AC Electrical Conductivity in Metals

(Brief discussion only, in the free & independent electron approximation)

Application to the Propagation of Electromagnetic Radiation in a Metal

Consider a time dependent electric field **E(t)** acting on a metal. Take the case when the wavelength of the field is large compared to the electron mean free path between collisions:

$$\lambda >> \ell$$

In this limit, the conduction electrons will "see" a ≈ homogeneous field when moving between collisions. Write:

$$\mathbf{E(t) = E(\omega)e^{-i\omega t}}$$

That is, assume a harmonic dependence on frequency.

Next is standard Junior-Senior physics major E&M!

Response to the electric field both in metals and dielectrics → mainly described by → historically used mainly for

electric field leads to:

electric current **j**	conductivity $\sigma = \mathbf{j}/\mathbf{E}$	metals
polarization **P**	polarizability $\chi = \mathbf{P}/\mathbf{E}$ dielectric function $\varepsilon = 1 + 4\pi\chi$	dielectrics

$$\mathbf{j} = \sum_i q_i \frac{d\mathbf{r}_i}{dt}$$

$$\mathbf{P} = \sum_i q_i \mathbf{r}_i$$

$$\longrightarrow \quad j = \frac{dP}{dt} = -i\omega P$$

$$\mathbf{E}(\omega,t) = \mathbf{E}(\omega)e^{-i\omega t}$$

$$\mathbf{j}(\omega,t) \sim \mathbf{j}(\omega)e^{-i\omega t}$$

$$\mathbf{P}(\omega,t) \sim \mathbf{P}(\omega)e^{-i\omega t}$$

$$\chi = \frac{P}{E} = \frac{1}{E}\frac{j}{-i\omega} = i\frac{\sigma}{\omega}$$

$$\varepsilon(\omega) = 1 + 4\pi\frac{P(\omega)}{E(\omega)} \longrightarrow \varepsilon(\omega) = 1 + i\frac{4\pi}{\omega}\sigma(\omega)$$

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \varepsilon\mathbf{E}$$

$$\mathrm{div}\mathbf{D} = 4\pi\rho_{ext}$$

$$\mathrm{div}\mathbf{E} = 4\pi\rho = 4\pi(\rho_{ext} + \rho_{ind})$$

ε(ω,0) describes the collective excitations of the electron gas – the plasmons

ε(0,k) describes the electrostatic screening

AC Electrical Conductivity of a Metal

Newton's 2nd Law Equation of Motion for the momentum of one electron in a time dependent electric field. Look for a steady state solution of the form:

$$\frac{d\mathbf{p}(\omega,t)}{dt} = -\frac{\mathbf{p}(\omega,t)}{\tau} - e\mathbf{E}(\omega,t)$$

$$\mathbf{E}(\omega,t) = \mathbf{E}(\omega)e^{-i\omega t}$$

$$\mathbf{p}(\omega,t) = \mathbf{p}(\omega)e^{-i\omega t}$$

$$\mathbf{p}(\omega) = \frac{e\mathbf{E}(\omega)}{1/\tau - i\omega}$$

$$\mathbf{j}(\omega) = -\frac{ne\mathbf{p}(\omega)}{m} = \frac{(ne^2/m)\mathbf{E}(\omega)}{1/\tau - i\omega}$$

$$\mathbf{j}(\omega) = \sigma(\omega)\mathbf{E}(\omega)$$

AC conductivity $$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$$

DC conductivity $$\sigma_0 = \frac{ne^2\tau}{m}$$

$$\mathrm{Re}\,\sigma(\omega) = \frac{\sigma_0}{1+\omega^2\tau^2}$$

$\omega\tau >> 1$

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$$

$$\varepsilon(\omega) = 1 + i\frac{4\pi}{\omega}\sigma(\omega)$$

$$\rightarrow \varepsilon(\omega) = 1 - \frac{4\pi ne^2}{m\omega^2}$$

$$\omega_p{}^2 = \frac{4\pi ne^2}{m}$$

$$\varepsilon(\omega) = 1 - \frac{\omega_p{}^2}{\omega^2}$$

Plasma Frequency

A **plasma** is a medium with positive & negative charges & at least one charge type is mobile.

