diverges (since $|r| = 5/4 > 1$). In series (1), there is some cancelation in the partial sums, which may be assisting the convergence property of the series. However, if we make all of the terms positive in series (1) to form the new series

$$5 + \frac{5}{4} + \frac{5}{16} + \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} \left| 5\left(\frac{-1}{4}\right)^n \right| = \sum_{n=0}^{\infty} 5\left(\frac{1}{4}\right)^n,$$

we see that it still converges. For a general series with both positive and negative terms, we can apply the tests for convergence studied before to the series of absolute values of its terms. In doing so, we are led naturally to the following concept.

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

So the geometric series (1) is absolutely convergent. We observed, too, that it is also convergent. This situation is always true: An absolutely convergent series is convergent as well, which we now prove.

Caution
Be careful when using Theorem 12. A convergent series need *not* converge absolutely, as you will see in the next section.

THEOREM 12—The Absolute Convergence Test

If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Proof For each n,

$$-|a_n| \le a_n \le |a_n|, \qquad \text{so} \qquad 0 \le a_n + |a_n| \le 2|a_n|.$$

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} 2|a_n|$ converges and, by the Direct Comparison Test, the nonnegative series $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. The equality $a_n = (a_n + |a_n|) - |a_n|$ now lets us express $\sum_{n=1}^{\infty} a_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ converges. ■

EXAMPLE 1 This example gives two series that converge absolutely.

(a) For $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$, the corresponding series of absolute values is the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

The original series converges because it converges absolutely.

(b) For $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \frac{\sin 3}{9} + \cdots$, which contains both positive and negative terms, the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \frac{|\sin 1|}{1} + \frac{|\sin 2|}{4} + \cdots,$$

which converges by comparison with $\sum_{n=1}^{\infty} (1/n^2)$ because $|\sin n| \le 1$ for every n. The original series converges absolutely; therefore it converges. ■

The Ratio Test

The Ratio Test measures the rate of growth (or decline) of a series by examining the ratio a_{n+1}/a_n. For a geometric series $\sum ar^n$, this rate is a constant $((ar^{n+1})/(ar^n) = r)$, and the series converges if and only if its ratio is less than 1 in absolute value. The Ratio Test is a powerful rule extending that result.

ρ is the Greek lowercase letter rho, which is pronounced "row."

THEOREM 13—The Ratio Test
Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

Proof

(a) $\boldsymbol{\rho < 1}$**.** Let r be a number between ρ and 1. Then the number $\varepsilon = r - \rho$ is positive. Since

$$\left|\frac{a_{n+1}}{a_n}\right| \to \rho,$$

$|a_{n+1}/a_n|$ must lie within ε of ρ when n is large enough, say, for all $n \geq N$. In particular,

$$\left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon = r, \qquad \text{when } n \geq N.$$

Hence

$$\begin{aligned}
|a_{N+1}| &< r|a_N|, \\
|a_{N+2}| &< r|a_{N+1}| < r^2|a_N|, \\
|a_{N+3}| &< r|a_{N+2}| < r^3|a_N|, \\
&\vdots \\
|a_{N+m}| &< r|a_{N+m-1}| < r^m|a_N|.
\end{aligned}$$

Therefore,

$$\sum_{m=N}^{\infty}|a_m| = \sum_{m=0}^{\infty}|a_{N+m}| \leq \sum_{m=0}^{\infty}|a_N|\, r^m = |a_N|\sum_{m=0}^{\infty} r^m.$$

The geometric series on the right-hand side converges because $0 < r < 1$, so the series of absolute values $\sum_{m=N}^{\infty}|a_m|$ converges by the Direct Comparison Test. Because adding or deleting finitely many terms in a series does not affect its convergence or divergence property, the series $\sum_{n=1}^{\infty}|a_n|$ also converges. That is, the series $\sum a_n$ is absolutely convergent.

(b) $\boldsymbol{1 < \rho \leq \infty}$**.** From some index M on,

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \qquad \text{and} \qquad |a_M| < |a_{M+1}| < |a_{M+2}| < \cdots.$$

The terms of the series do not approach zero as n becomes infinite, and the series diverges by the nth-Term Test.

(c) $\boldsymbol{\rho = 1}$**.** The two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

show that some other test for convergence must be used when $\rho = 1$.

$$\text{For } \sum_{n=1}^{\infty} \frac{1}{n}: \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{1/(n+1)}{1/n} = \frac{n}{n+1} \to 1.$$

$$\text{For } \sum_{n=1}^{\infty} \frac{1}{n^2}: \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{1/(n+1)^2}{1/n^2} = \left(\frac{n}{n+1} \right)^2 \to 1^2 = 1.$$

In both cases, $\rho = 1$, yet the first series diverges, whereas the second converges. ■

The Ratio Test is often effective when the terms of a series contain factorials of expressions involving n or expressions raised to a power involving n.

