# 10

# Infinite Sequences and Series

OVERVIEW In this chapter we introduce the topic of *infinite series*. Such series give us precise ways to express many numbers and functions, both familiar and new, as arithmetic sums with infinitely many terms. For example, we will learn that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

and

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^8}{40{,}320} - \cdots.$$

We need to develop a method to make sense of such expressions. Everyone knows how to add two numbers together, or even several. But how do you add together infinitely many numbers? Or, when adding together functions, how do you add infinitely many powers of $x$? In this chapter we answer these questions, which are part of the theory of infinite sequences and series. As with the differential and integral calculus, limits play a major role in the development of infinite series.

One common and important application of series occurs when making computations with complicated functions. A hard-to-compute function is replaced by an expression that looks like an "infinite degree polynomial," an infinite series in powers of $x$, as we see with the cosine function given above. Using the first few terms of this infinite series can allow for highly accurate approximations of functions by polynomials, enabling us to work with more general functions than those we encountered before. These new functions are commonly obtained as solutions to differential equations arising in important applications of mathematics to science and engineering.

The terms "sequence" and "series" are sometimes used interchangeably in spoken language. In mathematics, however, each has a distinct meaning. A sequence is a type of infinite list, whereas a series is an infinite sum. To understand the infinite sums described by series, we are led to first study infinite sequences.

## 10.1 Sequences

HISTORICAL ESSAY
**Sequences and Series**
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Sequences are fundamental to the study of infinite series and to many aspects of mathematics. We saw one example of a sequence when we studied Newton's Method in Section 4.6. Newton's Method produces a sequence of approximations $x_n$ that become closer and closer to the root of a differentiable function. Now we will explore general sequences of numbers and the conditions under which they converge to a finite number.

## Representing Sequences

A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order. Each of $a_1$, $a_2$, $a_3$ and so on represents a number. These are the **terms** of the sequence. For example, the sequence

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$$

has first term $a_1 = 2$, second term $a_2 = 4$, and $n$th term $a_n = 2n$. The integer $n$ is called the **index** of $a_n$, and indicates where $a_n$ occurs in the list. Order is important. The sequence 2, 4, 6, 8 . . . is not the same as the sequence 4, 2, 6, 8 . . . .

We can think of the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

as a function that sends 1 to $a_1$, 2 to $a_2$, 3 to $a_3$, and in general sends the positive integer $n$ to the $n$th term $a_n$. More precisely, an **infinite sequence** of numbers is a function whose domain is the set of positive integers. For example, the function associated with the sequence

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots$$

sends 1 to $a_1 = 2$, 2 to $a_2 = 4$, and so on. The general behavior of this sequence is described by the formula $a_n = 2n$.

We can change the index to start at any given number $n$. For example, the sequence

$$12, 14, 16, 18, 20, 22 \ldots$$

is described by the formula $a_n = 10 + 2n$, if we start with $n = 1$. It can also be described by the simpler formula $b_n = 2n$, where the index $n$ starts at 6 and increases. To allow such simpler formulas, we let the first index of the sequence be any appropriate integer. In the sequence above, $\{a_n\}$ starts with $a_1$ while $\{b_n\}$ starts with $b_6$.

Sequences can be described by writing rules that specify their terms, such as

$$a_n = \sqrt{n}, \qquad b_n = (-1)^{n+1}\frac{1}{n}, \qquad c_n = \frac{n-1}{n}, \qquad d_n = (-1)^{n+1},$$

or by listing terms:

$$\begin{aligned}
\{a_n\} &= \left\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\right\}\\
\{b_n\} &= \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots, (-1)^{n+1}\frac{1}{n}, \ldots\right\}\\
\{c_n\} &= \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\right\}\\
\{d_n\} &= \{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}.
\end{aligned}$$

We also sometimes write a sequence using its rule, as with

$$\{a_n\} = \left\{\sqrt{n}\right\}_{n=1}^{\infty}$$

and

$$\{b_n\} = \left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{\infty}.$$

Figure 10.1 shows two ways to represent sequences graphically. The first marks the first few points from $a_1, a_2, a_3, \ldots, a_n, \ldots$ on the real axis. The second method shows the graph of the function defining the sequence. The function is defined only on integer inputs, and the graph consists of some points in the $xy$-plane located at $(1, a_1)$, $(2, a_2), \ldots, (n, a_n), \ldots$ .

