Linear Programming: A Graphic Approach

13.1 GRAPHIC SOLUTIONS

The objective of linear programming is to determine the optimal allocation of scarce resources among competing products or activities. Economic situations frequently call for optimizing a function subject to several inequality constraints. For optimization subject to a single inequality constraint, the Lagrangian method (see Section 6.7) is relatively simple. When more than one inequality constraint is involved, linear programming is easier. If the constraints, however numerous, are limited to two variables, the easiest solution is the graphic approach. The graphic approach for maximization and minimization is demonstrated in Examples 1 and 2, respectively.

EXAMPLE 1. A manufacturer produces tables x_1 and desks x_2. Each table requires 2.5 hours for assembling A, 3 hours for buffing B, and 1 hour for crating C. Each desk requires 1 hour for assembling, 3 hours for buffing, and 2 hours for crating. The firm can use no more than 20 hours for assembling, 30 hours for buffing, and 16 hours for crating each week. Its profit margin is $3 per table and $4 per desk.

The graphic approach is used below to find the output mix that will maximize the firm's weekly profits. It is demonstrated in four easy steps.

1. Express the data as equations or inequalities. The function to be optimized, the *objective function*, becomes

$$\Pi = 3x_1 + 4x_2 \tag{13.1}$$

subject to the constraints

Constraint from A:	$2.5x_1 + x_2 \leq 20$
Constraint from B:	$3x_1 + 3x_2 \leq 30$
Constraint from C:	$x_1 + 2x_2 \leq 16$
Nonnegativity constraint:	$x_1, x_2 \geq 0$

The first three inequalities are *technical constraints* determined by the state of technology and the availability of inputs; the fourth is a *nonnegativity constraint* imposed on every problem to preclude negative (hence unacceptable) values from the solution.

2. Treat the three inequality constraints as equations, solve each one for x_2 in terms of x_1, and graph. Thus,

From A,	$x_2 = 20 - 2.5x_1$
From B,	$x_2 = 10 - x_1$
From C,	$x_2 = 8 - 0.5x_1$

The graph of the original "less than or equal to" inequality will include all the points *on the line and to the left of it*. See Fig. 13-1(*a*). The nonnegativity constraints $x_1, x_2 \geq 0$ are represented by the vertical and horizontal axes, respectively. The shaded area is called the *feasible region*. It contains all the points that satisfy all three constraints plus the nonnegativity constraints. The variables x_1 and x_2 are called *decision* or *structural variables*.

