Row space, Column Space, Null Space

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Row Space:

Def: If A is a $m \times n$ matrix, then the subspace of $\mathbb{R}^n$ spanned by the row vector of A is called Row Space of A

Notation: $Row(A)$

Column Space:

Def: The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called column space of A.

Notation: $Col(A)$

Remember:

★ Basis for $Row(A) \rightarrow$ non-zero rows in RREF.

★ Basis for $Col(A) \rightarrow$ Pivot-Element in RREF.

No. of Basis element of $row(A)$ = row rank of A

" " " " " $Col(A)$ = col rank of A

Row rank = Column rank = Rank of A.

Null-Space:

Def: The solution space of the system of eq's $Ax = 0$, which is the subspace of $\mathbb{R}^n$ is called Null-Space of A.

Notation: $null(A)$

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$$A = \begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \\ \vdots \\ R_n \rightarrow \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ C_1 & C_2 & C_3 & C_n \end{array}$$

$R_1 = (a_{11}, a_{12}, a_{13}, \cdots, a_{1n}) \in R^n$

$R_2 = (a_{21}, a_{22}, a_{23}, \cdots, a_{2n}) \in R^n$

$R_3 = (a_{31}\ a_{32}, a_{33}, \cdots\ a_{3n}) \in R^n$

$\vdots$

$R_n = (a_{m1}, a_{n2}, a_{m3}, \cdots a_{mn})$

Row Vectors

$$\underbrace{S = \{R_1, R_2, \cdots, R_m\}}_{\text{spanned by } S} \subset R^n \longrightarrow \text{Row space.}$$

$$C_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad C_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \quad C_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in R^m$$

$$S = \{\underbrace{C_1, C_2, C_3, \cdots, C_n}_{\text{spaned by } S}\} \subset \mathbb{R}^m \longrightarrow \text{Column Space}$$

Rank of matrix:

Def: ★ The dimension of Row (column) of a matrix A is called the rank of A.

★ Both rank of row space and column space are equal.

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Example: Find the basis for row(A) & Col(A).

where
$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \\ \to R_3 \\ \end{array}$$

Sol: step-I: Find the RRE-form of the given matrix.

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \sim R_2 - 2R_1 \\ \sim R_3 - 2R_1 \\ \sim R_4 + R_1 \\ \end{array}$$

$$\begin{bmatrix} 1 & -3 & 0 & -14 & 13 & 28 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \sim R_1 - 4R_2 \\ \sim R_3 - R_2 \\ \\ \end{array}$$

$$\begin{array}{c} R_1 \to \\ R_2 \to \\ R_3 \to \\ \end{array} \begin{bmatrix} \boxed{1} & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & \boxed{1} & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & \boxed{1} & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \sim R_2 + 2R_3 \\ \sim R_1 - 13R_3 \\ \\ \end{array}$$
$$\begin{array}{cccccc} C_1 & & C_3 & & C_5 & \end{array}$$

which is the RRE-form of the given matrix.

Basis for row(A) $= B_1 = \{R_1, R_2, R_3\} \subset \mathbb{R}^6$

where
$R_1 = (1, -3, 4, -2, 5, 4)$
$R_2 = (2, -6, 9, -1, 8, 2)$
$R_3 = (2, -6, 9, -1, 9, 7)$

Basis for column(A) $= B_2 = \{C_1, C_3, C_5\} \subset \mathbb{R}^4$

$$C_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

Row rank $= 3$

Column rank $= 3$

Example: Find the null-space of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Here

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}_{3\times 4} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times 1}$$

Augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & -3 & 0 \end{array}\right] \begin{array}{l} \sim R_2 - R_1 \\ \sim R_3 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \sim R_1 - R_2 \\ \sim R_3 + R_1 \end{array}$$

which is RRE-form.

$$x_1 + 0x_2 - x_3 - 2x_4 = 0$$

$$0x_1 + x_2 - 2x_3 + 3x_4 = 0$$

$$\Rightarrow \quad x_1 = x_3 + 2x_4$$

$$x_2 = -2x_3 - 3x_4$$

Now,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$N(A) = \left\{ x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis of N(A) $= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Nulity:

Number of Basis elements of $N(A)$ = Nulity of A.

$N(A) = 2$

Example:

Find the Null space, row-space, column space of a given matrix?

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 6 & 1 & 8 \\ 2 & -2 & -2 & 6 \end{bmatrix}$$

RREF :

$$\begin{bmatrix} 1 & 0 & 1/3 & 5/3 \\ 0 & 1 & 4/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Column space:

$$\left\{ C_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\}$$

Row-space:

$$\left\{ R_1 = \left(1, 0, \tfrac{1}{3}, \tfrac{5}{3}\right), R_2 = \left(0, 1, \tfrac{4}{3}, -\tfrac{1}{3}\right) \right\}$$

$$x_1 + \frac{1}{3}x_3 + \frac{5}{3}x_4 = 0 \qquad\qquad x_3 = t$$

$$x_1 + \frac{1}{3}t + \frac{5}{3}s = 0 \qquad\qquad x_4 = s$$

$$x_1 = -\frac{1}{3}t - \frac{5}{3}s$$

$$x_2 + \frac{4}{3}x_3 - \frac{1}{3}x_4 = 0$$

$$x_2 + \frac{4}{3}t - \frac{1}{3}s$$

$$x_2 = -\frac{4}{3}t + \frac{1}{3}s \quad , x_3 = t \quad , x_4 = t$$

Null Space $= \left\{ \left(-\frac{1}{3}, -\frac{4}{3}, 1, 0\right), \left(-\frac{5}{3}, \frac{1}{3}, 0, 1\right) \right\}$