EXAMPLE 2 Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

Solution We apply the Ratio Test to each series.

(a) For the series $\sum_{n=0}^{\infty} (2^n + 5)/3^n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}} \right) \to \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges absolutely (and thus converges) because $\rho = 2/3$ is less than 1. This does *not* mean that $2/3$ is the sum of the series. In fact,

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1 - (2/3)} + \frac{5}{1 - (1/3)} = \frac{21}{2}.$$

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!} \\ &= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \to 4. \end{aligned}$$

The series diverges because $\rho = 4$ is greater than 1.

(c) If $a_n = 4^n n! n!/(2n)!$, then

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \frac{4^{n+1}(n + 1)!(n + 1)!}{(2n + 2)(2n + 1)(2n)!} \cdot \frac{(2n)!}{4^n n! n!} \\ &= \frac{4(n + 1)(n + 1)}{(2n + 2)(2n + 1)} = \frac{2(n + 1)}{2n + 1} \to 1. \end{aligned}$$

Because the limit is $\rho = 1$, we cannot decide from the Ratio Test whether the series converges. However, when we notice that $a_{n+1}/a_n = (2n + 2)/(2n + 1)$, we conclude that a_{n+1} is always greater than a_n because $(2n + 2)/(2n + 1)$ is always greater than 1. Therefore, all terms are greater than or equal to $a_1 = 2$, and the nth term does not approach zero as $n \to \infty$. The series diverges. ■

The Root Test

The convergence tests we have so far for Σa_n work best when the formula for a_n is relatively simple. However, consider the series with the terms

$$a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even.} \end{cases}$$

To investigate convergence we write out several terms of the series:

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2^1} + \frac{1}{2^2} + \frac{3}{2^3} + \frac{1}{2^4} + \frac{5}{2^5} + \frac{1}{2^6} + \frac{7}{2^7} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{3}{8} + \frac{1}{16} + \frac{5}{32} + \frac{1}{64} + \frac{7}{128} + \cdots.$$

Clearly, this is not a geometric series. The nth term approaches zero as $n \to \infty$, so the nth-Term Test does not tell us if the series diverges. The Integral Test does not look promising. The Ratio Test produces

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \dfrac{1}{2n}, & n \text{ odd} \\[2ex] \dfrac{n+1}{2}, & n \text{ even} \end{cases}$$

As $n \to \infty$, the ratio is alternately small and large and therefore has no limit. However, we will see that the following test establishes that the series converges.

> **THEOREM 14—The Root Test**
> Let $\sum a_n$ be any series and suppose that
>
> $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$
>
> Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

Proof

(a) $\boldsymbol{\rho < 1}$**.** Choose an $\varepsilon > 0$ so small that $\rho + \varepsilon < 1$. Since $\sqrt[n]{|a_n|} \to \rho$, the terms $\sqrt[n]{|a_n|}$ eventually get to within ε of ρ. So there exists an index M such that

$$\sqrt[n]{|a_n|} < \rho + \varepsilon \qquad \text{when } n \geq M.$$

Then it is also true that

$$|a_n| < (\rho + \varepsilon)^n \qquad \text{for } n \geq M.$$

Now, $\sum_{n=M}^{\infty} (\rho + \varepsilon)^n$ is a geometric series with ratio $(\rho + \varepsilon) < 1$ and therefore converges. By the Direct Comparison Test, $\sum_{n=M}^{\infty} |a_n|$ converges. Adding finitely many terms to a series does not affect its convergence or divergence, so the series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + \cdots + |a_{M-1}| + \sum_{n=M}^{\infty} |a_n|$$

also converges. Therefore, $\sum a_n$ converges absolutely.

(b) $\boldsymbol{1 < \rho \leq \infty}$**.** For all indices beyond some integer M, we have $\sqrt[n]{|a_n|} > 1$, so that $|a_n| > 1$ for $n > M$. The terms of the series do not converge to zero. The series diverges by the nth-Term Test.

(c) $\boldsymbol{\rho = 1}$**.** The series $\sum_{n=1}^{\infty} (1/n)$ and $\sum_{n=1}^{\infty} (1/n^2)$ show that the test is not conclusive when $\rho = 1$. The first series diverges and the second converges, but in both cases $\sqrt[n]{|a_n|} \to 1$. ■

EXAMPLE 3 Consider again the series with terms $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even}. \end{cases}$ Does $\sum a_n$ converge?

