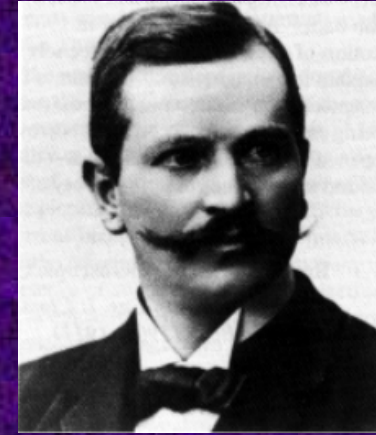


E. Electrical Resistivity of the FEG: A Derivation of Ohm's Law

The FEG model was developed by Paul Drude (1900) in order to describe the electrical and thermal conductivity of metals. This work greatly influenced the course of “solid-state physics” and it introduces basic concepts we still use today.



Since an electron that experienced a uniform electric field would accelerate indefinitely and imply an increasing current, Drude proposed a collision mechanism by which electrons make collisions every τ seconds. In each collision he assumed that all of the electron's forward velocity is reduced to zero and it must be accelerated again. The result is a constant average velocity:

$$\sum F_x = m \frac{d^2 x}{dt^2} = -eE + F_{coll}$$

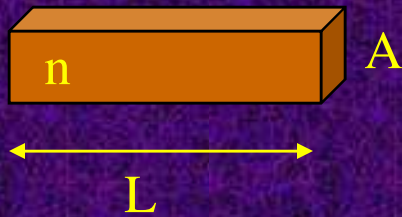
$$\bar{F}_{coll} = \frac{\Delta p_x}{\Delta t} \approx \frac{-mv}{\tau}$$

relaxation time

$$-eE - \frac{mv}{\tau} = 0 \quad \longrightarrow \quad v = \frac{-eE\tau}{m}$$

drift velocity
(opposite to E)

The current density in the FEG can easily be calculated assuming a simple sample geometry:



$$J \equiv \frac{I}{A} = \frac{1}{A} \frac{\Delta Q}{\Delta t} = \frac{1}{A} \frac{(-ne)(AL)}{L/v} = -nev$$

Combined with the earlier result

$$v = \frac{-eE\tau}{m}$$

we get

$$J = \left(\frac{ne^2\tau}{m} \right) E = \sigma E$$

Ohm's Law!

where we define the electrical conductivity

$$\sigma \equiv \left(\frac{ne^2\tau}{m} \right)$$

and the electrical resistivity is:

$$\rho \equiv \frac{1}{\sigma} = \left(\frac{m}{ne^2\tau} \right)$$

The question we now turn to is: How does the resistivity depend on temperature? What does the FEG model predict?

Temperature Dependence of the Electrical Resistivity

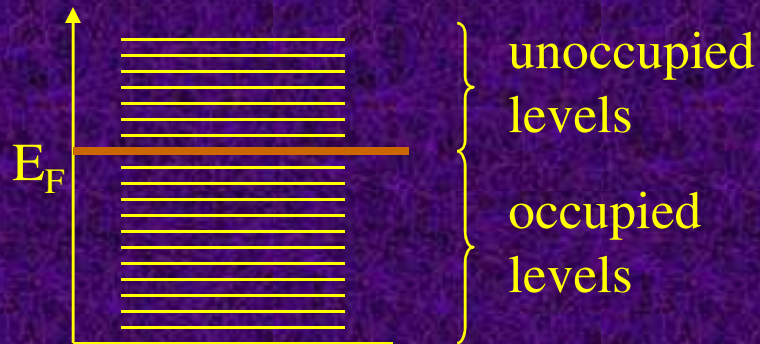
Clearly the temperature dependence of ρ enters through the relaxation time τ :

$$\tau = \frac{\Lambda}{v}$$

→ mean free path of electrons between collisions

→ mean velocity of electrons between collisions (not drift velocity)

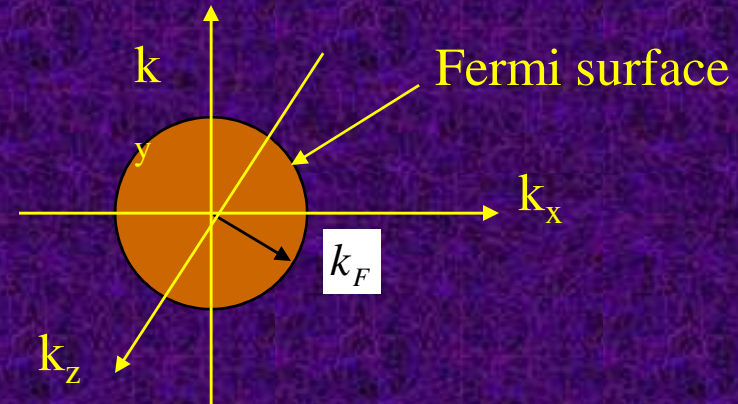
Consider once again the manifold of energy levels occupied by the FEG:



The occupied states have energies (and thus velocities) that are essentially independent of T . So even if we calculate an average velocity it will not depend on T . But we can easily show that only electrons near E_F contribute to the electrical conductivity.