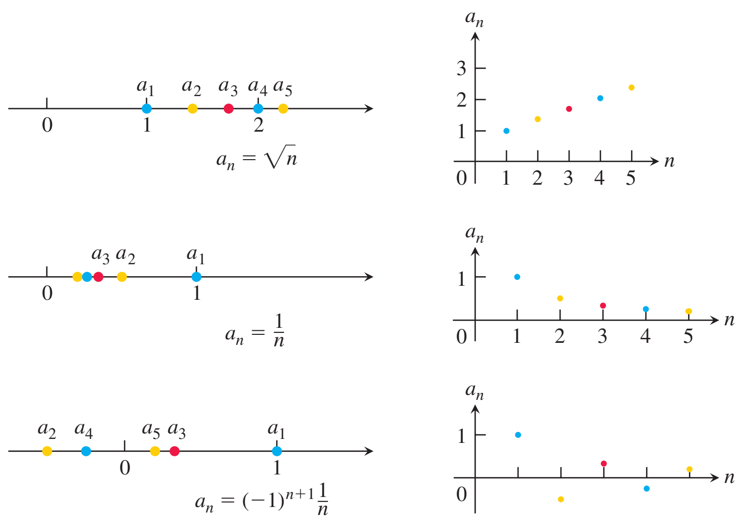



**FIGURE 10.1** Sequences can be represented as points on the real line or as points in the plane where the horizontal axis $n$ is the index number of the term and the vertical axis $a_n$ is its value.

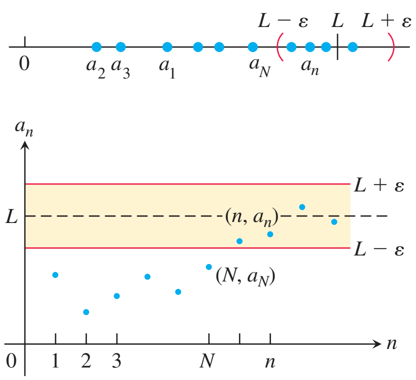



**FIGURE 10.2** In the representation of a sequence as points in the plane, $a_n \to L$ if $y = L$ is a horizontal asymptote of the sequence of points $\{(n, a_n)\}$. In this figure, all the $a_n$'s after $a_N$ lie within $\varepsilon$ of $L$.

## Convergence and Divergence

Sometimes the numbers in a sequence approach a single value as the index $n$ increases. This happens in the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$$

whose terms approach 0 as $n$ gets large, and in the sequence

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, 1 - \frac{1}{n}, \ldots\right\}$$

whose terms approach 1. On the other hand, sequences like

$$\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$$

have terms that get larger than any number as $n$ increases, and sequences like

$$\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$$

bounce back and forth between 1 and $-1$, never converging to a single value. The following definition captures the meaning of having a sequence converge to a limiting value. It says that if we go far enough out in the sequence, by taking the index $n$ to be larger than some value $N$, the difference between $a_n$ and the limit of the sequence becomes less than any preselected number $\varepsilon > 0$.

**DEFINITIONS** The sequence $\{a_n\}$ **converges** to the number $L$ if for every positive number $\varepsilon$ there corresponds an integer $N$ such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N.$$

If no such number $L$ exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to $L$, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call $L$ the **limit** of the sequence (Figure 10.2).

The definition is very similar to the definition of the limit of a function $f(x)$ as $x$ tends to $\infty$ ($\lim_{x\to\infty} f(x)$ in Section 2.6). We will exploit this connection to calculate limits of sequences.

**HISTORICAL BIOGRAPHY**
**Nicole Oresme**
**(ca. 1320–1382)**
www.goo.gl/r71S4z

**EXAMPLE 1** Show that

**(a)** $\lim_{n\to\infty} \frac{1}{n} = 0$ **(b)** $\lim_{n\to\infty} k = k$ (any constant $k$)

**Solution**

**(a)** Let $\varepsilon > 0$ be given. We must show that there exists an integer $N$ such that

$$\left|\frac{1}{n} - 0\right| < \varepsilon \quad \text{whenever} \quad n > N.$$

The inequality $|1/n - 0| < \varepsilon$ will hold if $1/n < \varepsilon$ or $n > 1/\varepsilon$. If $N$ is any integer greater than $1/\varepsilon$, the inequality will hold for all $n > N$. This proves that $\lim_{n\to\infty} 1/n = 0$.

**(b)** Let $\varepsilon > 0$ be given. We must show that there exists an integer $N$ such that

$$|k - k| < \varepsilon \quad \text{whenever} \quad n > N.$$

Since $k - k = 0$, we can use any positive integer for $N$ and the inequality $|k - k| < \varepsilon$ will hold. This proves that $\lim_{n\to\infty} k = k$ for any constant $k$. ■

**EXAMPLE 2** Show that the sequence $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ diverges.

**Solution** Suppose the sequence converges to some number $L$. Then the numbers in the sequence eventually get arbitrarily close to the limit $L$. This can't happen if they keep oscillating between 1 and $-1$. We can see this by choosing $\varepsilon = 1/2$ in the definition of the limit. Then all terms $a_n$ of the sequence with index $n$ larger than some $N$ must lie within $\varepsilon = 1/2$ of $L$. Since the number 1 appears repeatedly as every other term of the sequence, we must have that the number 1 lies within the distance $\varepsilon = 1/2$ of $L$. It follows that $|L - 1| < 1/2$, or equivalently, $1/2 < L < 3/2$. Likewise, the number $-1$ appears repeatedly in the sequence with arbitrarily high index. So we must also have that $|L - (-1)| < 1/2$, or equivalently, $-3/2 < L < -1/2$. But the number $L$ cannot lie in both of the intervals $(1/2, 3/2)$ and $(-3/2, -1/2)$ because they have no overlap. Therefore, no such limit $L$ exists and so the sequence diverges.