Solution We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even}. \end{cases}$$

Therefore,

$$\frac{1}{2} \le \sqrt[n]{|a_n|} \le \frac{\sqrt[n]{n}}{2}.$$

Since $\sqrt[n]{n} \to 1$ (Section 10.1, Theorem 5), we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges absolutely by the Root Test. ■

EXAMPLE 4 Which of the following series converge, and which diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

Solution We apply the Root Test to each series, noting that each series has positive terms.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\displaystyle\sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \frac{\left(\sqrt[n]{n}\right)^2}{2} \to \frac{1^2}{2} < 1.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges because $\displaystyle\sqrt[n]{\frac{2^n}{n^3}} = \frac{2}{\left(\sqrt[n]{n}\right)^3} \to \frac{2}{1^3} > 1.$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ converges because $\displaystyle\sqrt[n]{\left(\frac{1}{1+n}\right)^n} = \frac{1}{1+n} \to 0 < 1.$ ■

EXERCISES 10.5

Using the Ratio Test

In Exercises 1–8, use the Ratio Test to determine if each series converges absolutely or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$
2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{3^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{(-4)^n}$
6. $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+2}}{\ln n}$
7. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2(n+2)!}{n!\, 3^{2n}}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$

Using the Root Test

In Exercises 9–16, use the Root Test to determine if each series converges absolutely or diverges.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{7}{(2n+5)^n}$
10. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$
11. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$
12. $\displaystyle\sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{-8}{(3+(1/n))^{2n}}$
14. $\displaystyle\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$
15. $\displaystyle\sum_{n=1}^{\infty} (-1)^n\left(1 - \frac{1}{n}\right)^{n^2}$

    (*Hint:* $\lim_{n\to\infty} (1 + x/n)^n = e^x$)
16. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1+n}}$

Determining Convergence or Divergence

In Exercises 17–46, use any method to determine if the series converges or diverges. Give reasons for your answer.

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$
18. $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n}$
19. $\displaystyle\sum_{n=1}^{\infty} n!(-e)^{-n}$
20. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{10^n}$
21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$
22. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$

23. $\sum_{n=1}^{\infty} \frac{2 + (-1)^n}{1.25^n}$

24. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$

25. $\sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{3}{n}\right)^n$

26. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{3n}\right)^n$

27. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

28. $\sum_{n=1}^{\infty} \frac{(-\ln n)^n}{n^n}$

29. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$

30. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$

31. $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$

32. $\sum_{n=1}^{\infty} \frac{n \ln n}{(-2)^n}$

33. $\sum_{n=1}^{\infty} \frac{(n + 1)(n + 2)}{n!}$

34. $\sum_{n=1}^{\infty} e^{-n}(n^3)$

35. $\sum_{n=1}^{\infty} \frac{(n + 3)!}{3!n!3^n}$

36. $\sum_{n=1}^{\infty} \frac{n2^n(n + 1)!}{3^n n!}$

37. $\sum_{n=1}^{\infty} \frac{n!}{(2n + 1)!}$

38. $\sum_{n=1}^{\infty} \frac{n!}{(-n)^n}$

39. $\sum_{n=2}^{\infty} \frac{-n}{(\ln n)^n}$

40. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{(n/2)}}$

41. $\sum_{n=1}^{\infty} \frac{n! \ln n}{n(n + 2)!}$

42. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 2^n}$

43. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

44. $\sum_{n=1}^{\infty} \frac{(2n + 3)(2^n + 3)}{3^n + 2}$

45. $\sum_{n=3}^{\infty} \frac{2^n}{n^2}$

46. $\sum_{n=3}^{\infty} \frac{2^{n^2}}{n^{2^n}}$

Recursively Defined Terms Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 47–56 converge, and which diverge? Give reasons for your answers.

47. $a_1 = 2, \quad a_{n+1} = \frac{1 + \sin n}{n} a_n$

48. $a_1 = 1, \quad a_{n+1} = \frac{1 + \tan^{-1} n}{n} a_n$

49. $a_1 = \frac{1}{3}, \quad a_{n+1} = \frac{3n - 1}{2n + 5} a_n$

50. $a_1 = 3, \quad a_{n+1} = \frac{n}{n + 1} a_n$

51. $a_1 = 2, \quad a_{n+1} = \frac{2}{n} a_n$

52. $a_1 = 5, \quad a_{n+1} = \frac{\sqrt[n]{n}}{2} a_n$

53. $a_1 = 1, \quad a_{n+1} = \frac{1 + \ln n}{n} a_n$

54. $a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{n + \ln n}{n + 10} a_n$

55. $a_1 = \frac{1}{3}, \quad a_{n+1} = \sqrt[n]{a_n}$

56. $a_1 = \frac{1}{2}, \quad a_{n+1} = (a_n)^{n+1}$

Convergence or Divergence

Which of the series in Exercises 57–64 converge, and which diverge? Give reasons for your answers.