Electrical Conductivity in Reciprocal Space

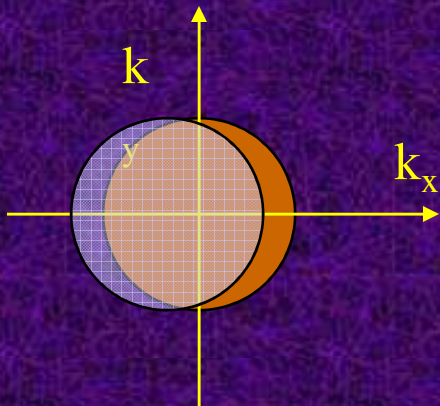
The Fermi sphere contains all occupied electron states in the FEG. In the absence of an electric field, there are the same number of electrons moving in the $\pm x$, $\pm y$, and $\pm z$ directions, so the net current is zero.



But when a field E is applied along the x -direction, the Fermi sphere is shifted by an amount related to the net change in momentum of the FEG:

$$\hbar \Delta k_x = \Delta p_x = m v_x = -e E_x \tau$$

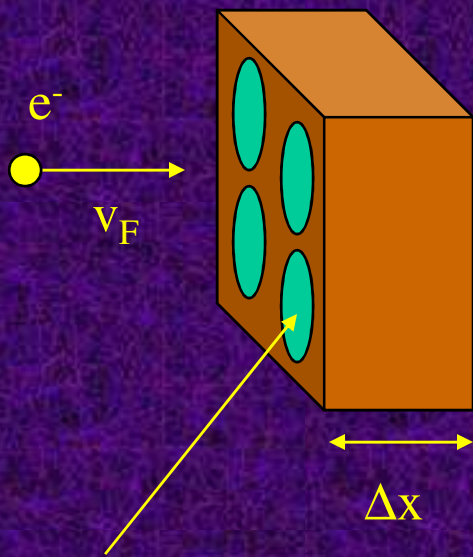
$$\Delta k_x = \Delta p_x = m v_x = -\frac{e E_x \tau}{\hbar}$$



The shift in Fermi sphere creates a net current flow since more electrons move in the $-x$ direction than the $+x$ direction. But the excess current carriers are only those very near the Fermi surface. So the current carriers have velocity v_F .

Analysis of Mean Free Path

Since the velocity of current-carrying electrons is essentially independent of T , we need to examine the behavior of the mean-free-path. Naively we might expect it to be some multiple of the distance between atoms in the solid. Let's dig a bit deeper:



collision cross-section = $\pi \langle r^2 \rangle$
cross-sectional area of slab = A
atomic density = n_a

The probability of a collision in a distance Δx is:

$$P = \frac{\text{total cross-sectional area for collision}}{\text{cross-sectional area of slab}}$$

$$P = \frac{n_a A \Delta x \pi \langle r^2 \rangle}{A} = n_a \Delta x \pi \langle r^2 \rangle$$

Now in a distance $\Delta x = \Lambda$, $P = 1$ is true, so we can solve for Λ :

$$\Lambda = \frac{1}{n_a \pi \langle r^2 \rangle}$$

Summary: $\rho(T)$

Now if we assume that the collision cross-section is due to vibrations of atoms about their equilibrium positions, then we can write:

$$\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle$$

And the thermal average potential energies can be written:

$$\left\langle \frac{1}{2} Cx^2 \right\rangle = \left\langle \frac{1}{2} Cy^2 \right\rangle = \frac{1}{2} kT$$

Therefore $\langle r^2 \rangle \propto T$ and $\Lambda \propto \frac{1}{T}$ from which $\rho \propto T$ is predicted.