Note that the same argument works for any positive number $\varepsilon$ smaller than 1, not just $1/2$. ■

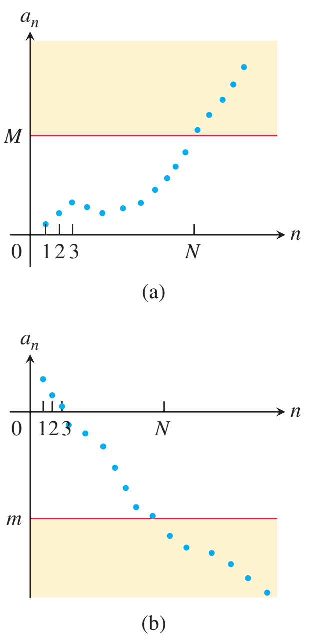



**FIGURE 10.3** (a) The sequence diverges to $\infty$ because no matter what number $M$ is chosen, the terms of the sequence after some index $N$ all lie in the yellow band above $M$. (b) The sequence diverges to $-\infty$ because all terms after some index $N$ lie below any chosen number $m$.

The sequence $\{\sqrt{n}\}$ also diverges, but for a different reason. As $n$ increases, its terms become larger than any fixed number. We describe the behavior of this sequence by writing

$$\lim_{n\to\infty} \sqrt{n} = \infty.$$

In writing infinity as the limit of a sequence, we are not saying that the differences between the terms $a_n$ and $\infty$ become small as $n$ increases. Nor are we asserting that there is some number infinity that the sequence approaches. We are merely using a notation that captures the idea that $a_n$ eventually gets and stays larger than any fixed number as $n$ gets large (see Figure 10.3a). The terms of a sequence might also decrease to negative infinity, as in Figure 10.3b.

**DEFINITION** The sequence $\{a_n\}$ **diverges to infinity** if for every number $M$ there is an integer $N$ such that for all $n$ larger than $N$, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty.$$

Similarly, if for every number $m$ there is an integer $N$ such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty.$$

A sequence may diverge without diverging to infinity or negative infinity, as we saw in Example 2. The sequences $\{1, -2, 3, -4, 5, -6, 7, -8, \ldots\}$ and $\{1, 0, 2, 0, 3, 0, \ldots\}$ are also examples of such divergence.

The convergence or divergence of a sequence is not affected by the values of any number of its initial terms (whether we omit or change the first 10, 1000, or even the first million terms does not matter). From Figure 10.2, we can see that only the part of the sequence that remains after discarding some initial number of terms determines whether the sequence has a limit and the value of that limit when it does exist.

## Calculating Limits of Sequences

Since sequences are functions with domain restricted to the positive integers, it is not surprising that the theorems on limits of functions given in Chapter 2 have versions for sequences.

**THEOREM 1** Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let $A$ and $B$ be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number $k$)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

The proof is similar to that of Theorem 1 of Section 2.2 and is omitted.

**EXAMPLE 3** By combining Theorem 1 with the limits of Example 1, we have:

**(a)** $\lim_{n\to\infty}\left(-\frac{1}{n}\right) = -1 \cdot \lim_{n\to\infty}\frac{1}{n} = -1 \cdot 0 = 0$ — Constant Multiple Rule and Example 1a

**(b)** $\lim_{n\to\infty}\left(\frac{n-1}{n}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n} = 1 - 0 = 1$ — Difference Rule and Example 1a

**(c)** $\lim_{n\to\infty}\frac{5}{n^2} = 5 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 5 \cdot 0 \cdot 0 = 0$ — Product Rule

**(d)** $\lim_{n\to\infty}\frac{4 - 7n^6}{n^6 + 3} = \lim_{n\to\infty}\frac{(4/n^6) - 7}{1 + (3/n^6)} = \frac{0 - 7}{1 + 0} = -7.$ — Divide numerator and denominator by $n^6$ and use the Sum and Quotient Rules.

Be cautious in applying Theorem 1. It does not say, for example, that each of the sequences $\{a_n\}$ and $\{b_n\}$ have limits if their sum $\{a_n + b_n\}$ has a limit. For instance, $\{a_n\} = \{1, 2, 3, \ldots\}$ and $\{b_n\} = \{-1, -2, -3, \ldots\}$ both diverge, but their sum $\{a_n + b_n\} = \{0, 0, 0, \ldots\}$ clearly converges to 0.

