10.7 Power Series

Now that we can test many infinite series of numbers for convergence, we can study sums that look like "infinite polynomials." We call these sums *power series* because they are defined as infinite series of powers of some variable, in our case x. Like polynomials, power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series. With power series we can extend the methods of calculus to a vast array of functions, making the techniques of calculus applicable in an even wider setting.

Power Series and Convergence

We begin with the formal definition, which specifies the notation and terminology used for power series.

DEFINITIONS **A power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \tag{1}$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{2}$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Equation (1) is the special case obtained by taking $a = 0$ in Equation (2). We will see that a power series defines a function $f(x)$ on a certain interval where it converges. Moreover, this function will be shown to be continuous and differentiable over the interior of that interval.

EXAMPLE 1 Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This is the geometric series with first term 1 and ratio x. It converges to $1/(1-x)$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1. \tag{3}$$

■

Power Series for $\frac{1}{1-x}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

Up to now, we have used Equation (3) as a formula for the sum of the series on the right. We now change the focus: We think of the partial sums of the series on the right as polynomials $P_n(x)$ that approximate the function on the left. For values of x near zero, we need take only a few terms of the series to get a good approximation. As we move toward $x = 1$, or -1, we must take more terms. Figure 10.17 shows the graphs of $f(x) = 1/(1-x)$ and the approximating polynomials $y_n = P_n(x)$ for $n = 0, 1, 2,$ and 8. The function $f(x) = 1/(1-x)$ is not continuous on intervals containing $x = 1$, where it has a vertical asymptote. The approximations do not apply when $x \geq 1$.

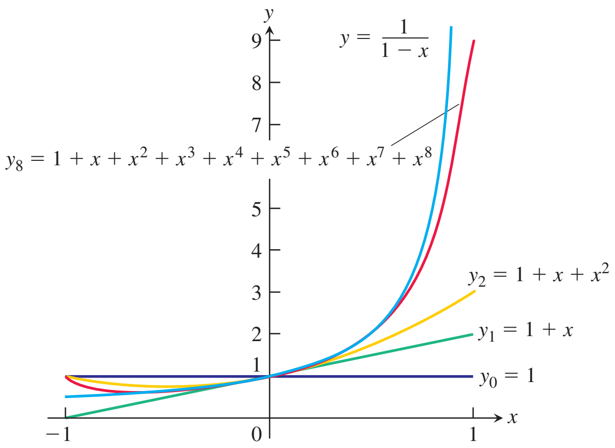


FIGURE 10.17 The graphs of $f(x) = 1/(1 - x)$ in Example 1 and four of its polynomial approximations.

EXAMPLE 2 The power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots \tag{4}$$

matches Equation (2) with $a = 2$, $c_0 = 1$, $c_1 = -1/2$, $c_2 = 1/4, \ldots, c_n = (-1/2)^n$. This is a geometric series with first term 1 and ratio $r = -\frac{x-2}{2}$. The series converges for $\left|\frac{x-2}{2}\right| < 1$, which simplifies to $0 < x < 4$. The sum is

$$\frac{1}{1-r} = \frac{1}{1 + \frac{x-2}{2}} = \frac{2}{x},$$

so

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \qquad 0 < x < 4.$$

Series (4) generates useful polynomial approximations of $f(x) = 2/x$ for values of x near 2:

$$P_0(x) = 1$$

$$P_1(x) = 1 - \frac{1}{2}(x-2) = 2 - \frac{x}{2}$$

$$P_2(x) = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 = 3 - \frac{3x}{2} + \frac{x^2}{4},$$

and so on (Figure 10.18). ■

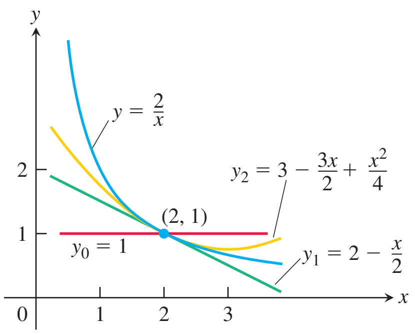


FIGURE 10.18 The graphs of $f(x) = 2/x$ and its first three polynomial approximations (Example 2).

The following example illustrates how we test a power series for convergence by using the Ratio Test to see where it converges and diverges.