Even more simplified: $\omega\tau >> 1$

No electron collisions (no frictional damping term)

Equation of motion of a **Free Electron**: $m\frac{d^2x}{dt^2} = -eE$

If **x** & **E** have harmonic time dependences **e**$^{-i\omega t}$

$$x = \frac{eE}{m\omega^2}$$

The polarization **P** is the dipole moment per unit volume:

$$P = -exn = -\frac{ne^2}{m\omega^2}E$$

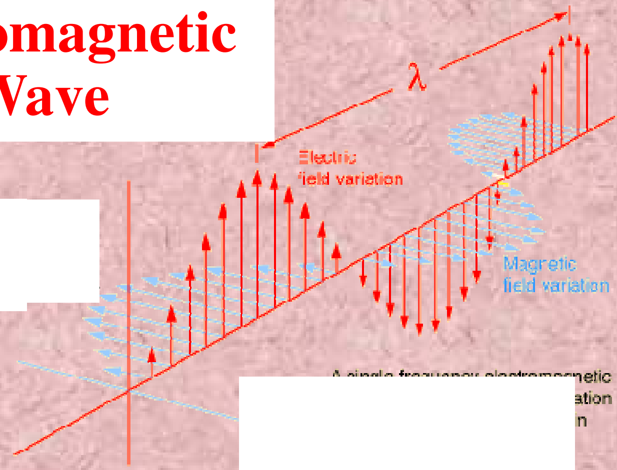
$$\varepsilon(\omega) = 1 + 4\pi\frac{P(\omega)}{E(\omega)} = 1 - \frac{4\pi ne^2}{m\omega^2}$$

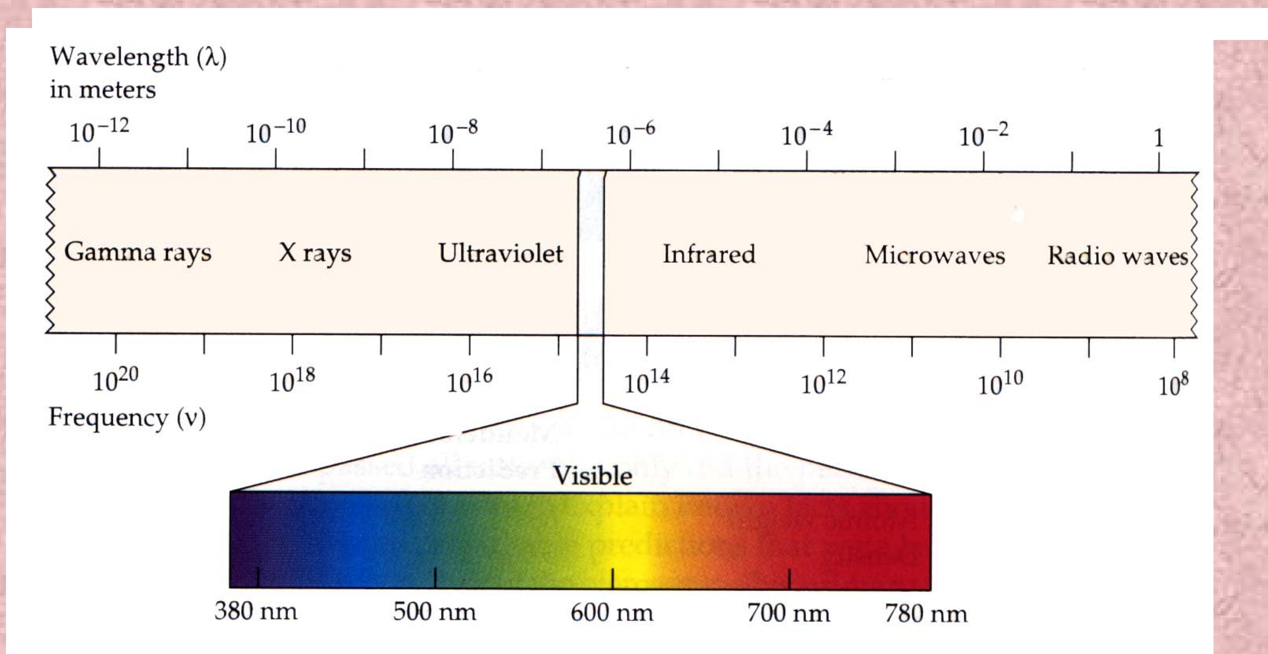
$$\omega_p^{\ 2} = \frac{4\pi ne^2}{m}$$