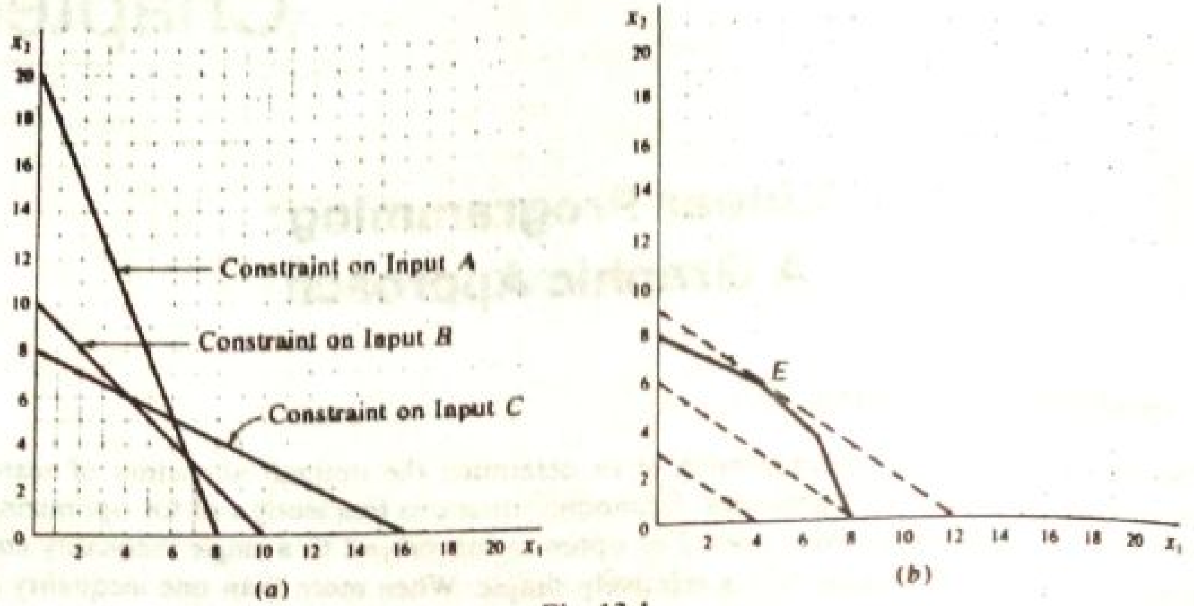

Fig. 13-1

3. To find the optimal solution within the feasible region, if it exists, graph the objective function as a series of isoprofit lines. From *(13.1)*,

$$x_2 = \frac{\Pi}{4} - \frac{3}{4}x_1$$

Thus, the isoprofit line has a slope of $-\frac{3}{4}$. Drawing a series of (dashed) isoprofit lines allowing for larger and larger profits, we find the isoprofit line representing the largest possible profit touches the feasible region at E, where $\bar{x}_1 = 4$ and $\bar{x}_2 = 6$. See Fig. 13-1(*b*). Substituting in *(13.1)*, $\bar{\Pi} = 3(4) + 4(6) = 36$.

4. Profit is maximized at the intersection of two constraints, called an *extreme point*.

13.2 THE EXTREME POINT THEOREM

The *extreme point theorem* states that if an optimal feasible value of the objective function exists, it will be found at one of the extreme (or corner) points of the boundary. Notice that there are 10 extreme points: (0, 20), (0, 10), (6, 5), (10, 0), (16, 0), (0, 8), (4, 6), $(6\frac{2}{3}, 3\frac{1}{3})$, (8, 0), and (0, 0) in Fig. 13-1(*a*), the last being the intersection of the nonnegativity constraints. All are called *basic solutions*, but only the last five are basic *feasible* solutions since they violate none of the constraints. Ordinarily only one of the basic feasible solutions will be optimal. At $(6\frac{2}{3}, 3\frac{1}{3})$, for instance, $\Pi = 3(6\frac{2}{3}) + 4(3\frac{1}{3}) = 33\frac{1}{3}$, which is lower than $\Pi = 36$ above.

EXAMPLE 2. A farmer wants to see that her herd gets the minimum daily requirement of three basic nutrients A, B, and C. Daily requirements are 14 for A, 12 for B, and 18 for C. Product y_1 has 2 units of A and 1 unit each of B and C; product y_2 has 1 unit each of A and B and 3 units of C. The cost of y_1 is \$2, and the cost of y_2 is \$4. The graphic method is used below to determine the least-cost combination of y_1 and y_2 that will fulfill all minimum requirements. Following the procedure used in Example 1,

1. The objective function to be minimized is

$$c = 2y_1 + 4y_2 \qquad (13.2)$$

subject to the constraints

Constraint from A:	$2y_1 + y_2 \geq 14$
Constraint from B:	$y_1 + y_2 \geq 12$
Constraint from C:	$y_1 + 3y_2 \geq 18$
Nonnegativity constraint:	$y_1, y_2 \geq 0$

where the technical constraints read $\geq$ since minimum requirements must be fulfilled but may be exceeded.

2. Treat the inequalities as equations, solve each one for y_2 in terms of y_1, and graph. The graph of the original "greater than or equal to" inequality will include all the points *on the line and to the right of it*. See Fig. 13-2(*a*). The shaded area is the feasible region containing all the points that satisfy all three requirements plus the nonnegativity constraint.