EXAMPLE 3 For what values of x do the following power series converge?

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

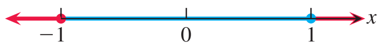
(c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(d) $\displaystyle\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

Solution Apply the Ratio Test to the series $\sum |u_n|$, where u_n is the nth term of the power series in question.

(a) $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x}\right| = \frac{n}{n+1}|x| \to |x|.$$

By the Ratio Test, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. At $x = 1$, we get the alternating harmonic series $1 - 1/2 + 1/3 - 1/4 + \cdots$, which converges. At $x = -1$, we get $-1 - 1/2 - 1/3 - 1/4 - \cdots$, the negative of the harmonic series, which diverges. Series (a) converges for $-1 < x \le 1$ and diverges elsewhere.



We will see in Example 6 that this series converges to the function $\ln(1 + x)$ on the interval $(-1, 1]$ (see Figure 10.19).

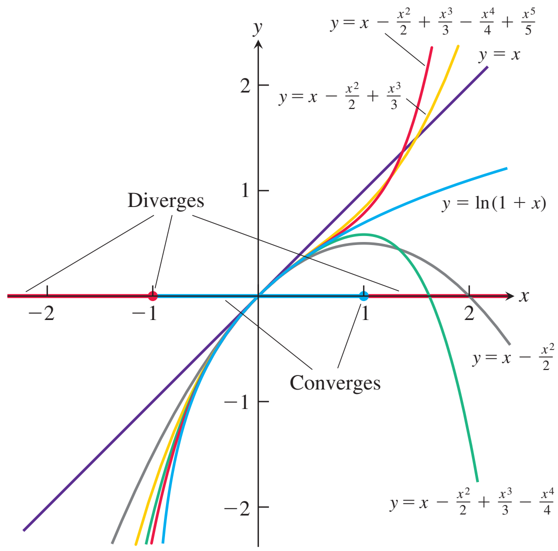
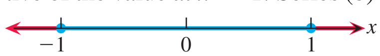


FIGURE 10.19 The power series $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$ converges on the interval $(-1, 1]$.

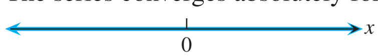
(b) $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2 \to x^2.$$ $2(n + 1) - 1 = 2n + 1$

By the Ratio Test, the series converges absolutely for $x^2 < 1$ and diverges for $x^2 > 1$. At $x = 1$ the series becomes $1 - 1/3 + 1/5 - 1/7 + \cdots$, which converges by the Alternating Series Theorem. It also converges at $x = -1$ because it is again an alternating series that satisfies the conditions for convergence. The value at $x = -1$ is the negative of the value at $x = 1$. Series (b) converges for $-1 \le x \le 1$ and diverges elsewhere.



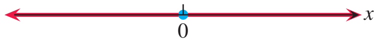
(c) $$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \to 0 \text{ for every } x.$$ $\dfrac{n!}{(n+1)!} = \dfrac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)}$

The series converges absolutely for all x.



(d) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \to \infty$ unless $x = 0$.

The series diverges for all values of x except $x = 0$.



The previous example illustrated how a power series might converge. The next result shows that if a power series converges at more than one value, then it converges over an entire interval of values. The interval might be finite or infinite and contain one, both, or none of its endpoints. We will see that each endpoint of a finite interval must be tested independently for convergence or divergence.

THEOREM 18—The Convergence Theorem for Power Series
If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \text{ converges at } x = c \neq 0, \text{ then it converges}$$

absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

Proof The proof uses the Direct Comparison Test, with the given series compared to a converging geometric series.

Suppose the series $\sum_{n=0}^{\infty} a_n c^n$ converges. Then $\lim_{n\to\infty} a_n c^n = 0$ by the nth-Term Test. Hence, there is an integer N such that $|a_n c^n| < 1$ for all $n > N$, so that

$$|a_n| < \frac{1}{|c|^n} \quad \text{for } n > N. \tag{5}$$

Now take any x such that $|x| < |c|$, so that $|x|/|c| < 1$. Multiplying both sides of Equation (5) by $|x|^n$ gives

$$|a_n||x|^n < \frac{|x|^n}{|c|^n} \quad \text{for } n > N.$$

Since $|x/c| < 1$, it follows that the geometric series $\sum_{n=0}^{\infty} |x/c|^n$ converges. By the Direct Comparison Test (Theorem 10), the series $\sum_{n=0}^{\infty} |a_n||x^n|$ converges, so the original power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $-|c| < x < |c|$ as claimed by the theorem. (See Figure 10.20.)