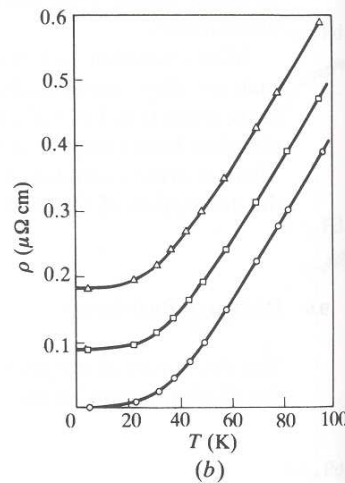
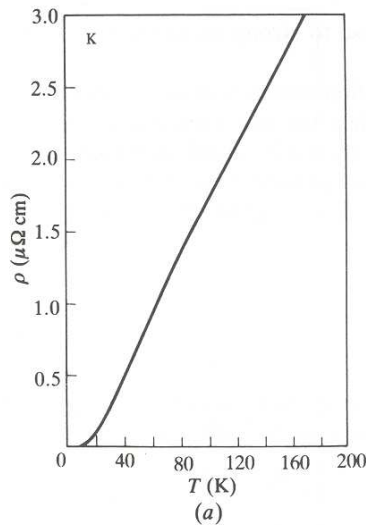
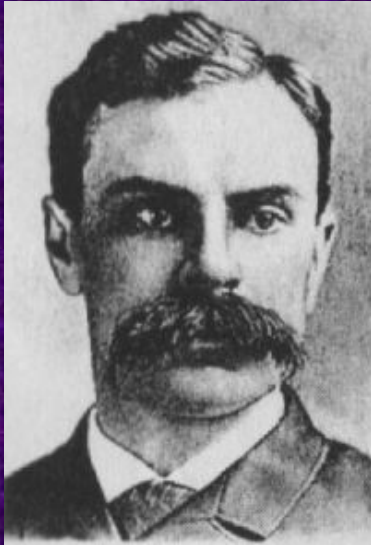


Figure 9.31 (a) The variation of the electrical resistance of pure potassium as a function of temperature. The behaviour is almost linear over a wide range of temperature, but below about 20 K the resistance becomes constant. At high temperatures, deviations from linearity may arise particularly for the transition metals. (b) The effect of small amounts of an alloying element (impurity) on the residual resistance of Ag: \circ , 'pure Ag'; \square , Ag + 0.02 at.% Sn; \triangle , Ag + 0.5 at.% Au. (After Dugdale 1977.)

Except for the very lowest of temperatures (where the classical treatment of the atomic vibrations breaks down), the linear behavior is closely obeyed.

F. The Hall Effect



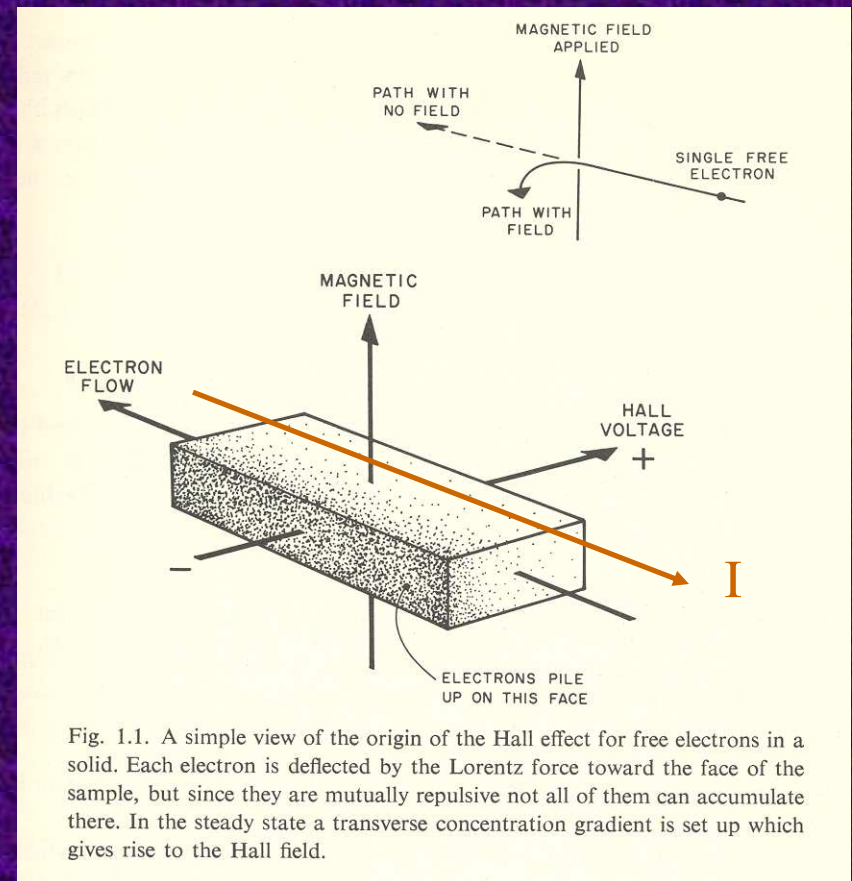
On a New Action of the Magnet on Electric Currents

E. H. Hall

American Journal of Mathematics, Vol. 2, No. 3 (Sep., 1879), 287-292.

