Logarithmic Function

Logarithms

The exponential function of the form

$$y = a^x$$

for all positive values of a, where a not equal to 1.

The equation defining the inverse of a function is found by interchanging x and y in the equation that defines the function. Starting with $y = a^x$ and interchanging x and y yields

$$x = a^y.$$

Here y is the exponent to which a must be raised in order to obtain x. We call this exponent a **logarithm,** symbolized by the abbreviation **"log."** The expression $\log_a x$ represents the logarithm in this discussion. The number a is the **base** of the logarithm, and x is the **argument** of the expression. It is read **"logarithm with base a of x,"** or **"logarithm of x with base a,"** or **"base a logarithm of x."**

If $a > 0$, $a \neq 1$, and $x > 0$, then the **logarithmic function with base a** is

$$f(x) = \log_a x.$$

Logarithm

For all real numbers y and all positive numbers a and x, where $a \neq 1$,

$$y = \log_a x \quad \text{is equivalent to} \quad x = a^y.$$

The expression $\log_a x$ represents the exponent to which the base a must be raised in order to obtain x.

Exponential and logarithmic functions are inverses of each other. To show this, we use the three steps for finding the inverse of a function.

	$f(x) = 2^x$	Exponential function with base 2
	$y = 2^x$	Let $y = f(x)$.
Step 1	$x = 2^y$	Interchange x and y.
Step 2	$y = \log_2 x$	Solve for y by writing in equivalent logarithmic form.
Step 3	$f^{-1}(x) = \log_2 x$	Replace y with $f^{-1}(x)$.

The graph of $f1x2 = 2_x$ has the x-axis as horizontal asymptote and is shown in red in **Figure 25.** Its inverse, $f_{-1}1x2 = \log_2 x$, has the y-axis as vertical asymptote and is shown in blue. The graphs are reflections of each other across the line $y = x$. As a result, their domains and ranges are interchanged.

x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

x	$f^{-1}(x) = \log_2 x$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2

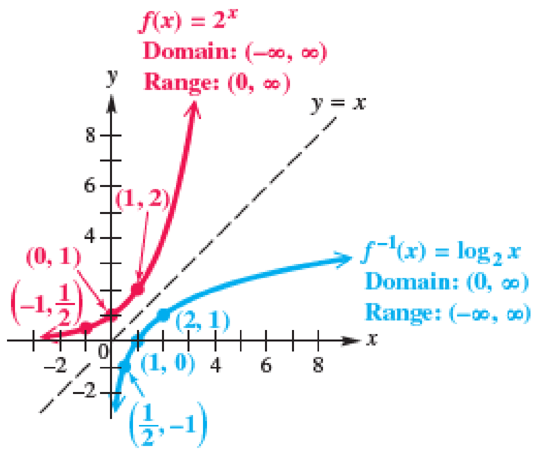


Figure 25

The domain of an exponential function is the set of all real numbers, so the range of a logarithmic function also will be the set of all real numbers. In the same way, both the range of an exponential function and the domain of a logarithmic function are the set of all positive real numbers.

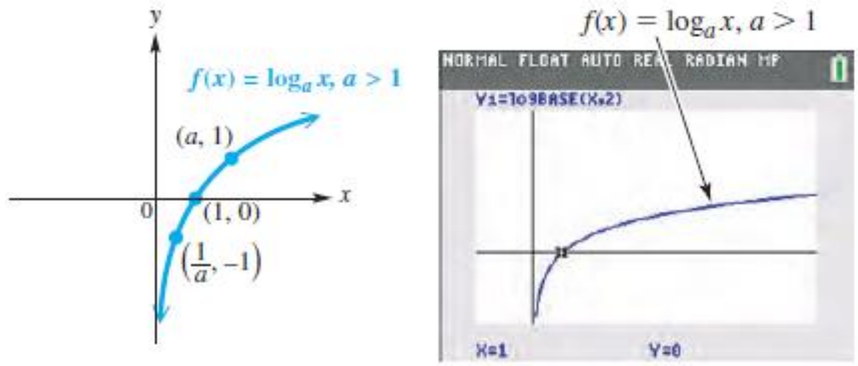
Thus, logarithms can be found for positive numbers only.