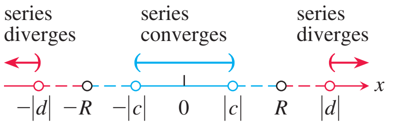


FIGURE 10.20 Convergence of $\sum a_n x^n$ at $x = c$ implies absolute convergence on the interval $-|c| < x < |c|$; divergence at $x = d$ implies divergence for $|x| > |d|$. The corollary to Theorem 18 asserts the existence of a radius of convergence $R \geq 0$. For $|x| < R$ the series converges absolutely and for $|x| > R$ it diverges.

Now suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ diverges at $x = d$. If x is a number with $|x| > |d|$ and the series converges at x, then the first half of the theorem shows that the series also converges at d, contrary to our assumption. So the series diverges for all x with $|x| > |d|$. ■

To simplify the notation, Theorem 18 deals with the convergence of series of the form $\sum a_n x^n$. For series of the form $\sum a_n (x - a)^n$ we can replace $x - a$ by x' and apply the results to the series $\sum a_n (x')^n$.

The Radius of Convergence of a Power Series

The theorem we have just proved and the examples we have studied lead to the conclusion that a power series $\sum c_n (x - a)^n$ behaves in one of three possible ways. It might converge only at $x = a$, or converge everywhere, or converge on some interval of radius R centered

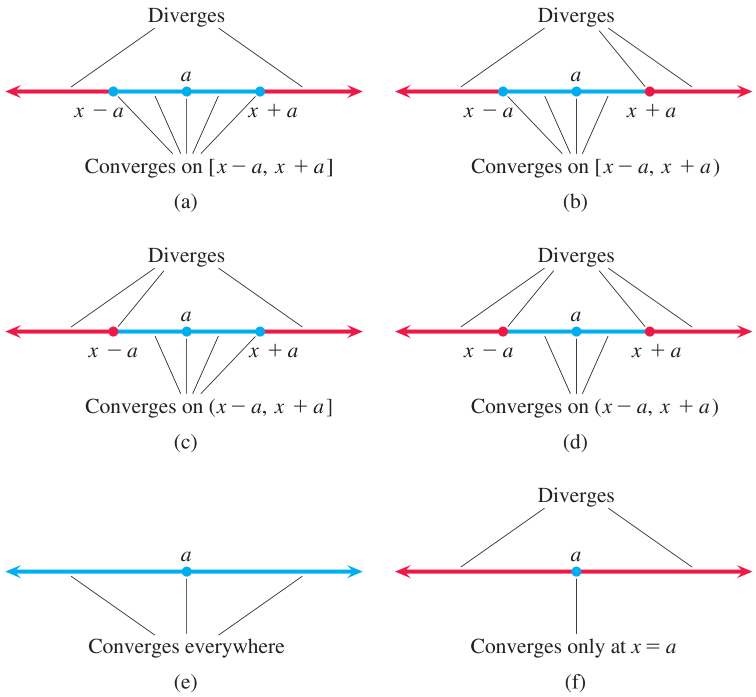


FIGURE 10.21 The six possibilities for an interval of convergence.

at $x = a$. We prove this as a Corollary to Theorem 18. When we also consider the convergence at the endpoints of an interval, there are six different possibilities. These are shown in Figure 10.21.

Corollary to Theorem 18

The convergence of the series $\sum c_n(x - a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

Proof We first consider the case where $a = 0$, so that we have a power series $\sum_{n=0}^{\infty} c_n x^n$ centered at 0. If the series converges everywhere we are in Case 2. If it converges only at $x = 0$ then we are in Case 3. Otherwise there is a nonzero number d such that $\sum_{n=0}^{\infty} c_n d^n$ diverges. Let S be the set of values of x for which $\sum_{n=0}^{\infty} c_n x^n$ converges. The set S does not include any x with $|x| > |d|$, since Theorem 18 implies the series diverges at all such values. So the set S is bounded. By the Completeness Property of the Real Numbers (Appendix 6) S has a least upper bound R. (This is the smallest number with the property that all elements of S are less than or equal to R.) Since we are not in Case 3, the series converges at some number $b \neq 0$ and, by Theorem 18, also on the open interval $(-|b|, |b|)$. Therefore, $R > 0$.