One consequence of Theorem 1 is that every nonzero multiple of a divergent sequence $\{a_n\}$ diverges. Suppose, to the contrary, that $\{ca_n\}$ converges for some number $c \neq 0$. Then, by taking $k = 1/c$ in the Constant Multiple Rule in Theorem 1, we see that the sequence

$$\left\{\frac{1}{c} \cdot ca_n\right\} = \{a_n\}$$

converges. Thus, $\{ca_n\}$ cannot converge unless $\{a_n\}$ also converges. If $\{a_n\}$ does not converge, then $\{ca_n\}$ does not converge.

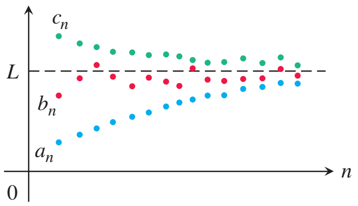



**FIGURE 10.4** The terms of sequence $\{b_n\}$ are sandwiched between those of $\{a_n\}$ and $\{c_n\}$, forcing them to the same common limit $L$.

The next theorem is the sequence version of the Sandwich Theorem in Section 2.2. You are asked to prove the theorem in Exercise 119. (See Figure 10.4.)

> **THEOREM 2—The Sandwich Theorem for Sequences**
> Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all $n$ beyond some index $N$, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

An immediate consequence of Theorem 2 is that, if $|b_n| \le c_n$ and $c_n \to 0$, then $b_n \to 0$ because $-c_n \le b_n \le c_n$. We use this fact in the next example.

**EXAMPLE 4** Since $1/n \to 0$, we know that

**(a)** $\dfrac{\cos n}{n} \to 0$ because $-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$;

**(b)** $\dfrac{1}{2^n} \to 0$ because $0 \le \dfrac{1}{2^n} \le \dfrac{1}{n}$;

**(c)** $(-1)^n \dfrac{1}{n} \to 0$ because $-\dfrac{1}{n} \le (-1)^n \dfrac{1}{n} \le \dfrac{1}{n}$.

**(d)** If $|a_n| \to 0$, then $a_n \to 0$ because $-|a_n| \le a_n \le |a_n|$. ■

The application of Theorems 1 and 2 is broadened by a theorem stating that applying a continuous function to a convergent sequence produces a convergent sequence. We state the theorem, leaving the proof as an exercise (Exercise 120).

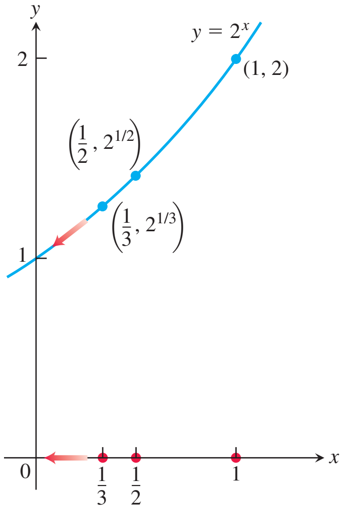



**FIGURE 10.5** As $n \to \infty$, $1/n \to 0$ and $2^{1/n} \to 2^0$ (Example 6). The terms of $\{1/n\}$ are shown on the $x$-axis; the terms of $\{2^{1/n}\}$ are shown as the $y$-values on the graph of $f(x) = 2^x$.

> **THEOREM 3—The Continuous Function Theorem for Sequences**
> Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if $f$ is a function that is continuous at $L$ and defined at all $a_n$, then $f(a_n) \to f(L)$.

**EXAMPLE 5** Show that $\sqrt{(n+1)/n} \to 1$.

**Solution** We know that $(n+1)/n \to 1$. Taking $f(x) = \sqrt{x}$ and $L = 1$ in Theorem 3 gives $\sqrt{(n+1)/n} \to \sqrt{1} = 1$. ■

**EXAMPLE 6** The sequence $\{1/n\}$ converges to 0. By taking $a_n = 1/n$, $f(x) = 2^x$, and $L = 0$ in Theorem 3, we see that $2^{1/n} = f(1/n) \to f(L) = 2^0 = 1$. The sequence $\{2^{1/n}\}$ converges to 1 (Figure 10.5). ■

## Using L'Hôpital's Rule

The next theorem formalizes the connection between $\lim_{n\to\infty} a_n$ and $\lim_{x\to\infty} f(x)$. It enables us to use l'Hôpital's Rule to find the limits of some sequences.