$$\varepsilon(\omega) = 1 - \frac{\omega_p^{\ 2}}{\omega^2}$$

Application to the Propagation of Electromagnetic Radiation in a Metal

Transverse Electromagnetic Wave





Application to the Propagation of Electromagnetic Radiation in a Metal

The electromagnetic wave equation in a nonmagnetic isotropic medium. Look for a solution with the **dispersion relation for electromagnetic waves**

$$\varepsilon(\omega,\mathbf{K})\partial^2\mathbf{E}/\partial t^2 = c^2\nabla^2\mathbf{E}$$

$$\mathbf{E} \propto \exp(-i\omega t + i\mathbf{K}\cdot\mathbf{r})$$

$$\varepsilon(\omega,\mathbf{K})\omega^2 = c^2K^2$$

(1) ε real & **> 0** → for ω real, **K** is real & the transverse electromagnetic wave propagates with the phase velocity $\mathbf{v_{ph} = c/\varepsilon^{1/2}}$

(2) ε real & **< 0** → for ω real, **K** is imaginary & the wave is damped with a characteristic length **1/|K|:**

$$E \propto e^{-|K|r}$$

(3) ε complex → for ω real, **K** is complex & the wave is damped in space

(4) **ε = → ∞** → The system has a final response in the absence of an applied force (at **E = 0**); the poles of **ε(ω,K)** define the frequencies of the free oscillations of the medium

(5) **ε = 0** longitudinally polarized waves are possible

Transverse optical modes in a plasma

$$\omega_p^{\,2} = \frac{4\pi n e^2}{m}$$

$$\varepsilon(\omega) = 1 - \frac{\omega_p^{\,2}}{\omega^2}$$

Dispersion relation for electromagnetic waves

$$\varepsilon(\omega, \mathbf{K})\omega^2 = c^2 K^2$$

$$\omega^2 - \omega_p^{\,2} = c^2 K^2$$

(1) For **$\omega > \omega_p \rightarrow K^2 > 0$, K** is real, waves with **$\omega > \omega_p$** propagate in the media with the dispersion relation: $\omega^2 = \omega_p^{\,2} + c^2 K^2$

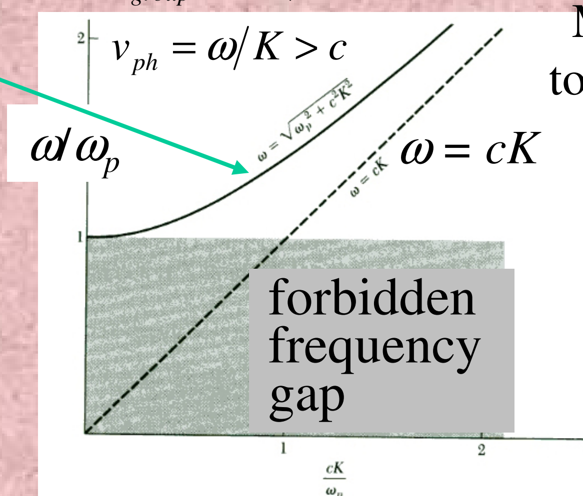
The electron gas is transparent.