If $|x| < R$ then there is a number c in S with $|x| < c < R$, since otherwise R would not be the least upper bound for S. The series converges at c since $c \in S$, so by Theorem 18 the series converges absolutely at x.

Now suppose $|x| > R$. If the series converges at x, then Theorem 18 implies it converges absolutely on the open interval $(-|x|, |x|)$, so that S contains this interval. Since R is an upper bound for S, it follows that $|x| \le R$, which is a contradiction. So if $|x| > R$ then the series diverges. This proves the theorem for power series centered at $a = 0$.

For a power series centered at an arbitrary point $x = a$, set $x' = x - a$ and repeat the argument above, replacing x with x'. Since $x' = 0$ when $x = a$, convergence of the series $\sum_{n=0}^{\infty} |c_n(x')^n|$ on a radius R open interval centered at $x' = 0$ corresponds to convergence of the series $\sum_{n=0}^{\infty} |c_n(x - a)^n|$ on a radius R open interval centered at $x = a$. ■

R is called the **radius of convergence** of the power series, and the interval of radius R centered at $x = a$ is called the **interval of convergence**. The interval of convergence may be open, closed, or half-open, depending on the particular series. At points x with $|x - a| < R$, the series converges absolutely. If the series converges for all values of x, we say its radius of convergence is infinite. If it converges only at $x = a$, we say its radius of convergence is zero.

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the largest open interval where the series converges absolutely,

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. If R is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. If R is finite, the series diverges for $|x - a| > R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

Operations on Power Series

On the intersection of their intervals of convergence, two power series can be added and subtracted term by term just like series of constants (Theorem 8). They can be multiplied just as we multiply polynomials, but we often limit the computation of the product to the first few terms, which are the most important. The following result gives a formula for the coefficients in the product, but we omit the proof. (Power series can also be divided in a way similar to division of polynomials, but we do not give a formula for the general coefficient here.)

THEOREM 19—Series Multiplication for Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Finding the general coefficient c_n in the product of two power series can be very tedious and the term may be unwieldy. The following computation provides an illustration of a product where we find the first few terms by multiplying the terms of the second series by each term of the first series:

$$\left(\sum_{n=0}^{\infty} x^n\right)\cdot\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}\right)$$

$$= (1 + x + x^2 + \cdots)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) \qquad \text{Multiply second series . . .}$$

$$= \underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right)}_{\text{by } 1} + \underbrace{\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \cdots\right)}_{\text{by } x} + \underbrace{\left(x^3 - \frac{x^4}{2} + \frac{x^5}{3} - \cdots\right)}_{\text{by } x^2} + \cdots$$

$$= x + \frac{x^2}{2} + \frac{5x^3}{6} - \frac{x^4}{6}\cdots. \qquad \text{and gather the first four powers.}$$

We can also substitute a function $f(x)$ for x in a convergent power series.

THEOREM 20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$ and f is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points x where $|f(x)| < R$.

Since $1/(1 - x) = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$, it follows from Theorem 20 that $1/(1 - 4x^2) = \sum_{n=0}^{\infty} (4x^2)^n$ converges absolutely when x satisfies $|4x^2| < 1$ or equivalently when $|x| < 1/2$.

Theorem 21 says that a power series can be differentiated term by term at each interior point of its interval of convergence. A proof is outlined in Exercise 64.

THEOREM 21—Term-by-Term Differentiation
If $\sum c_n(x - a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \qquad \text{on the interval} \qquad a - R < x < a + R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n - 1)c_n(x - a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

EXAMPLE 4 Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

Solution We differentiate the power series on the right term by term:

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$
$$= \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1;$$
$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$
$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1.$$

Caution Term-by-term differentiation might not work for other kinds of series. For example, the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$$

converges for all x. But if we differentiate term by term we get the series

$$\sum_{n=1}^{\infty} \frac{n!\cos(n!x)}{n^2},$$

which diverges for all x. This is not a power series since it is not a sum of positive integer powers of x.