57. $\sum_{n=1}^{\infty} \frac{2^n n! n!}{(2n)!}$

58. $\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!}{n!(n + 1)!(n + 2)!}$

59. $\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

60. $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^n}{n^{(n^2)}}$

61. $\sum_{n=1}^{\infty} \frac{n^n}{2^{(n^2)}}$

62. $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

63. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n - 1)}{4^n 2^n n!}$

64. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \cdots \cdot (2n - 1)}{[2 \cdot 4 \cdot \cdots \cdot (2n)](3^n + 1)}$

65. Assume that b_n is a sequence of positive numbers converging to $4/5$. Determine if the following series converge or diverge.

a. $\sum_{n=1}^{\infty} (b_n)^{1/n}$

b. $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n (b_n)$

c. $\sum_{n=1}^{\infty} (b_n)^n$

d. $\sum_{n=1}^{\infty} \frac{1000^n}{n! + b_n}$

66. Assume that b_n is a sequence of positive numbers converging to $1/3$. Determine if the following series converge or diverge.

a. $\sum_{n=1}^{\infty} \frac{b_{n+1} b_n}{n 4^n}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{n!\, b_1^2 b_2^2 \cdots b_n^2}$

Theory and Examples

67. Neither the Ratio Test nor the Root Test helps with p-series. Try them on

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and show that both tests fail to provide information about convergence.

68. Show that neither the Ratio Test nor the Root Test provides information about the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p} \qquad (p \text{ constant}).$$

69. Let $a_n = \begin{cases} n/2^n, & \text{if } n \text{ is a prime number} \\ 1/2^n, & \text{otherwise.} \end{cases}$

Does $\sum a_n$ converge? Give reasons for your answer.

70. Show that $\sum_{n=1}^{\infty} 2^{(n^2)}/n!$ diverges. Recall from the Laws of Exponents that $2^{(n^2)} = (2^n)^n$.

10.6 Alternating Series and Conditional Convergence

A series in which the terms are alternately positive and negative is an **alternating series**. Here are three examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \tag{1}$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \tag{2}$$

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + (-1)^{n+1}n + \cdots \tag{3}$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n+1}u_n \qquad \text{or} \qquad a_n = (-1)^n u_n$$

where $u_n = |a_n|$ is a positive number.

Series (1), called the **alternating harmonic series**, converges, as we will see in a moment. Series (2), a geometric series with ratio $r = -1/2$, converges to $-2/[1 + (1/2)] = -4/3$. Series (3) diverges because the nth term does not approach zero.

We prove the convergence of the alternating harmonic series by applying the Alternating Series Test. This test is for *convergence* of an alternating series and cannot be used to conclude that such a series diverges. If we multiply $(u_1 - u_2 + u_3 - u_4 + \cdots)$ by -1, we see that the test is also valid for the alternating series $-u_1 + u_2 - u_3 + u_4 - \cdots$, as with the one in Series (2) given above.

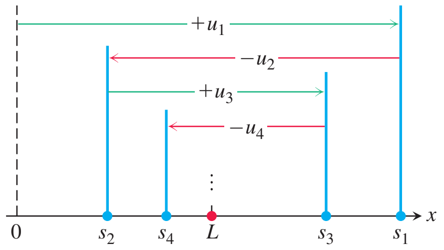

FIGURE 10.15 The partial sums of an alternating series that satisfies the hypotheses of Theorem 15 for $N = 1$ straddle the limit from the beginning.

THEOREM 15—The Alternating Series Test

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if the following conditions are satisfied:

1. The u_n's are all positive.
2. The u_n's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

Proof We look at the case where $u_1, u_2, u_3, u_4, \ldots$ is nonincreasing, so that $N = 1$. If n is an even integer, say $n = 2m$, then the sum of the first n terms is

$$\begin{aligned} s_{2m} &= (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \\ &= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}. \end{aligned}$$

The first equality shows that s_{2m} is the sum of m nonnegative terms, since each term in parentheses is positive or zero. Hence $s_{2m+2} \geq s_{2m}$, and the sequence $\{s_{2m}\}$ is nondecreasing. The second equality shows that $s_{2m} \leq u_1$. Since $\{s_{2m}\}$ is nondecreasing and bounded from above, it has a limit, say

$$\lim_{m\to\infty} s_{2m} = L. \qquad \text{Theorem 6} \tag{4}$$

If n is an odd integer, say $n = 2m + 1$, then the sum of the first n terms is $s_{2m+1} = s_{2m} + u_{2m+1}$. Since $u_n \to 0$,

$$\lim_{m\to\infty} u_{2m+1} = 0$$

and, as $m \to \infty$,

$$s_{2m+1} = s_{2m} + u_{2m+1} \to L + 0 = L. \tag{5}$$

Combining the results of Equations (4) and (5) gives $\lim_{n\to\infty} s_n = L$ (Section 10.1, Exercise 143). ■

EXAMPLE 1 The alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

clearly satisfies the three requirements of Theorem 15 with $N = 1$; it therefore converges by the Alternating Series Test. Notice that the test gives no information about what the sum of the series might be. Figure 10.16 shows histograms of the partial sums of the divergent harmonic series and those of the convergent alternating harmonic series. It turns out that the alternating harmonic series converges to ln 2. ■