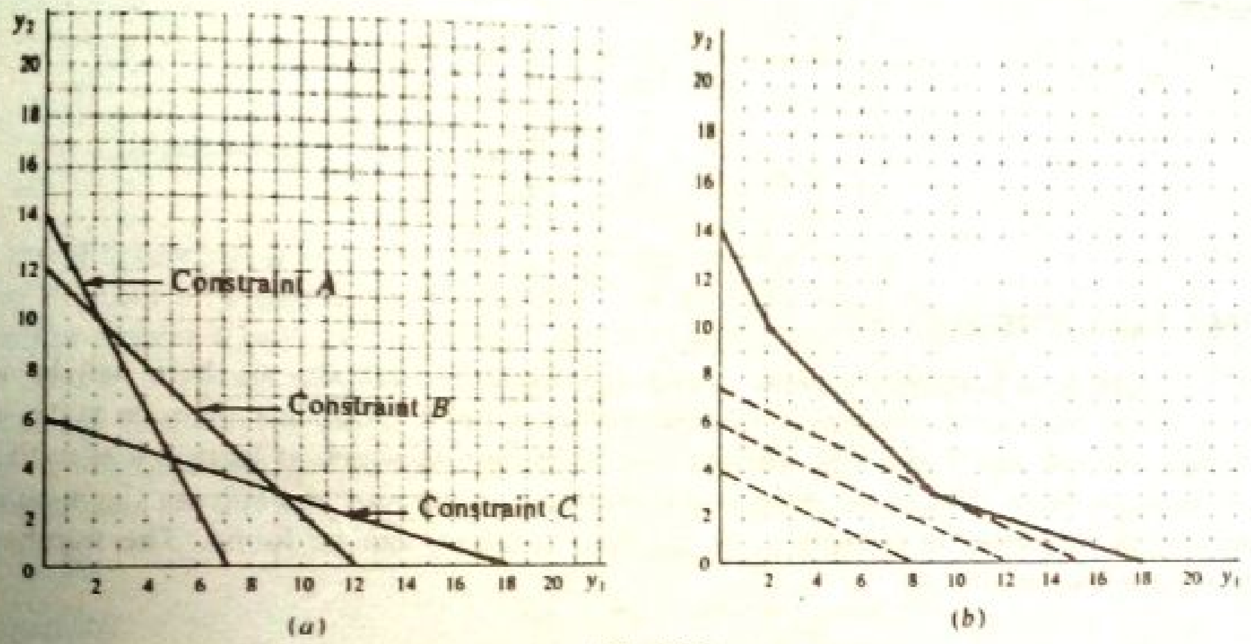

Fig. 13-2

3. To find the optimal solution, graph the objective function as a series of (dashed) isocost lines. From (*13.2*),

$$y_2 = \frac{c}{4} - \frac{1}{2} y_1$$

The lowest isocost line that will touch the feasible region is tangent at $\bar{y}_1 = 9$ and $\bar{y}_2 = 3$ in Fig. 13-2(*b*). Thus, $\bar{c} = 2(9) + 4(3) = 30$, which represents a cost lower than at any other feasible extreme point. For example, at (2, 10), $c = 2(2) + 4(10) = 44$. [For minimization problems, (0, 0) is not in the feasible region.]

Linear Programming: The Dual

15.1 THE DUAL

Every maximization (minimization) problem in linear programming has a corresponding minimization (maximization) problem. The original problem is called the *primal*; the corresponding problem is called the *dual*. The relationship between the two can best be expressed through the use of the parameters they share in common. [For similar properties in Lagrangian functions, see Problems 12.29(*c*) and 12.30(*c*).]

