Dimension of 4-Subspaces

For dimension of the four subspaces, we have to find bases of

Row space	$R(A)$
Column space	$C(A)$
Null space	$N(A)$
Left null-space	$N(A^t)$

Example: Find the dimension and a bases for the four fundamental subspaces, for,

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \begin{matrix} \sim R_2 - 2R_1 \\ \sim R_3 + R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim R_1 - R_2$$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim R_2/3$$

which is required RREF.

Bases of $R(A) = \{(R_1, R_2)\}$

Row space of A =

$R_1 = (1 \quad 3 \quad 0 \quad -1)$

$R_2 = (0 \quad 0 \quad 1 \quad 1)$

Row space

or

$R_1 = (1 \quad 3 \quad 3 \quad 2)$

$R_2 = (2 \quad 6 \quad 9 \quad 7)$

Dimension of $R(A) = 2$

Bases of $C(A) = \{(C_1, C_3)\}$

column-space of A =

$C_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix}$

Col-space

Dimension of $C(A) = 2$

Null - Space

The null space of A is the set of all X. such that $AX = 0$

$$\therefore \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 3x_2 + 0x_3 - x_4 = 0$

$x_3 + x_4 = 0$

$x_1 = -3x_2 + x_4$

$x_3 = -x_4$

Let

$x_2 = t$

$x_4 = s$

Put in above, we get

$x_1 = -3t + s$

$x_2 = t$

$x_3 = -s$

$x_4 = s$

$$= \begin{bmatrix} -3t+s \\ t \\ -s \\ s \end{bmatrix} = t\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ Bases for null space of A is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

dimension of $N(A) = 2$

Left Null-Space:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

then $$A^t = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 3 & 9 & 3 \\ 2 & 7 & 4 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix} \quad \begin{matrix} \sim R_2 - 3R_1 \\ \sim R_3 - 3R_1 \\ \sim R_4 - 2R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \sim R_2 \longleftrightarrow R_4$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \sim R_3 - R_2$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \sim R_2/3$$

$$A^t = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \sim R_1 - 2R_2$$

The left null space of A is the null space of A^t.

$N(A^T)$ is the set of all y.

$$\therefore \quad A^T y = 0$$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_1 + 0y_2 - 5y_3 = 0 \quad \Rightarrow \quad y_1 = 5y_3$$

$$y_2 + 2y_3 = 0 \quad \Rightarrow \quad y_2 = -2y_3$$

Let $y_3 = t$

$$\Rightarrow \quad \begin{matrix} y_1 = 5t \\ y_2 = -2t \\ y_3 = t \end{matrix} \quad \Rightarrow \quad t \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

Bases of $N(A^T) = \left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \right\}$

Dimension of $N(A^T) = 1$.