> **THEOREM 4** Suppose that $f(x)$ is a function defined for all $x \ge n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \ge n_0$. Then
> $$\lim_{n\to\infty} a_n = L \qquad \text{whenever} \qquad \lim_{x\to\infty} f(x) = L.$$

**Proof** Suppose that $\lim_{x\to\infty} f(x) = L$. Then for each positive number $\varepsilon$ there is a number $M$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad x > M.$$

Let $N$ be an integer greater than $M$ and greater than or equal to $n_0$. Since $a_n = f(n)$, it follows that for all $n > N$ we have

$$|a_n - L| = |f(n) - L| < \varepsilon.$$ ■

**EXAMPLE 7** Show that

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

**Solution** The function $(\ln x)/x$ is defined for all $x \geq 1$ and agrees with the given sequence at positive integers. Therefore, by Theorem 4, $\lim_{n\to\infty}(\ln n)/n$ will equal $\lim_{x\to\infty}(\ln x)/x$ if the latter exists. A single application of l'Hôpital's Rule shows that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \frac{0}{1} = 0.$$

We conclude that $\lim_{n\to\infty}(\ln n)/n = 0$. ■

When we use l'Hôpital's Rule to find the limit of a sequence, we often treat $n$ as a continuous real variable and differentiate directly with respect to $n$. This saves us from having to rewrite the formula for $a_n$ as we did in Example 7.

**EXAMPLE 8** Does the sequence whose $n$th term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

**Solution** The limit leads to the indeterminate form $1^\infty$. We can apply l'Hôpital's Rule if we first change the form to $\infty \cdot 0$ by taking the natural logarithm of $a_n$:

$$\ln a_n = \ln\left(\frac{n+1}{n-1}\right)^n = n \ln\left(\frac{n+1}{n-1}\right).$$

Then,

$$\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) && \infty \cdot 0 \text{ form} \\
&= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} && \frac{0}{0} \text{ form} \\
&= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} && \text{L'Hôpital's Rule: differentiate numerator and denominator.} \\
&= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2. && \text{Simplify and evaluate.}
\end{aligned}$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to $e^2$. ■

## Commonly Occurring Limits

The next theorem gives some limits that arise frequently.

**Factorial Notation**
The notation $n!$ ("$n$ factorial") means the product $1 \cdot 2 \cdot 3 \cdots n$ of the integers from 1 to $n$. Notice that $(n+1)! = (n+1) \cdot n!$. Thus, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ and $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5 \cdot 4! = 120$. We define $0!$ to be 1. Factorials grow even faster than exponentials, as the table suggests. The values in the table are rounded.

| $n$ | $e^n$ | $n!$ |
|---|---|---|
| 1 | 3 | 1 |
| 5 | 148 | 120 |
| 10 | 22,026 | 3,628,800 |
| 20 | $4.9 \times 10^8$ | $2.4 \times 10^{18}$ |

**THEOREM 5** The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), $x$ remains fixed as $n \to \infty$.

**Proof** The first limit was computed in Example 7. The next two can be proved by taking logarithms and applying Theorem 4 (Exercises 117 and 118). The remaining proofs are given in Appendix 5. ■

**EXAMPLE 9** These are examples of the limits in Theorem 5.

(a) $\frac{\ln(n^2)}{n} = \frac{2 \ln n}{n} \to 2 \cdot 0 = 0$ Formula 1

(b) $\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \to (1)^2 = 1$ Formula 2

(c) $\sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \to 1 \cdot 1 = 1$ Formula 3 with $x = 3$ and Formula 2

(d) $\left(-\frac{1}{2}\right)^n \to 0$ Formula 4 with $x = -\frac{1}{2}$

(e) $\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \to e^{-2}$ Formula 5 with $x = -2$

(f) $\frac{100^n}{n!} \to 0$ Formula 6 with $x = 100$ ■

## Recursive Definitions

So far, we have calculated each $a_n$ directly from the value of $n$. But sequences are often defined **recursively** by giving

1. The value(s) of the initial term or terms, and
2. A rule, called a **recursion formula**, for calculating any later term from terms that precede it.

**EXAMPLE 10**

(a) The statements $a_1 = 1$ and $a_n = a_{n-1} + 1$ for $n > 1$ define the sequence $1, 2, 3, \ldots, n, \ldots$ of positive integers. With $a_1 = 1$, we have $a_2 = a_1 + 1 = 2$, $a_3 = a_2 + 1 = 3$, and so on.

(b) The statements $a_1 = 1$ and $a_n = n \cdot a_{n-1}$ for $n > 1$ define the sequence $1, 2, 6, 24, \ldots, n!, \ldots$ of factorials. With $a_1 = 1$, we have $a_2 = 2 \cdot a_1 = 2$, $a_3 = 3 \cdot a_2 = 6$, $a_4 = 4 \cdot a_3 = 24$, and so on.

**(c)** The statements $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n > 2$ define the sequence 1, 1, 2, 3, 5, . . . of **Fibonacci numbers**. With $a_1 = 1$ and $a_2 = 1$, we have $a_3 = 1 + 1 = 2$, $a_4 = 2 + 1 = 3$, $a_5 = 3 + 2 = 5$, and so on.

