 1995.

Chapters 2 and 3 provide a detailed development of preference relations and their representation by utility functions.

Stigler, G. "The Development of Utility Theory." *Journal of Political Economy* 59, pts. 1–2 (August/October 1950): 307–27, 373–96.

A lucid and complete survey of the history of utility theory. Has many interesting insights and asides.

EXTENSIONS Special Preferences

The utility function concept is a general one that can be adapted to a large number of special circumstances. Discovery of ingenious functional forms that reflect the essential aspects of some problem can provide a number of insights that would not be readily apparent with a more literary approach. Here we look at four aspects of preferences that economists have tried to model: (1) threshold effects, (2) quality, (3) habits and addiction, and (4) second-party preferences. In Chapters 7 and 17, we illustrate a number of additional ways of capturing aspects of preferences.

E3.1 Threshold effects

The model of utility that we developed in this chapter implies an individual will always prefer commodity bundle A to bundle B, provided $U(A) > U(B)$. There may be events that will cause people to shift quickly from consuming bundle A to consuming B. In many cases, however, such a lightning-quick response seems unlikely. People may in fact be "set in their ways" and may require a rather large change in circumstances to change what they do. For example, individuals may not have especially strong opinions about what precise brand of toothpaste they choose and may stick with what they know despite a proliferation of new (and perhaps better) brands. Similarly, people may stick with an old favorite TV show even though it has declined in quality. One way to capture such behavior is to assume individuals make decisions as though they faced thresholds of preference. In such a situation, commodity bundle A might be chosen over B only when

$$U(A) > U(B) + \epsilon, \quad \textbf{(i)}$$

where ϵ is the threshold that must be overcome. With this specification, indifference curves then may be rather thick and even fuzzy, rather than the distinct contour lines shown in this chapter. Threshold models of this type are used extensively in marketing. The theory behind such models is presented in detail in Aleskerov and Monjardet (2002). There, the authors consider a number of ways of specifying the threshold so that it might depend on the characteristics of the bundles being considered or on other contextual variables.

Alternative fuels

Vedenov, Duffield, and Wetzstein (2006) use the threshold idea to examine the conditions under which individuals will shift from gasoline to other fuels (primarily ethanol) for powering their cars. The authors point out that the main disadvantage of using gasoline in recent years has been the excessive price volatility of the product relative to other fuels. They conclude that switching to ethanol blends is efficient (especially during periods of increased gasoline price volatility), provided that the blends do not decrease fuel efficiency.

E3.2 Quality

Because many consumption items differ widely in quality, economists have an interest in incorporating such differences into models of choice. One approach is simply to regard items of different quality as totally separate goods that are relatively close substitutes. But this approach can be unwieldy because of the large number of goods involved. An alternative approach focuses on quality as a direct item of choice. Utility might in this case be reflected by

$$\text{utility} = U(q, Q), \quad \textbf{(ii)}$$

where q is the quantity consumed and Q is the quality of that consumption. Although this approach permits some examination of quality–quantity trade-offs, it encounters difficulty when the quantity consumed of a commodity (e.g., wine) consists of a variety of qualities. Quality might then be defined as an average (see Theil,[1] 1952), but that approach may not be appropriate when the quality of new goods is changing rapidly (e.g., as in the case of personal computers). A more general approach (originally suggested by Lancaster, 1971) focuses on a well-defined set of attributes of goods and assumes that those attributes provide utility. If a good q provides two such attributes, a_1 and a_2, then utility might be written as

$$\text{utility} = U[q, a_1(q), a_2(q)], \quad \textbf{(iii)}$$

and utility improvements might arise either because this individual chooses a larger quantity of the good or because a given quantity yields a higher level of valuable attributes.

Personal computers

This is the practice followed by economists who study demand in such rapidly changing industries as personal computers. In this case it would clearly be incorrect to focus only on the quantity of personal computers purchased each year

[1]Theil also suggests measuring quality by looking at correlations between changes in consumption and the income elasticities of various goods.

because new machines are much better than old ones (and, presumably, provide more utility). For example, Berndt, Griliches, and Rappaport (1995) find that personal computer quality has been increasing about 30 percent per year over a relatively long period, primarily because of improved attributes such as faster processors or better hard drives. A person who spends, say, \$2,000 for a personal computer today buys much more utility than did a similar consumer 5 years ago.

E3.3 Habits and addiction

Because consumption occurs over time, there is the possibility that decisions made in one period will affect utility in later periods. Habits are formed when individuals discover they enjoy using a commodity in one period and this increases their consumption in subsequent periods. An extreme case is addiction (be it to drugs, cigarettes, or Marx Brothers movies), where past consumption significantly increases the utility of present consumption. One way to portray these ideas mathematically is to assume that utility in period t depends on consumption in period t and the total of all previous consumption of the habit-forming good (say, X):

$$\text{utility} = U_t(x_t, y_t, s_t), \qquad \textbf{(iv)}$$

where

$$s_t = \sum_{i=1}^{\infty} x_{t-i}.$$

In empirical applications, however, data on all past levels of consumption usually do not exist. Therefore, it is common to model habits using only data on current consumption (x_t) and on consumption in the previous period (x_{t-1}). A common way to proceed is to assume that utility is given by

$$\text{utility} = U_t\left(x_t^*, y_t\right), \qquad \textbf{(v)}$$

where x_t^* is some simple function of x_t and x_{t-1}, such as $x_t^* = x_t - x_{t-1}$ or $x_t^* = x_t / x_{t-1}$. Such functions imply that, ceteris paribus, the higher x_{t-1}, the more x_t will be chosen in the current period.

Modeling habits

These approaches to modeling habits have been applied to a wide variety of topics. Stigler and Becker (1977) use such models to explain why people develop a "taste" for going to operas or playing golf. Becker, Grossman, and Murphy (1994) adapt the models to studying cigarette smoking and other addictive behavior. They show that reductions in smoking early in life can have large effects on eventual cigarette consumption because of the dynamics in individuals' utility functions. Whether addictive behavior is "rational" has been extensively studied by economists. For example, Gruber and Koszegi (2001) show that smoking can be approached as a rational, although time-inconsistent,[2] choice.

[2]For more on time inconsistency, see Chapter 17.

E3.4 Second-party preferences

Individuals clearly care about the well-being of other individuals. Phenomena such as making charitable contributions or making bequests to children cannot be understood without recognizing the interdependence that exists among people. Second-party preferences can be incorporated into the utility function of person i, say, by

$$\text{utility} = U_i\left(x_i, y_i, U_j\right), \qquad \textbf{(vi)}$$

where U_j is the utility of someone else.

If $\partial U_i / \partial U_j > 0$ then this person will engage in altruistic behavior, whereas if $\partial U_i / \partial U_j < 0$ then he or she will demonstrate the malevolent behavior associated with envy. The usual case of $\partial U_i / \partial U_j = 0$ is then simply a middle ground between these alternative preference types. Gary Becker has been a pioneer in the study of these possibilities and has written on a variety of topics, including the general theory of social interactions (1976) and the importance of altruism in the theory of the family (1981).

Evolutionary biology and genetics

Biologists have suggested a particular form for the utility function in Equation vi, drawn from the theory of genetics. In this case

$$\text{utility} = U_i(x_i, y_i) + \sum_j r_j U_j, \qquad \textbf{(vii)}$$

where r_j measures closeness of the genetic relationship between person i and person j. For parents and children, for example, $r_j = 0.5$, whereas for cousins $r_j = 0.125$. Bergstrom (1996) describes a few of the conclusions about evolutionary behavior that biologists have drawn from this particular functional form.

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