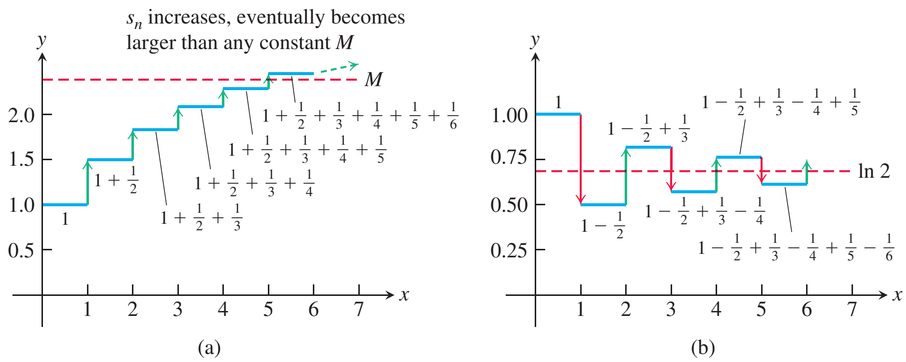

FIGURE 10.16 (a) The harmonic series diverges, with partial sums that eventually exceed any constant. (b) The alternating harmonic series converges to ln 2 ≈ .693.

Rather than directly verifying the definition $u_n \geq u_{n+1}$, a second way to show that the sequence $\{u_n\}$ is nonincreasing is to define a differentiable function $f(x)$ satisfying $f(n) = u_n$. That is, the values of f match the values of the sequence at every positive integer n. If $f'(x) \leq 0$ for all x greater than or equal to some positive integer N, then $f(x)$ is nonincreasing for $x \geq N$. It follows that $f(n) \geq f(n+1)$, or $u_n \geq u_{n+1}$, for $n \geq N$.

EXAMPLE 2 We show that the sequence $u_n = 10n/(n^2 + 16)$ is eventually nonincreasing. Define $f(x) = 10x/(x^2 + 16)$. Then from the Derivative Quotient Rule,

$$f'(x) = \frac{10(16 - x^2)}{(x^2 + 16)^2} \leq 0 \qquad \text{whenever } x \geq 4.$$

It follows that $u_n \geq u_{n+1}$ for $n \geq 4$. That is, the sequence $\{u_n\}$ is nonincreasing for $n \geq 4$. ■

A graphical interpretation of the partial sums (Figure 10.15) shows how an alternating series converges to its limit L when the three conditions of Theorem 15 are satisfied with $N = 1$. Starting from the origin of the x-axis, we lay off the positive distance $s_1 = u_1$. To find the point corresponding to $s_2 = u_1 - u_2$, we back up a distance equal to u_2. Since $u_2 \leq u_1$, we do not back up any farther than the origin. We continue in this seesaw fashion, backing up or going forward as the signs in the series demand. But for $n \geq N$, each forward or backward step is shorter than (or at most the same size as) the preceding step because $u_{n+1} \leq u_n$. And since the nth term approaches zero as n increases, the size of step we take forward or backward gets smaller and smaller. We oscillate back and forth across the limit L, and the amplitude of oscillation approaches zero. The limit L lies between any two successive sums s_n and s_{n+1} and hence differs from s_n by an amount less than u_{n+1}.

Because

$$|L - s_n| < u_{n+1} \qquad \text{for } n \geq N,$$

we can make useful estimates of the sums of convergent alternating series.

THEOREM 16—The Alternating Series Estimation Theorem
If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

We leave the verification of the sign of the remainder for Exercise 87.

EXAMPLE 3 We try Theorem 16 on a series whose sum we know:

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \frac{1}{64} - \frac{1}{128} + \frac{1}{256} - \cdots.$$

The theorem says that if we truncate the series after the eighth term, we throw away a total that is positive and less than $1/256$. The sum of the first eight terms is $s_8 = 0.6640625$ and the sum of the first nine terms is $s_9 = 0.66796875$. The sum of the geometric series is

$$\frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{2}{3},$$

and we note that $0.6640625 < (2/3) < 0.66796875$. The difference, $(2/3) - 0.6640625 = 0.0026041666\ldots$, is positive and is less than $(1/256) = 0.00390625$. ■

Conditional Convergence

If we replace all the negative terms in the alternating series in Example 3, changing them to positive terms instead, we obtain the geometric series $\sum 1/2^n$. The original series and the new series of absolute values both converge (although to different sums). For an absolutely convergent series, changing infinitely many of the negative terms in the series to positive values does not change its property of still being a convergent series. Other convergent series may behave differently. The convergent alternating harmonic series has infinitely many negative terms, but if we change its negative terms to positive values, the resulting series is the divergent harmonic series. So the presence of infinitely many negative terms is essential to the convergence of the alternating harmonic series. The following terminology distinguishes these two types of convergent series.