$E \propto e^{-|K|r}$

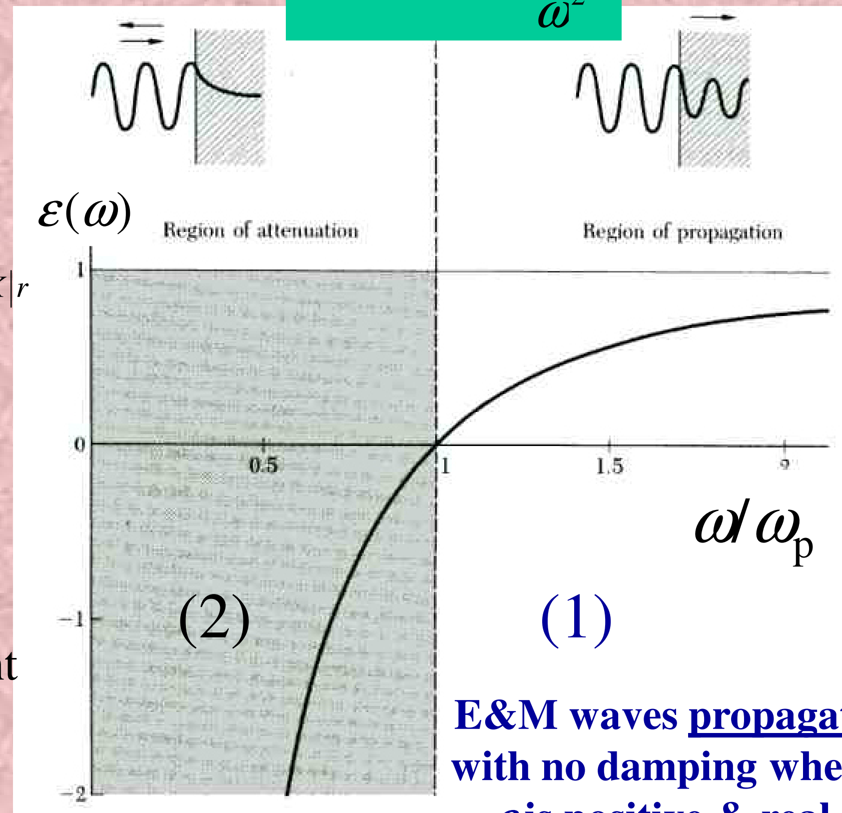
(2) For **$\omega < \omega_p \rightarrow K^2 < 0$, K** is imaginary, waves with **$\omega < \omega_p$** incident on the medium do not propagate, but are ***totally reflected***

$v_{group} = d\omega / dK < c$

Metals are shiny due to the reflection of light



cK/ω_p



E&M waves <u>propagate</u> with no damping when ε is positive & real

E&M waves are totally <u>reflected</u> from the medium when ε is negative

$v_{ph} > c \rightarrow v_{ph}$

This does not correspond to the velocity of the propagation of any quantity!!