**(d)** As we can see by applying Newton's method (see Exercise 145), the statements $x_0 = 1$ and $x_{n+1} = x_n - [(\sin x_n - x_n^2)/(\cos x_n - 2x_n)]$ for $n > 0$ define a sequence that, when it converges, gives a solution to the equation $\sin x - x^2 = 0$. ■

## Bounded Monotonic Sequences

Two concepts that play a key role in determining the convergence of a sequence are those of a *bounded* sequence and a *monotonic* sequence.

**DEFINITION** A sequence $\{a_n\}$ is **bounded from above** if there exists a number $M$ such that $a_n \leq M$ for all $n$. The number $M$ is an **upper bound** for $\{a_n\}$. If $M$ is an upper bound for $\{a_n\}$ but no number less than $M$ is an upper bound for $\{a_n\}$, then $M$ is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number $m$ such that $a_n \geq m$ for all $n$. The number $m$ is a **lower bound** for $\{a_n\}$. If $m$ is a lower bound for $\{a_n\}$ but no number greater than $m$ is a lower bound for $\{a_n\}$, then $m$ is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.

**EXAMPLE 11**

**(a)** The sequence 1, 2, 3, . . . , $n$, . . . has no upper bound because it eventually surpasses every number $M$. However, it is bounded below by every real number less than or equal to 1. The number $m = 1$ is the greatest lower bound of the sequence.

**(b)** The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is bounded above by every real number greater than or equal to 1. The upper bound $M = 1$ is the least upper bound (Exercise 137). The sequence is also bounded below by every number less than or equal to $\frac{1}{2}$, which is its greatest lower bound. ■

**Convergent sequences are bounded**

If a sequence $\{a_n\}$ converges to the number $L$, then by definition there is a number $N$ such that $|a_n - L| < 1$ if $n > N$. That is,

$$L - 1 < a_n < L + 1 \quad \text{for } n > N.$$

If $M$ is a number larger than $L + 1$ and all of the finitely many numbers $a_1, a_2, \ldots, a_N$, then for every index $n$ we have $a_n \leq M$ so that $\{a_n\}$ is bounded from above. Similarly, if $m$ is a number smaller than $L - 1$ and all of the numbers $a_1, a_2, \ldots, a_N$, then $m$ is a lower bound of the sequence. Therefore, all convergent sequences are bounded.

Although it is true that every convergent sequence is bounded, there are bounded sequences that fail to converge. One example is the bounded sequence $\{(-1)^{n+1}\}$ discussed in Example 2. The problem here is that some bounded sequences bounce around in the band determined by any lower bound $m$ and any upper bound $M$ (Figure 10.6). An important type of sequence that does not behave that way is one for which each term is at least as large, or at least as small, as its predecessor.

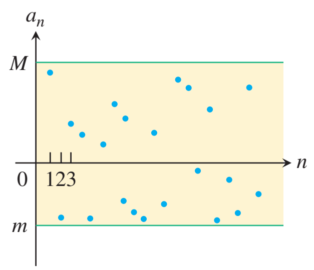



**FIGURE 10.6** Some bounded sequences bounce around between their bounds and fail to converge to any limiting value.

**DEFINITIONS** A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1}$ for all $n$. That is, $a_1 \leq a_2 \leq a_3 \leq \ldots$. The sequence is **nonincreasing** if $a_n \geq a_{n+1}$ for all $n$. The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

**EXAMPLE 12**

**(a)** The sequence $1, 2, 3, \ldots, n, \ldots$ is nondecreasing.

**(b)** The sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is nondecreasing.

**(c)** The sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots$ is nonincreasing.

**(d)** The constant sequence $3, 3, 3, \ldots, 3, \ldots$ is both nondecreasing and nonincreasing.

**(e)** The sequence $1, -1, 1, -1, 1, -1, \ldots$ is not monotonic.

A nondecreasing sequence that is bounded from above always has a least upper bound. Likewise, a nonincreasing sequence bounded from below always has a greatest lower bound. These results are based on the *completeness property* of the real numbers, discussed in Appendix 6. We now prove that if $L$ is the least upper bound of a nondecreasing sequence then the sequence converges to $L$, and that if $L$ is the greatest lower bound of a nonincreasing sequence then the sequence converges to $L$.

**THEOREM 6—The Monotonic Sequence Theorem**
If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

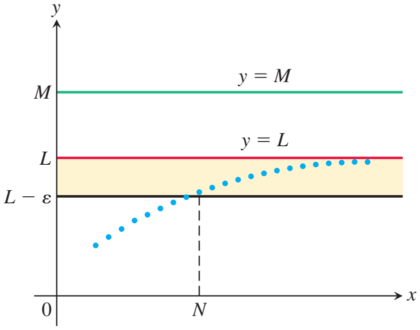



**FIGURE 10.7** If the terms of a nondecreasing sequence have an upper bound $M$, they have a limit $L \le M$.

**Proof** Suppose $\{a_n\}$ is nondecreasing, $L$ is its least upper bound, and we plot the points $(1, a_1), (2, a_2), \ldots, (n, a_n), \ldots$ in the $xy$-plane. If $M$ is an upper bound of the sequence, all these points will lie on or below the line $y = M$ (Figure 10.7). The line $y = L$ is the lowest such line. None of the points $(n, a_n)$ lies above $y = L$, but some do lie above any lower line $y = L - \varepsilon$, if $\varepsilon$ is a positive number (because $L - \varepsilon$ is not an upper bound). The sequence converges to $L$ because