It is also true that a power series can be integrated term by term throughout its interval of convergence. The proof is outlined in Exercise 65.

THEOREM 22—Term-by-Term Integration
Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

converges for $a - R < x < a + R \, (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

EXAMPLE 5 Identify the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1.$$

Solution We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1. \qquad \text{Theorem 21}$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

We can now integrate $f'(x) = 1/(1+x^2)$ to get

$$\int f'(x)\,dx = \int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

The series for $f(x)$ is zero when $x = 0$, so $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \qquad -1 < x < 1. \tag{6}$$

It can be shown that the series also converges to $\tan^{-1} x$ at the endpoints $x = \pm 1$, but we omit the proof. ■

The Number π as a Series

$$\frac{\pi}{4} = \tan^{-1} 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Notice that the original series in Example 5 converges at both endpoints of the original interval of convergence, but Theorem 22 can only guarantee the convergence of the differentiated series inside the interval.

EXAMPLE 6 The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \qquad \text{Theorem 22}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x < 1.$$

Alternating Harmonic Series Sum

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

It can also be shown that the series converges at $x = 1$ to the number $\ln 2$, but that was not guaranteed by the theorem. A proof of this is outlined in Exercise 61. ■

EXERCISES 10.7

Intervals of Convergence

In Exercises 1–36, **(a)** find the series' radius and interval of convergence. For what values of x does the series converge **(b)** absolutely, **(c)** conditionally?

1. $\sum_{n=0}^{\infty} x^n$
2. $\sum_{n=0}^{\infty} (x+5)^n$
3. $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$
4. $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$
5. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$
6. $\sum_{n=0}^{\infty} (2x)^n$
7. $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$
8. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$
9. $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n}$
10. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$
11. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$
12. $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$
13. $\sum_{n=1}^{\infty} \frac{4^n x^{2n}}{n}$
14. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$
15. $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$
16. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n+3}}$

17. $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

18. $\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$

19. $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$

20. $\sum_{n=1}^{\infty} \sqrt[n]{n}(2x+5)^n$

21. $\sum_{n=1}^{\infty} (2+(-1)^n)\cdot(x+1)^{n-1}$

22. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n}(x-2)^n}{3n}$

23. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n x^n$

24. $\sum_{n=1}^{\infty} (\ln n)x^n$

25. $\sum_{n=1}^{\infty} n^n x^n$

26. $\sum_{n=0}^{\infty} n!(x-4)^n$

27. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$

28. $\sum_{n=0}^{\infty} (-2)^n(n+1)(x-1)^n$

29. $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$ Get the information you need about $\sum 1/(n(\ln n)^2)$ from Section 10.3, Exercise 61.

30. $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$ Get the information you need about $\sum 1/(n \ln n)$ from Section 10.3, Exercise 60.

31. $\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$

32. $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

33. $\sum_{n=1}^{\infty} \frac{1}{2\cdot 4\cdot 6\cdots(2n)} x^n$

34. $\sum_{n=1}^{\infty} \frac{3\cdot 5\cdot 7\cdots(2n+1)}{n^2\cdot 2^n} x^{n+1}$

35. $\sum_{n=1}^{\infty} \frac{1+2+3+\cdots+n}{1^2+2^2+3^2+\cdots+n^2} x^n$

36. $\sum_{n=1}^{\infty} (\sqrt{n+1}-\sqrt{n})(x-3)^n$

In Exercises 37–40, find the series' radius of convergence.

37. $\sum_{n=1}^{\infty} \frac{n!}{3\cdot 6\cdot 9\cdots 3n} x^n$

38. $\sum_{n=1}^{\infty} \left(\frac{2\cdot 4\cdot 6\cdots(2n)}{2\cdot 5\cdot 8\cdots(3n-1)}\right)^2 x^n$

39. $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^n(2n)!} x^n$

40. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$

(*Hint:* Apply the Root Test.)

In Exercises 41–48, use Theorem 20 to find the series' interval of convergence and, within this interval, the sum of the series as a function of x.