EXAMPLE 1. Given an original or primal problem,

Maximize $$\Pi = g_1x_1 + g_2x_2 + g_3x_3$$

subject to
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le b_3$$
$$x_1, x_2, x_3 \ge 0$$

the related dual problem is

Minimize $$c = b_1z_1 + b_2z_2 + b_3z_3$$

subject to
$$a_{11}z_1 + a_{21}z_2 + a_{31}z_3 \ge g_1$$
$$a_{12}z_1 + a_{22}z_2 + a_{32}z_3 \ge g_2$$
$$a_{13}z_1 + a_{23}z_2 + a_{33}z_3 \ge g_3$$
$$z_1, z_2, z_3 \ge 0$$

15.2 RULES OF TRANSFORMATION TO OBTAIN THE DUAL

In the formulation of a dual from a primal problem,

1. The direction of optimization is reversed. Maximization in the primal becomes minimization in the dual and vice versa.
2. The inequality signs of the technical constraints are reversed, but the nonnegativity restraint on decision variables is always maintained.
3. The rows of the coefficient matrix of the constraints in the primal are transposed to columns for the coefficient matrix of constraints in the dual.
4. The row vector of coefficients in the objective function in the primal is transposed to a column vector of constants for the dual constraints.
5. The column vector of constants from the primal constraints is transposed to a row vector of coefficients for the objective function in the dual.
6. Primal decision variables x_j are replaced by dual decision variables z_i.

EXAMPLE 2. The dual of the linear programming problem

Maximize $$\Pi = 5x_1 + 3x_2$$

subject to
$$6x_1 + 2x_2 \le 36$$
$$5x_1 + 5x_2 \le 40 \qquad x_1, x_2 \ge 0$$
$$2x_1 + 4x_2 \le 28$$

is

Minimize $$c = 36z_1 + 40z_2 + 28z_3$$

subject to $$6z_1 + 5z_2 + 2z_3 \geq 5 \qquad 2z_1 + 5z_2 + 4z_3 \geq 3 \qquad z_1, z_2, z_3 \geq 0$$

EXAMPLE 3. The dual of the linear programming problem

Minimize $$c = 20z_1 + 30z_2 + 16z_3$$

subject to $$2.5z_1 + 3z_2 + z_3 \geq 3$$
$$z_1 + 3z_2 + 2z_3 \geq 4 \qquad z_1, z_2, z_3 \geq 0$$

is

Maximize $$\Pi = 3x_1 + 4x_2$$

subject to $$2.5x_1 + x_2 \leq 20 \qquad x_1 + 2x_2 \leq 16$$
$$3x_1 + 3x_2 \leq 30 \qquad x_1, x_2 \geq 0$$

Note that if the dual of the dual were taken here or in the examples above, the corresponding primal would be obtained.

15.3 THE DUAL THEOREMS

Two dual theorems are of extreme importance for linear programming. They state:

1. The optimal value of the primal objective function always equals the optimal value of the dual objective function, provided an optimal feasible solution exists.
2. If in the optimal feasible solution
   i. A decision variable in the primal program has a nonzero value, the corresponding slack (or surplus) variable in the dual program must have an optimal value of zero.
   ii. A slack (or surplus) variable in the primal has a nonzero value, the corresponding decision variable in the dual program must have an optimal value of zero.

EXAMPLE 4. Given the following linear programming problem

Maximize: $$\Pi = 14x_1 + 12x_2 + 18x_3$$

subject to $$2x_1 + x_2 + x_3 \leq 2$$
$$x_1 + x_2 + 3x_3 \leq 4 \qquad x_1, x_2, x_3 \geq 0$$

The dual theorems are used as follows to find the optimal value of (1) the primal objective function and (2) the primal decision variables. The dual program is

Minimize $$c = 2z_1 + 4z_2$$

subject to $$2z_1 + z_2 \geq 14$$
$$z_1 + z_2 \geq 12$$
$$z_1 + 3z_2 \geq 18 \qquad z_1, z_2 \geq 0$$

1. The optimal values of the dual program were found graphically in Chapter 13, Example 2: $\bar{z}_1 = 9$, $\bar{z}_2 = 3$, and $\bar{c} = 30$. With the optimal value of the dual equal to 30, it is clear from the first dual theorem that $\bar{\Pi}$ must also equal 30.
2. To find the optimal values of the primal decision variables, convert the inequality constraints to equations by adding slack variables to the primal (I) and subtracting surplus variables from the dual (II). To distinguish the slack variables of the primal from the surplus variables of the dual, s_i is used for the primal and t_i for the dual.