**a.** $a_n \le L$ for *all* values of $n$, and

**b.** given any $\varepsilon > 0$, there exists at least one integer $N$ for which $a_N > L - \varepsilon$.

The fact that $\{a_n\}$ is nondecreasing tells us further that

$$a_n \ge a_N > L - \varepsilon \qquad \text{for all } n \ge N.$$

Thus, *all* the numbers $a_n$ beyond the $N$th number lie within $\varepsilon$ of $L$. This is precisely the condition for $L$ to be the limit of the sequence $\{a_n\}$.

The proof for nonincreasing sequences bounded from below is similar.

It is important to realize that Theorem 6 does not say that convergent sequences are monotonic. The sequence $\{(-1)^{n+1}/n\}$ converges and is bounded, but it is not monotonic since it alternates between positive and negative values as it tends toward zero. What the theorem does say is that a nondecreasing sequence converges when it is bounded from above, but it diverges to infinity otherwise.

## EXERCISES 10.1

### Finding Terms of a Sequence

Each of Exercises 1–6 gives a formula for the $n$th term $a_n$ of a sequence $\{a_n\}$. Find the values of $a_1$, $a_2$, $a_3$, and $a_4$.

**1.** $a_n = \dfrac{1-n}{n^2}$

**2.** $a_n = \dfrac{1}{n!}$

**3.** $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

**4.** $a_n = 2 + (-1)^n$

**5.** $a_n = \dfrac{2^n}{2^{n+1}}$

**6.** $a_n = \dfrac{2^n - 1}{2^n}$

Each of Exercises 7–12 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

**7.** $a_1 = 1, \quad a_{n+1} = a_n + (1/2^n)$

**8.** $a_1 = 1, \quad a_{n+1} = a_n/(n+1)$

9. $a_1 = 2, \quad a_{n+1} = (-1)^{n+1}a_n/2$

10. $a_1 = -2, \quad a_{n+1} = na_n/(n+1)$

11. $a_1 = a_2 = 1, \quad a_{n+2} = a_{n+1} + a_n$

12. $a_1 = 2, \quad a_2 = -1, \quad a_{n+2} = a_{n+1}/a_n$

## Finding a Sequence's Formula

In Exercises 13–30, find a formula for the $n$th term of the sequence.

13. $1, -1, 1, -1, 1, \ldots$ — 1's with alternating signs

14. $-1, 1, -1, 1, -1, \ldots$ — 1's with alternating signs

15. $1, -4, 9, -16, 25, \ldots$ — Squares of the positive integers, with alternating signs

16. $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$ — Reciprocals of squares of the positive integers, with alternating signs

17. $\frac{1}{9}, \frac{2}{12}, \frac{2^2}{15}, \frac{2^3}{18}, \frac{2^4}{21}, \ldots$ — Powers of 2 divided by multiples of 3

18. $-\frac{3}{2}, -\frac{1}{6}, \frac{1}{12}, \frac{3}{20}, \frac{5}{30}, \ldots$ — Integers differing by 2 divided by products of consecutive integers

19. $0, 3, 8, 15, 24, \ldots$ — Squares of the positive integers diminished by 1

20. $-3, -2, -1, 0, 1, \ldots$ — Integers, beginning with $-3$

21. $1, 5, 9, 13, 17, \ldots$ — Every other odd positive integer

22. $2, 6, 10, 14, 18, \ldots$ — Every other even positive integer

23. $\frac{5}{1}, \frac{8}{2}, \frac{11}{6}, \frac{14}{24}, \frac{17}{120}, \ldots$ — Integers differing by 3 divided by factorials

24. $\frac{1}{25}, \frac{8}{125}, \frac{27}{625}, \frac{64}{3125}, \frac{125}{15{,}625}, \ldots$ — Cubes of positive integers divided by powers of 5

25. $1, 0, 1, 0, 1, \ldots$ — Alternating 1's and 0's

26. $0, 1, 1, 2, 2, 3, 3, 4, \ldots$ — Each positive integer repeated

27. $\frac{1}{2} - \frac{1}{3}, \frac{1}{3} - \frac{1}{4}, \frac{1}{4} - \frac{1}{5}, \frac{1}{5} - \frac{1}{6}, \ldots$

28. $\sqrt{5} - \sqrt{4}, \sqrt{6} - \sqrt{5}, \sqrt{7} - \sqrt{6}, \sqrt{8} - \sqrt{7}, \ldots$

29. $\sin\left(\frac{\sqrt{2}}{1+4}\right), \sin\left(\frac{\sqrt{3}}{1+9}\right), \sin\left(\frac{\sqrt{4}}{1+16}\right), \sin\left(\frac{\sqrt{5}}{1+25}\right), \ldots$

30. $\sqrt{\frac{5}{8}}, \sqrt{\frac{7}{11}}, \sqrt{\frac{9}{14}}, \sqrt{\frac{11}{17}}, \ldots$

## Convergence and Divergence

Which of the sequences $\{a_n\}$ in Exercises 31–100 converge, and which diverge? Find the limit of each convergent sequence.