41. $\sum_{n=0}^{\infty} 3^n x^n$

42. $\sum_{n=0}^{\infty} (e^x-4)^n$

43. $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$

44. $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

45. $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2}-1\right)^n$

46. $\sum_{n=0}^{\infty} (\ln x)^n$

47. $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3}\right)^n$

48. $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$

Using the Geometric Series

49. In Example 2 we represented the function $f(x) = 2/x$ as a power series about $x = 2$. Use a geometric series to represent $f(x)$ as a power series about $x = 1$, and find its interval of convergence.

50. Use a geometric series to represent each of the given functions as a power series about $x = 0$, and find their intervals of convergence.

a. $f(x) = \frac{5}{3-x}$ **b.** $g(x) = \frac{3}{x-2}$

51. Represent the function $g(x)$ in Exercise 50 as a power series about $x = 5$, and find the interval of convergence.

52. a. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{8}{4^{n+2}} x^n.$$

b. Represent the power series in part (a) as a power series about $x = 3$ and identify the interval of convergence of the new series. (Later in the chapter you will understand why the new interval of convergence does not necessarily include all of the numbers in the original interval of convergence.)

Theory and Examples

53. For what values of x does the series

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$$

converge? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What is its sum?

54. If you integrate the series in Exercise 53 term by term, what new series do you get? For what values of x does the new series converge, and what is another name for its sum?

55. The series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

converges to $\sin x$ for all x.

a. Find the first six terms of a series for $\cos x$. For what values of x should the series converge?

b. By replacing x by $2x$ in the series for $\sin x$, find a series that converges to $\sin 2x$ for all x.

c. Using the result in part (a) and series multiplication, calculate the first six terms of a series for $2 \sin x \cos x$. Compare your answer with the answer in part (b).

56. The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

converges to e^x for all x.

a. Find a series for $(d/dx)e^x$. Do you get the series for e^x? Explain your answer.

b. Find a series for $\int e^x\,dx$. Do you get the series for e^x? Explain your answer.

c. Replace x by $-x$ in the series for e^x to find a series that converges to e^{-x} for all x. Then multiply the series for e^x and e^{-x} to find the first six terms of a series for $e^{-x}\cdot e^x$.

57. The series

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$$

converges to $\tan x$ for $-\pi/2 < x < \pi/2$.

a. Find the first five terms of the series for $\ln|\sec x|$. For what values of x should the series converge?

b. Find the first five terms of the series for $\sec^2 x$. For what values of x should this series converge?

c. Check your result in part (b) by squaring the series given for $\sec x$ in Exercise 58.

58. The series

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots$$

converges to $\sec x$ for $-\pi/2 < x < \pi/2$.

a. Find the first five terms of a power series for the function $\ln|\sec x + \tan x|$. For what values of x should the series converge?

b. Find the first four terms of a series for $\sec x \tan x$. For what values of x should the series converge?

c. Check your result in part (b) by multiplying the series for $\sec x$ by the series given for $\tan x$ in Exercise 57.

59. Uniqueness of convergent power series

a. Show that if two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are convergent and equal for all values of x in an open interval $(-c, c)$, then $a_n = b_n$ for every n. (*Hint:* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. Differentiate term by term to show that a_n and b_n both equal $f^{(n)}(0)/(n!)$.)

b. Show that if $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x in an open interval $(-c, c)$, then $a_n = 0$ for every n.

60. The sum of the series $\sum_{n=0}^{\infty}(n^2/2^n)$ To find the sum of this series, express $1/(1-x)$ as a geometric series, differentiate both sides of the resulting equation with respect to x, multiply both sides of the result by x, differentiate again, multiply by x again, and set x equal to $1/2$. What do you get?

61. The sum of the alternating harmonic series This exercise will show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

Let h_n be the nth partial sum of the harmonic series, and let s_n be the nth partial sum of the alternating harmonic series.

a. Use mathematical induction or algebra to show that

$$s_{2n} = h_{2n} - h_n.$$

b. Use the results in Exercise 63 in Section 10.3 to conclude that

$$\lim_{n\to\infty} (h_n - \ln n) = \gamma$$

and

$$\lim_{n\to\infty} (h_{2n} - \ln 2n) = \gamma,$$

where γ is Euler's constant.

c. Use these facts to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{n\to\infty} s_{2n} = \ln 2.$$

62. Assume that the series $\sum a_n x^n$ converges for $x = 4$ and diverges for $x = 7$. Answer true (T), false (F), or not enough information given (N) for the following statements about the series.