This phenomenon, discovered in 1879 by American physics graduate student (!) Edwin Hall, is important because it allows us to measure the free-electron concentration n for metals (and semiconductors!) and compare to predictions of the FEG model.

The Hall effect is quite simple to understand. Consider a B field applied transverse to a thin metal sample carrying a current:



Hall Effect Measurements

A hypothetical charge carrier of charge q experiences a Lorentz force in the lateral direction:

$$F_B = qvB$$

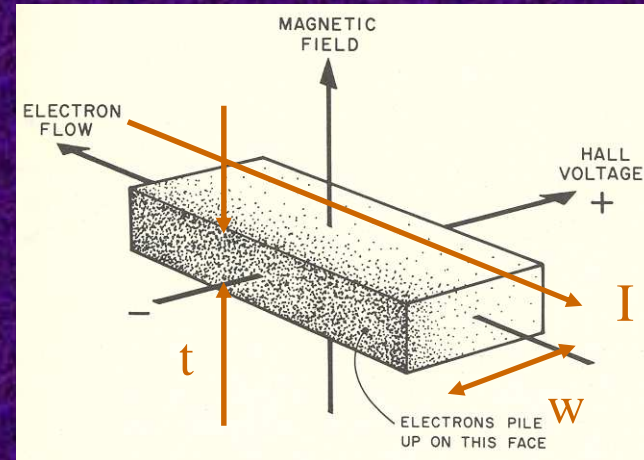
As more and more carriers are deflected, the accumulation of charge produces a “Hall field” E_H that imparts a force opposite to the Lorentz force:

$$F_E = qE_H$$

Equilibrium is reached when these two opposing forces are equal in magnitude, which allows us to determine the drift speed:

From this we can write the current density:

And it is customary to define the Hall coefficient in terms of the measured quantities:



$$qvB = qE_H$$



$$v = \frac{E_H}{B}$$

$$J = nqv = \frac{nqE_H}{B}$$

$$R_H = \frac{E_H}{JB} = \frac{1}{nq}$$

Hall Effect Results!

In the lab we actually measure the Hall voltage V_H and the current I , which gives us a more useful way to write R_H :

$$V_H = E_H w$$

$$I = JA = Jwt$$

$$R_H = \frac{E_H}{JB} = \frac{V_H / w}{(I / wt)B} = \frac{V_H t}{IB} = \frac{1}{nq}$$

If we calculate R_H from our measurements and assume $|q| = e$ (which Hall did not know!) we can find n . Also, the sign of V_H and thus R_H tells us the sign of q !

The discrepancies between the FEG predictions and expt. nearly vanish when liquid metals are compared. This reveals clearly that the source of these discrepancies lies in the electron-lattice interaction. But the results for Be and Zn are puzzling. How can we have $q > 0$???

Stay tuned.....

		R_H ($10^{-11} \text{ m}^3/\text{As}$)		
Metal	n_0	solid	liquid	FEG value
Na	1	-25	-25.5	-25.5
Cu	1	-5.5	-8.25	-8.25
Ag	1	-9.0	-12.0	-12.0
Au	1	-7.2	-11.8	-11.8
* Be	2	+24.4	-2.6	-2.53
* Zn	2	+3.3	-5	-5.1
Al	3	-3.5	-3.9	-3.9

G. Thermal Conductivity of Metals

In metals at all but the lowest temperatures, the electronic contribution to κ far outweighs the contribution of the lattice. So we can write:

$$\kappa \cong \kappa_{el} = \frac{1}{3} C_{el} v \Lambda$$

The electron mean-free path can be rewritten in terms of the collision time:

$$\Lambda = v \tau$$



$$\kappa = \frac{1}{3} C_{el} v^2 \tau$$

Also, the electrons that can absorb thermal energy and therefore contribute to the heat capacity have energies very near E_F , so they essentially all have velocity v_F . This gives:

$$\kappa = \frac{1}{3} C_{el} v_F^2 \tau$$

From our earlier discussion the electronic heat capacity is:

$$C_{el} = \frac{\pi^2}{3} k^2 N(E_F) T$$

It is easy to show that $N(E_F)$ can be expressed:

$$N(E_F) = \frac{3N}{2E_F}$$

Or per unit volume of sample,

$$N'(E_F) = \frac{3n}{2E_F}$$

Which gives the heat capacity per unit volume:

$$C'_{el} = \frac{\pi^2}{2} k^2 n \frac{T}{E_F}$$

Wiedemann-Franz Law and Lorenz Number

Now the thermal conductivity per unit volume is:

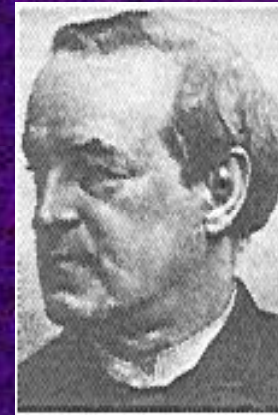
$$\kappa' = \frac{1}{3} \frac{\pi^2}{2} k^2 n \frac{T}{E_F} v_F^2 \tau = \frac{1}{3} \frac{\pi^2}{2} k^2 n \frac{T}{\frac{1}{2} m v_F^2} v_F^2 \tau$$

Finally...

$$\kappa' = \frac{\pi^2 k^2 n \tau}{3m} T$$

Now long before Drude's time, Gustav Wiedemann and Rudolf Franz published a paper in 1853 claiming that the ratio of thermal and electrical conductivities of all metals has nearly the same value at a given T:

$$\frac{\kappa}{\sigma} = \text{constant}$$



Gustav Wiedemann

Not long after (1872) Ludwig Lorenz (not Lorentz!) realized that this ratio scaled linearly with temperature, and thus a Lorenz number L can be defined:

$$\frac{\kappa}{\sigma T} \equiv L$$

very nearly constant for all metals (at room T and above)

The Experimental Test!

We can readily compare the prediction of the FEG model to the results of experiment:

$$L_{FEG} = \frac{\kappa}{\sigma T} = \frac{\frac{\pi^2 k^2 n \tau}{3m} T}{\frac{ne^2 \tau}{m} T} = \frac{\pi^2 k^2}{3e^2}$$

$$L_{FEG} = 2.45 \times 10^{-8} \left(\frac{J}{CK} \right)^2$$

This is remarkable...it is independent of n , m , and even τ !!

	$L = \kappa/\sigma T \cdot 10^{-8} \text{ (J/CK)}^2$	
Metal	0 ° C	100 ° C
Cu	2.23	2.33
Ag	2.31	2.37
Au	2.35	2.40
Zn	2.31	2.33
Cd	2.42	2.43
Mo	2.61	2.79
Pb	2.47	2.56

Agreement with experiment is quite good, although the value of L is about a factor of 10 less at temperatures near 10 K...Can you speculate about the reason?

An Historical Footnote

Drude of course used classical values for the electron velocity v and heat capacity C_{el} . By a tremendous coincidence, the error in each term was about two orders of magnitude...in the opposite direction! So the classical Drude model gives the prediction:

$$L_{Drude} = 1.12 \times 10^{-8} \left(\frac{J}{CK} \right)^2$$

But in Drude's original paper, he inserted an erroneous factor of two, due to a mistake in the calculation of the electrical conductivity. So he originally reported:

$$L = 2.24 \times 10^{-8} \left(\frac{J}{CK} \right)^2 \quad !!!$$

So although Drude's predicted electronic heat capacity was far too high, this prediction of L made the FEG model seem even more impressive than it really was, and led to general acceptance of the model.

H. Limitations of the FEG Model—and Beyond

The FEG model of Drude, augmented by the results of quantum mechanics in the years after 1926, was extremely successful in accounting for many of the basic properties of metals. However, strict quantitative agreement with experiment was not achieved. We can summarize the flawed assumptions behind the FEG model as follows:

1. The free-electron approximation

The positive ions act only as scattering centers and is assumed to have no effect on the motion of electrons between collisions.

2. The independent electron approximation

Interactions between electrons are ignored.

3. The relaxation time approximation

The outcome of a electron collision is assumed to be independent of the state of motion of an electron before the collision.

A comprehensive theory of metals would require abandoning these rather crude approximations. However, a remarkable amount of progress comes from abandoning only the free-electron approximation in order to take into account the effect of the lattice on the conduction electrons.

Unanswered Questions

In addition to quantitative discrepancies between the predictions of the quantum-mechanical FEG model and experiment (heat capacity, resistivity, thermal conductivity, Hall effect, etc.), the FEG model is unable to answer two simple and very important questions. A more comprehensive theory of the solid state should be able to come to grips with these.

1. What determines the number of conduction electrons in a metal?

Why should all valence electrons be “free”? What about elements with more than one valence?

2. Why are some elements metals and others non-metals?

One form of C (diamond) vs. another (graphite); In the same family, B vs. Al