DEFINITION A series that is convergent but not absolutely convergent is called **conditionally convergent**.

The alternating harmonic series is conditionally convergent, or **converges conditionally**. The next example extends that result to the alternating p-series.

EXAMPLE 4 If p is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore, the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

converges.

If $p > 1$, the series converges absolutely as an ordinary p-series. If $0 < p \leq 1$, the series converges conditionally by the alternating series test. For instance,

$$\text{Absolute convergence } \left(p = 3/2\right): \quad 1 - \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} - \frac{1}{4^{3/2}} + \cdots$$

$$\text{Conditional convergence } \left(p = 1/2\right): \quad 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

We need to be careful when using a conditionally convergent series. We have seen with the alternating harmonic series that altering the signs of infinitely many terms of a conditionally convergent series can change its convergence status. Even more, simply changing the order of occurrence of infinitely many of its terms can also have a significant effect, as we now discuss.

Rearranging Series

We can always rearrange the terms of a *finite* collection of numbers without changing their sum. The same result is true for an infinite series that is absolutely convergent (see Exercise 96 for an outline of the proof).

THEOREM 17—The Rearrangement Theorem for Absolutely Convergent Series
If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

On the other hand, if we rearrange the terms of a conditionally convergent series, we can get different results. In fact, for any real number r, a given conditionally convergent series can be rearranged so its sum is equal to r. (We omit the proof of this fact.) Here's an example of summing the terms of a conditionally convergent series with different orderings, with each ordering giving a different value for the sum.

EXAMPLE 5 We know that the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ converges to some number L. Moreover, by Theorem 16, L lies between the successive partial sums $s_2 = 1/2$ and $s_3 = 5/6$, so $L \neq 0$. If we multiply the series by 2 we obtain

$$\begin{aligned} 2L = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right) \\ &= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \cdots. \end{aligned}$$

Now we change the order of this last sum by grouping each pair of terms with the same odd denominator, but leaving the negative terms with the even denominators as they are placed (so the denominators are the positive integers in their natural order). This rearrangement gives

$$\begin{aligned}(2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \left(\frac{2}{7} - \frac{1}{7}\right) - \frac{1}{8} + \cdots \\ = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \cdots\right) \\ = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = L.\end{aligned}$$

So by rearranging the terms of the conditionally convergent series $\sum_{n=1}^{\infty} 2(-1)^{n+1}/n$, the series becomes $\sum_{n=1}^{\infty} (-1)^{n+1}/n$, which is the alternating harmonic series itself. If the two series are the same, it would imply that $2L = L$, which is clearly false since $L \neq 0$. ■

Example 5 shows that we cannot rearrange the terms of a conditionally convergent series and expect the new series to be the same as the original one. When we use a conditionally convergent series, the terms must be added together in the order they are given to obtain a correct result. In contrast, Theorem 17 guarantees that the terms of an absolutely convergent series can be summed in any order without affecting the result.

Summary of Tests to Determine Convergence or Divergence

We have developed a variety of tests to determine convergence or divergence for an infinite series of constants. There are other tests we have not presented which are sometimes given in more advanced courses. Here is a summary of the tests we have considered.

1. **The *n*th-Term Test for Divergence:** Unless $a_n \to 0$, the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Direct Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

EXERCISES 10.6

Determining Convergence or Divergence

In Exercises 1–14, determine if the alternating series converges or diverges. Some of the series do not satisfy the conditions of the Alternating Series Test.

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{3/2}}$

3. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n3^n}$

4. $\sum_{n=2}^{\infty} (-1)^{n} \frac{4}{(\ln n)^2}$

5. $\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n^2+1}$

6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+5}{n^2+4}$

7. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n^2}$

8. $\sum_{n=1}^{\infty} (-1)^{n} \frac{10^n}{(n+1)!}$

9. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^n$

10. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$

11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$

12. $\sum_{n=1}^{\infty}(-1)^{n}\ln\left(1+\frac{1}{n}\right)$

13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n+1}$

14. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$

Absolute and Conditional Convergence

Which of the series in Exercises 15–48 converge absolutely, which converge, and which diverge? Give reasons for your answers.