a. Converges absolutely for $x = -4$

b. Diverges for $x = 5$

c. Converges absolutely for $x = -8.5$

d. Converges for $x = -2$

e. Diverges for $x = 8$

f. Diverges for $x = -6$

g. Converges absolutely for $x = 0$

h. Converges absolutely for $x = -7.1$

63. Assume that the series $\sum a_n(x-2)^n$ converges for $x = -1$ and diverges for $x = 6$. Answer true (T), false (F), or not enough information given (N) for the following statements about the series.

a. Converges absolutely for $x = 1$

b. Diverges for $x = -6$

c. Diverges for $x = 2$

d. Converges for $x = 0$

e. Converges absolutely for $x = 5$

f. Diverges for $x = 4.9$

g. Diverges for $x = 5.1$

h. Converges absolutely for $x = 4$

64. Proof of Theorem 21 Assume that $a = 0$ in Theorem 21 and that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $-R < x < R$. Let $g(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$. This exercise will prove that $f'(x) = g(x)$, that is, $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = g(x)$.

a. Use the Ratio Test to show that $g(x)$ converges for $-R < x < R$.

b. Use the Mean Value Theorem to show that

$$\frac{(x+h)^n - x^n}{h} = n c_n^{n-1}$$

for some c_n between x and $x + h$ for $n = 1, 2, 3, \ldots$.

c. Show that

$$\left| g(x) - \frac{f(x+h) - f(x)}{h} \right| = \left| \sum_{n=2}^{\infty} n a_n \left(x^{n-1} - c_n^{n-1}\right) \right|$$

d. Use the Mean Value Theorem to show that

$$\frac{x^{n-1} - c_n^{n-1}}{x - c_n} = (n-1)\,d_{n-1}^{n-2}$$

for some d_{n-1} between x and c_n for $n = 2, 3, 4, \ldots$.

e. Explain why $|x - c_n| < h$ and why $|d_{n-1}| \leq \alpha = \max\{|x|, |x + h|\}$.

f. Show that

$$\left| g(x) - \frac{f(x+h) - f(x)}{h} \right| \leq |h| \sum_{n=2}^{\infty} |n(n-1)a_n \alpha^{n-2}|$$

g. Show that $\sum_{n=2}^{\infty} n(n-1)\alpha^{n-2}$ converges for $-R < x < R$.

h. Let $h \to 0$ in part (f) to conclude that

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = g(x).$$

65. Proof of Theorem 22 Assume that $a = 0$ in Theorem 22 and that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $-R < x < R$. Let $g(x) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$. This exercise will prove that $g'(x) = f(x)$.

a. Use the Ratio Test to show that $g(x)$ converges for $-R < x < R$.

b. Use Theorem 21 to show that $g'(x) = f(x)$, that is,

$$\int f(x)\,dx = g(x) + C.$$

10.8 Taylor and Maclaurin Series

We have seen how geometric series can be used to generate a power series for functions such as $f(x) = 1/(1 - x)$ or $g(x) = 3/(x - 2)$. Now we expand our capability to represent a function with a power series. This section shows how functions that are infinitely differentiable generate power series called *Taylor series*. In many cases, these series provide useful polynomial approximations of the original functions. Because approximation by polynomials is extremely useful to both mathematicians and scientists, Taylor series are an important application of the theory of infinite series.

Series Representations

We know from Theorem 21 that within its interval of convergence I the sum of a power series is a continuous function with derivatives of all orders. But what about the other way around? If a function $f(x)$ has derivatives of all orders on an interval, can it be expressed as a power series on at least part of that interval? And if it can, what are its coefficients?

We can answer the last question readily if we assume that $f(x)$ is the sum of a power series about $x = a$,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\ &= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \end{aligned}$$

with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence I, we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots, \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \cdots, \\ f'''(x) &= 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^2 + \cdots, \end{aligned}$$

with the nth derivative being

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } (x-a) \text{ as a factor.}$$

Since these equations all hold at $x = a$, we have

$$f'(a) = a_1, \qquad f''(a) = 1 \cdot 2a_2, \qquad f'''(a) = 1 \cdot 2 \cdot 3a_3,$$

and, in general,

$$f^{(n)}(a) = n!a_n.$$