Ultraviolet Transparency of Metals

$$\omega_p^{\,2} = \frac{4\pi n e^2}{m}$$

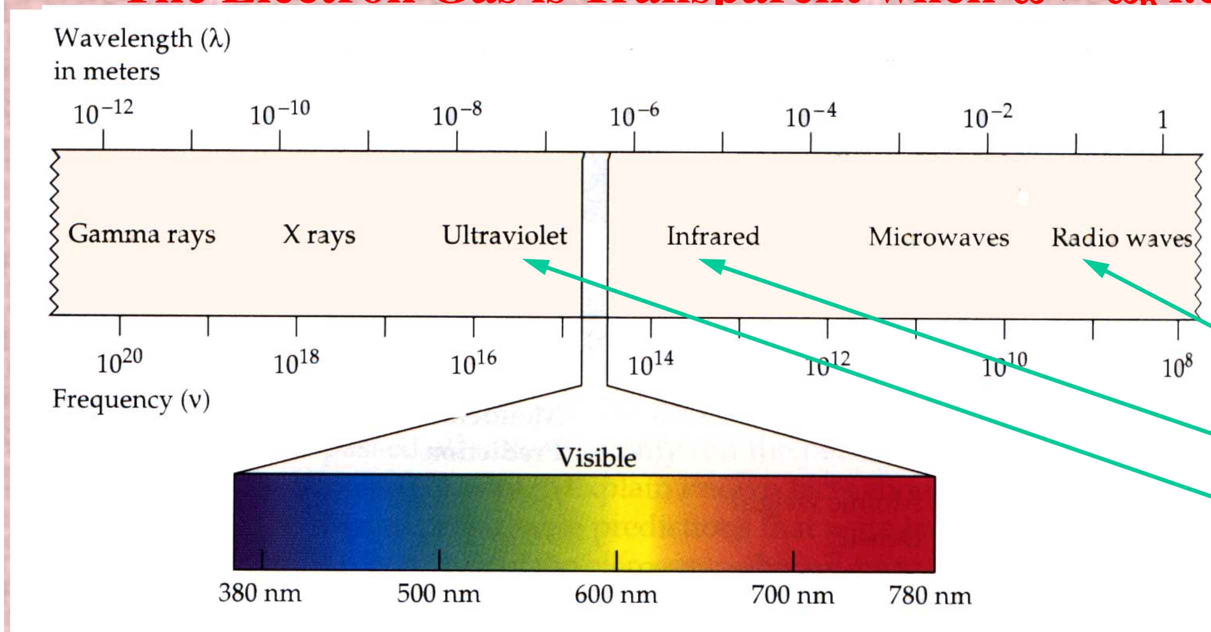
$$\varepsilon(\omega) = 1 - \frac{\omega_p^{\,2}}{\omega^2}$$

Plasma Frequency ω_p & Free Space Wavelength $\lambda_p = 2\pi c/\omega_p$

Range	Metals	Semiconductors	Ionosphere
n, cm^{-3}	10^{22}	10^{18}	10^{10}
ω_p, Hz	5.7×10^{15}	5.7×10^{13}	5.7×10^{9}
λ_p, cm	3.3×10^{-5}	3.3×10^{-3}	33
spectral range	UV	IF	radio

The reflection of light from a metal is similar to the reflection of radio waves from the Ionosphere!

The Electron Gas is Transparent when $\omega > \omega_p$ i.e. $\lambda < \lambda_p$



Plasma Frequency
Ionosphere
Semiconductors
Metals

	reflects	transparent for
metal	visible	UV
ionosphere	radio	visible → ☼

Skin Effect

When $\omega < \omega_p$ the electromagnetic wave is reflected.

It is damped with a characteristic length $\delta = 1/|K|$:

$$E \propto e^{-r/\delta} = e^{-|K|r}$$

The wave penetration – **the skin effect**

The penetration depth δ – **the skin depth**

$$K^2 = \frac{\omega^2}{c^2}\varepsilon = \frac{\omega^2}{c^2}\left(1 + i\frac{4\pi}{\omega}\sigma\right) \approx i\frac{\omega^2}{c^2}\frac{4\pi}{\omega}\sigma$$

$$K = \frac{(2\pi\sigma\omega)^{1/2}}{c}(1+i)$$

$$E \propto \exp(-i\omega t + iKr) \propto \exp\left(-\frac{(2\pi\sigma\omega)^{1/2}}{c}r\right)$$

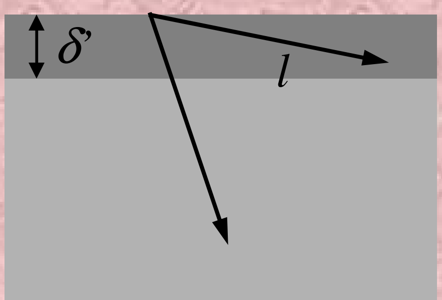
$$\delta_{cl} = \frac{c}{(2\pi\sigma\omega)^{1/2}}$$