Logarithmic Function $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

For $f(x) = \log_2 x$:

x	$f(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



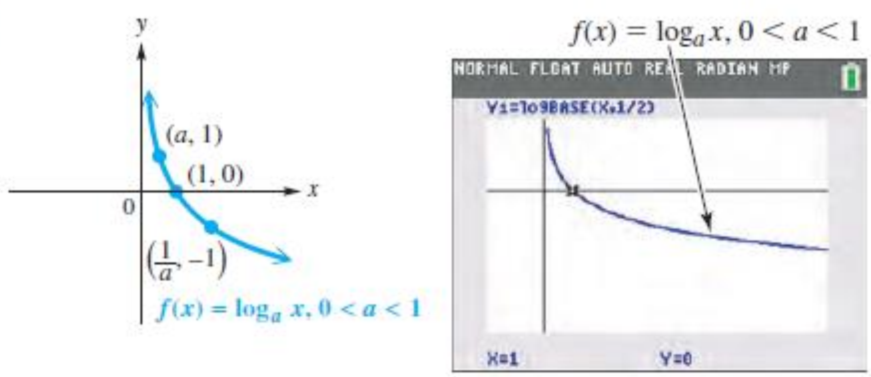
This is the general behavior seen on a calculator graph for **any base *a*, for $a > 1$.**

Figure 26

- **$f(x) = \log_a x$, for $a > 1$,** is increasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.

For $f(x) = \log_{1/2} x$:

x	$f(x)$
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2
8	-3



This is the general behavior seen on a calculator graph for **any base *a*, for $0 < a < 1$.**

Figure 27

- **$f(x) = \log_a x$, for $0 < a < 1$,** is decreasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.

Graphing Logarithmic Functions

Graph each function.

(a) $f(x) = \log_{1/2} x$ **(b)** $f(x) = \log_3 x$

SOLUTION

(a) One approach is to first graph $y = \left(\frac{1}{2}\right)^x$, which defines the inverse function of f, by plotting points. Some ordered pairs are given in the table with the graph shown in red in **Figure 28.**

The graph of $f(x) = \log_{1/2} x$ is the reflection of the graph of $y = \left(\frac{1}{2}\right)^x$ across the line $y = x$. The ordered pairs for $y = \log_{1/2} x$ are found by interchanging the x- and y-values in the ordered pairs for $y = \left(\frac{1}{2}\right)^x$. See the graph in blue in **Figure 28.**

x	$y = \left(\frac{1}{2}\right)^x$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
4	$\frac{1}{16}$

x	$f(x) = \log_{1/2} x$
4	-2
2	-1
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2
$\frac{1}{16}$	4

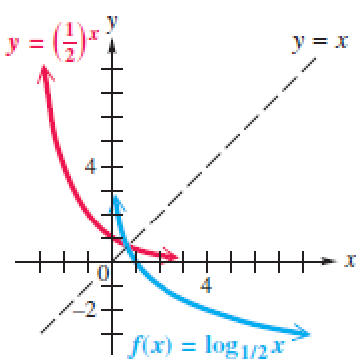


Figure 28

x	$f(x) = \log_3 x$
$\frac{1}{3}$	-1
1	0
3	1
9	2

Think: $x = 3^y$

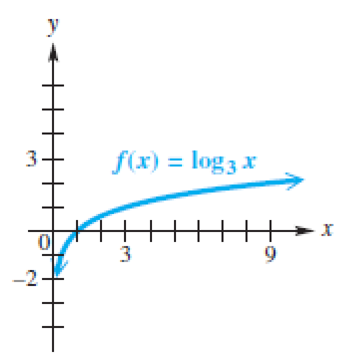


Figure 29

(b) Another way to graph a logarithmic function is to write $f(x) = y = \log_3 x$ in exponential form as $x = 3^y$, and then select y-values and calculate corresponding x-values. Several selected ordered pairs are shown in the table for the graph in **Figure 29.**