15. $\sum_{n=1}^{\infty}(-1)^{n+1}(0.1)^n$

16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.1)^n}{n}$

17. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\sqrt{n}}$

18. $\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$

19. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^3+1}$

20. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n!}{2^n}$

21. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n+3}$

22. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\sin n}{n^2}$

23. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3+n}{5+n}$

24. $\sum_{n=1}^{\infty}\frac{(-2)^{n+1}}{n+5^n}$

25. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1+n}{n^2}$

26. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt[n]{10}\right)$

27. $\sum_{n=1}^{\infty}(-1)^{n}n^2(2/3)^n$

28. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$

29. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\tan^{-1}n}{n^2+1}$

30. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\ln n}{n-\ln n}$

31. $\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1}$

32. $\sum_{n=1}^{\infty}(-5)^{-n}$

33. $\sum_{n=1}^{\infty}\frac{(-100)^n}{n!}$

34. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2+2n+1}$

35. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n\sqrt{n}}$

36. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}$

37. $\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)^n}{(2n)^n}$

38. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n!)^2}{(2n)!}$

39. $\sum_{n=1}^{\infty}(-1)^{n}\frac{(2n)!}{2^n n! n}$

40. $\sum_{n=1}^{\infty}(-1)^{n}\frac{(n!)^2 3^n}{(2n+1)!}$

41. $\sum_{n=1}^{\infty}(-1)^{n}\left(\sqrt{n+1}-\sqrt{n}\right)$

42. $\sum_{n=1}^{\infty}(-1)^{n}\left(\sqrt{n^2+n}-n\right)$

43. $\sum_{n=1}^{\infty}(-1)^{n}\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$

44. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+\sqrt{n+1}}$

45. $\sum_{n=1}^{\infty}(-1)^{n}\operatorname{sech} n$

46. $\sum_{n=1}^{\infty}(-1)^{n}\operatorname{csch} n$

47. $\frac{1}{4}-\frac{1}{6}+\frac{1}{8}-\frac{1}{10}+\frac{1}{12}-\frac{1}{14}+\cdots$

48. $1+\frac{1}{4}-\frac{1}{9}-\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}+\cdots$

Error Estimation

In Exercises 49–52, estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

49. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

50. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{10^n}$

51. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.01)^n}{n}$ As you will see in Section 10.7, the sum is ln (1.01).

52. $\frac{1}{1+t}=\sum_{n=0}^{\infty}(-1)^n t^n,\quad 0<t<1$

In Exercises 53–56, determine how many terms should be used to estimate the sum of the entire series with an error of less than 0.001.

53. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n^2+3}$

54. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$

55. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\left(n+3\sqrt{n}\right)^3}$

56. $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{\ln(\ln(n+2))}$

In Exercises 57–82, use any method to determine whether the series converges or diverges. Give reasons for your answer.

57. $\sum_{n=1}^{\infty}\frac{3^n}{n^n}$

58. $\sum_{n=1}^{\infty}\frac{3^n}{n^3}$

59. $\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$

60. $\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+2}\right)$

61. $\sum_{n=0}^{\infty}(-1)^{n}\frac{(n+2)!}{(2n)!}$

62. $\sum_{n=2}^{\infty}\frac{(3n)!}{(n!)^3}$

63. $\sum_{n=1}^{\infty}n^{-2/\sqrt{5}}$

64. $\sum_{n=2}^{\infty}\frac{3}{10+n^{4/3}}$

65. $\sum_{n=1}^{\infty}\left(1-\frac{2}{n}\right)^{n^2}$

66. $\sum_{n=0}^{\infty}\left(\frac{n+1}{n+2}\right)^{n}$

67. $\sum_{n=1}^{\infty}\frac{n-2}{n^2+3n}\left(-\frac{2}{3}\right)^{n}$

68. $\sum_{n=0}^{\infty}\frac{n+1}{(n+2)!}\left(\frac{3}{2}\right)^{n}$

69. $\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\cdots$

70. $1-\frac{1}{8}+\frac{1}{64}-\frac{1}{512}+\frac{1}{4096}-\cdots$

71. $\sum_{n=3}^{\infty}\sin\left(\frac{1}{\sqrt{n}}\right)$

72. $\sum_{n=1}^{\infty}\tan\left(n^{1/n}\right)$

73. $\sum_{n=2}^{\infty}\frac{n}{\ln n}$

74. $\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$

75. $\sum_{n=2}^{\infty}\ln\left(\frac{n+2}{n+1}\right)$

76. $\sum_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^{3}$

77. $\sum_{n=2}^{\infty}\frac{1}{1+2+2^2+\cdots+2^n}$

78. $\sum_{n=2}^{\infty}\frac{1+3+3^2+\cdots+3^{n-1}}{1+2+3+\cdots+n}$

79. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{e^n}{e^n+e^{n^2}}$

80. $\displaystyle\sum_{n=0}^{\infty}\frac{(2n+3)(2^n+3)}{3^n+2}$

81. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2 3^n}{3\cdot5\cdot7\cdots(2n+1)}$

82. $\displaystyle\sum_{n=1}^{\infty}\frac{4\cdot6\cdot8\cdots(2n)}{5^{n+1}(n+2)!}$

T Approximate the sums in Exercises 83 and 84 with an error of magnitude less than 5×10^{-6}.

83. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}$ As you will see in Section 10.9, the sum is cos 1, the cosine of 1 radian.

84. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}$ As you will see in Section 10.9 the sum is e^{-1}.

Theory and Examples

85. a. The series

$$\frac{1}{3}-\frac{1}{2}+\frac{1}{9}-\frac{1}{4}+\frac{1}{27}-\frac{1}{8}+\cdots+\frac{1}{3^n}-\frac{1}{2^n}+\cdots$$

does not meet one of the conditions of Theorem 14. Which one?

b. Use Theorem 17 to find the sum of the series in part (a).

T **86.** The limit L of an alternating series that satisfies the conditions of Theorem 15 lies between the values of any two consecutive partial sums. This suggests using the average

$$\frac{s_n+s_{n+1}}{2}=s_n+\frac{1}{2}(-1)^{n+2}a_{n+1}$$

to estimate L. Compute

$$s_{20}+\frac{1}{2}\cdot\frac{1}{21}$$

as an approximation to the sum of the alternating harmonic series. The exact sum is $\ln 2 = 0.69314718\ldots$.

87. The sign of the remainder of an alternating series that satisfies the conditions of Theorem 15 Prove the assertion in Theorem 16 that whenever an alternating series satisfying the conditions of Theorem 15 is approximated with one of its partial sums, then the remainder (sum of the unused terms) has the same sign as the first unused term. (*Hint:* Group the remainder's terms in consecutive pairs.)

88. Show that the sum of the first $2n$ terms of the series

$$1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\cdots$$

is the same as the sum of the first n terms of the series

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+\frac{1}{5\cdot6}+\cdots.$$

Do these series converge? What is the sum of the first $2n+1$ terms of the first series? If the series converge, what is their sum?

89. Show that if $\sum_{n=1}^{\infty}a_n$ diverges, then $\sum_{n=1}^{\infty}|a_n|$ diverges.

90. Show that if $\sum_{n=1}^{\infty}a_n$ converges absolutely, then

$$\left|\sum_{n=1}^{\infty}a_n\right|\le\sum_{n=1}^{\infty}|a_n|.$$

91. Show that if $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ both converge absolutely, then so do the following.

a. $\displaystyle\sum_{n=1}^{\infty}(a_n+b_n)$ **b.** $\displaystyle\sum_{n=1}^{\infty}(a_n-b_n)$

c. $\displaystyle\sum_{n=1}^{\infty}ka_n$ (k any number)

92. Show by example that $\sum_{n=1}^{\infty}a_nb_n$ may diverge even if $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ both converge.

93. If $\sum a_n$ converges absolutely, prove that $\sum a_n{}^2$ converges.

94. Does the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)$$

converge or diverge? Justify your answer.

T **95.** In the alternating harmonic series, suppose the goal is to arrange the terms to get a new series that converges to $-1/2$. Start the new arrangement with the first negative term, which is $-1/2$. Whenever you have a sum that is less than or equal to $-1/2$, start introducing positive terms, taken in order, until the new total is greater than $-1/2$. Then add negative terms until the total is less than or equal to $-1/2$ again. Continue this process until your partial sums have been above the target at least three times and finish at or below it. If s_n is the sum of the first n terms of your new series, plot the points (n, s_n) to illustrate how the sums are behaving.

96. Outline of the proof of the Rearrangement Theorem (Theorem 17)

a. Let ε be a positive real number, let $L=\sum_{n=1}^{\infty}a_n$, and let $s_k=\sum_{n=1}^{k}a_n$. Show that for some index N_1 and for some index $N_2\ge N_1$,

$$\sum_{n=N_1}^{\infty}|a_n|<\frac{\varepsilon}{2}\quad\text{and}\quad|s_{N_2}-L|<\frac{\varepsilon}{2}.$$

Since all the terms $a_1, a_2, \ldots, a_{N_2}$ appear somewhere in the sequence $\{b_n\}$, there is an index $N_3\ge N_2$ such that if $n\ge N_3$, then $\left(\sum_{k=1}^{n}b_k\right)-s_{N_2}$ is at most a sum of terms a_m with $m\ge N_1$. Therefore, if $n\ge N_3$,

$$\left|\sum_{k=1}^{n}b_k-L\right|\le\left|\sum_{k=1}^{n}b_k-s_{N_2}\right|+|s_{N_2}-L|$$

$$\le\sum_{k=N_1}^{\infty}|a_k|+|s_{N_2}-L|<\varepsilon.$$

b. The argument in part (a) shows that if $\sum_{n=1}^{\infty}a_n$ converges absolutely then $\sum_{n=1}^{\infty}b_n$ converges and $\sum_{n=1}^{\infty}b_n=\sum_{n=1}^{\infty}a_n$. Now show that because $\sum_{n=1}^{\infty}a_n$ converges, $\sum_{n=1}^{\infty}b_n$ converges to $\sum_{n=1}^{\infty}a_n$.