$$\text{I.}\quad \begin{aligned} 2x_1 + x_2 + x_3 + s_1 &= 2 \\ x_1 + x_2 + 3x_3 + s_2 &= 4 \end{aligned} \tag{15.1}$$

$$\text{II.}\quad \begin{aligned} 2z_1 + z_2 - t_1 &= 14 \\ z_1 + z_2 - t_2 &= 12 \\ z_1 + 3z_2 - t_3 &= 18 \end{aligned} \tag{15.2}$$

Substitute $\bar{z}_1 = 9$, $\bar{z}_2 = 3$ in (*15.2*) to find $\bar{t}_1, \bar{t}_2, \bar{t}_3$ as follows:

$$2(9) + 3 - t_1 = 14 \qquad \bar{t}_1 = 7$$
$$9 + 3 - t_2 = 12 \qquad \bar{t}_2 = 0$$
$$9 + 3(3) - t_3 = 18 \qquad \bar{t}_3 = 0$$

With the surplus variables $\bar{t}_2, \bar{t}_3$ for the second and third dual constraints equal to zero, according to the second dual theorem, the corresponding primal decision variables $\bar{x}_2, \bar{x}_3$ must be nonzero. With $\bar{t}_1 \neq 0$, the corresponding decision variable $\bar{x}_1$ must equal zero. Therefore $\bar{x}_1 = 0$.

The second dual theorem also states that if the optimal dual decision variables $\bar{z}_1, \bar{z}_2$ do not equal zero in the dual, the corresponding primal slack variables $\bar{s}_1, \bar{s}_2$ in the primary must equal zero. Substituting $\bar{s}_1 = \bar{s}_2 = 0$ in (*15.1*), and recalling that $\bar{x}_1 = 0$, (*15.1*) reduces to

$$x_2 + x_3 = 2 \qquad x_2 + 3x_3 = 4$$

Solving simultaneously by Cramer's rule, $\bar{x}_2 = 1$ and $\bar{x}_3 = 1$. Thus the optimal decision variables are $\bar{x}_1 = 0$, $\bar{x}_2 = 1$, and $\bar{x}_3 = 1$, which can be easily checked by substitution into the objective function: $\Pi = 14(0) + 12(1) + 18(1) = 30$.

EXAMPLE 5. The dual in Example 4 was solved as a primal in Chapter 14, Example 3. Converting x_i to z_i and s_i to t_i for the dual form, the final tableau reads

Final tableau:

z_1	z_2	t_1	t_2	t_3	A_1	A_2	A_3	Constant
1	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	$-\frac{1}{2}$	9
0	0	1	$-\frac{5}{2}$	$\frac{1}{2}$	-1	$\frac{5}{2}$	$-\frac{1}{2}$	7
0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	3
0	0	[0	-1	-1]	$-M$	$-M+1$	$-M+1$	30

The final tableau of the dual can be used to determine the optimal values of (1) the primal objective function and (2) the primal decision variables.

1. The optimal value of the primal objective function, just as the optimal value for the dual objective function, is indicated by the last element of the last row: 30.
2. The optimal values for the primal decision variables can also be read directly from the dual tableau. They are given by the absolute values of the indicators in the columns under the corresponding dual surplus variables. Since t_1 is the surplus variable for the first dual constraint and it corresponds to x_1 in the primal, $\bar{x}_1 = 0$. Since t_2 is the dual surplus variable for the second constraint and it corresponds to x_2 in the primal, $\bar{x}_2 = 1$. Similarly, $\bar{x}_3 = 1$. The dual indicators whose absolute values give the optimal values of the primal decision variables are boxed. The indicators in the artificial variable columns have no economic meaning.