The classical skin depth

$\delta >> \ell$

The **classical skin effect**

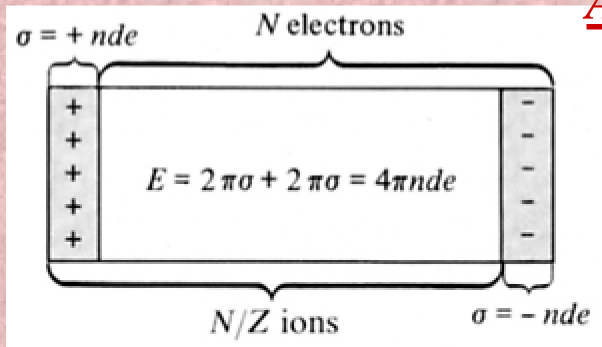
$\delta << \ell$: The anomalous skin effect (pure metals at low temperatures) the usual theory of electrical conductivity is no longer valid; the electric field varies rapidly over ℓ . Further, not all electrons are participating in the wave absorption & reflection.



Only electrons moving inside the skin depth for most of the mean free path ℓ are capable of picking up much energy from the electric field. Only a fraction of the electrons δ'/ℓ contribute to the conductivity

$$\delta' = \frac{c}{(2\pi\sigma'\omega)^{1/2}} \approx \frac{c}{\left(2\pi\frac{\delta'}{l}\sigma\omega\right)^{1/2}} \longrightarrow \delta' = \left(\frac{lc^2}{2\pi\sigma\omega}\right)^{1/3}$$

Longitudinal Plasma Oscillations



A charge density oscillation, or a longitudinal plasma oscillation, or a plasmon

The Nature of Plasma Oscillations: Correspond to a displacement of the entire electron gas a distance **d** with respect to the positive ion background. This creates surface charges **σ = nde** & thus an electric field **$E = 4\pi nde$.**

Equation of Motion

$$Nm\frac{\partial^2 d}{\partial t^2} = -NeE = -Ne(4\pi nde)$$

$$\frac{\partial^2 d}{\partial t^2} + \omega_p{}^2 d = 0 \longrightarrow$$

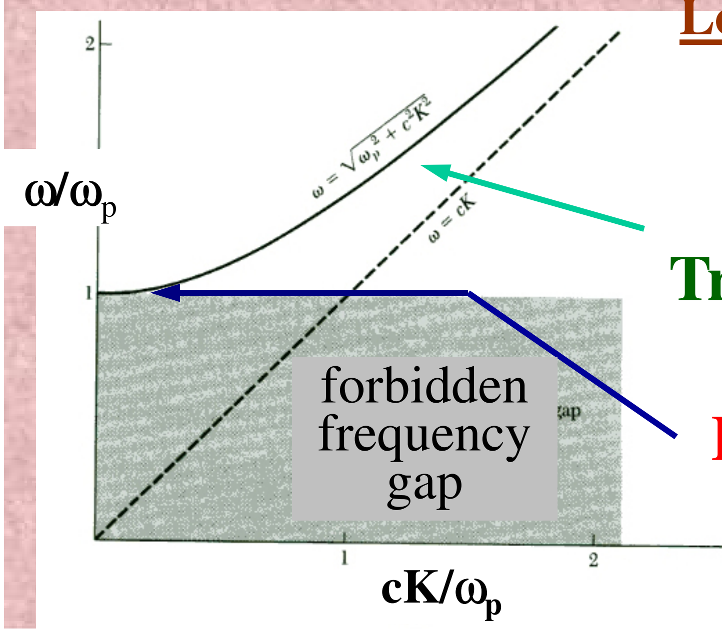
Oscillations at the Plasma Frequency

$$\omega_p{}^2 = \frac{4\pi ne^2}{m}$$

Longitudinal Plasma Oscillations

$\omega_L = \omega_p$

$$\varepsilon(\omega_L) = 1 - \frac{\omega_L{}^2}{\omega_p{}^2} = 0$$



Transverse Electromagnetic Waves

Longitudinal Plasma Oscillations