31. $a_n = 2 + (0.1)^n$

32. $a_n = \dfrac{n + (-1)^n}{n}$

33. $a_n = \dfrac{1 - 2n}{1 + 2n}$

34. $a_n = \dfrac{2n + 1}{1 - 3\sqrt{n}}$

35. $a_n = \dfrac{1 - 5n^4}{n^4 + 8n^3}$

36. $a_n = \dfrac{n + 3}{n^2 + 5n + 6}$

37. $a_n = \dfrac{n^2 - 2n + 1}{n - 1}$

38. $a_n = \dfrac{1 - n^3}{70 - 4n^2}$

39. $a_n = 1 + (-1)^n$

40. $a_n = (-1)^n\left(1 - \dfrac{1}{n}\right)$

41. $a_n = \left(\dfrac{n+1}{2n}\right)\left(1 - \dfrac{1}{n}\right)$

42. $a_n = \left(2 - \dfrac{1}{2^n}\right)\left(3 + \dfrac{1}{2^n}\right)$

43. $a_n = \dfrac{(-1)^{n+1}}{2n - 1}$

44. $a_n = \left(-\dfrac{1}{2}\right)^n$

45. $a_n = \sqrt{\dfrac{2n}{n + 1}}$

46. $a_n = \dfrac{1}{(0.9)^n}$

47. $a_n = \sin\left(\dfrac{\pi}{2} + \dfrac{1}{n}\right)$

48. $a_n = n\pi \cos(n\pi)$

49. $a_n = \dfrac{\sin n}{n}$

50. $a_n = \dfrac{\sin^2 n}{2^n}$

51. $a_n = \dfrac{n}{2^n}$

52. $a_n = \dfrac{3^n}{n^3}$

53. $a_n = \dfrac{\ln(n+1)}{\sqrt{n}}$

54. $a_n = \dfrac{\ln n}{\ln 2n}$

55. $a_n = 8^{1/n}$

56. $a_n = (0.03)^{1/n}$

57. $a_n = \left(1 + \dfrac{7}{n}\right)^n$

58. $a_n = \left(1 - \dfrac{1}{n}\right)^n$

59. $a_n = \sqrt[n]{10n}$

60. $a_n = \sqrt[n]{n^2}$

61. $a_n = \left(\dfrac{3}{n}\right)^{1/n}$

62. $a_n = (n + 4)^{1/(n+4)}$

63. $a_n = \dfrac{\ln n}{n^{1/n}}$

64. $a_n = \ln n - \ln(n + 1)$

65. $a_n = \sqrt[n]{4^n n}$

66. $a_n = \sqrt[n]{3^{2n+1}}$

67. $a_n = \dfrac{n!}{n^n}$ (*Hint:* Compare with $1/n$.)

68. $a_n = \dfrac{(-4)^n}{n!}$

69. $a_n = \dfrac{n!}{10^{6n}}$

70. $a_n = \dfrac{n!}{2^n \cdot 3^n}$

71. $a_n = \left(\dfrac{1}{n}\right)^{1/(\ln n)}$

72. $a_n = \dfrac{(n+1)!}{(n+3)!}$

73. $a_n = \dfrac{(2n+2)!}{(2n-1)!}$

74. $a_n = \dfrac{3e^n + e^{-n}}{e^n + 3e^{-n}}$

75. $a_n = \dfrac{e^{-2n} - 2e^{-3n}}{e^{-2n} - e^{-n}}$

76. $a_n = \left(1 - \dfrac{1}{2}\right) + \left(\dfrac{1}{2} - \dfrac{1}{3}\right) + \left(\dfrac{1}{3} - \dfrac{1}{4}\right) + \cdots + \left(\dfrac{1}{n-2} - \dfrac{1}{n-1}\right) + \left(\dfrac{1}{n-1} - \dfrac{1}{n}\right)$

77. $a_n = (\ln 3 - \ln 2) + (\ln 4 - \ln 3) + (\ln 5 - \ln 4) + \cdots + (\ln(n-1) - \ln(n-2)) + (\ln n - \ln(n-1))$

78. $a_n = \ln\left(1 + \dfrac{1}{n}\right)^n$

79. $a_n = \left(\dfrac{3n + 1}{3n - 1}\right)^n$

80. $a_n = \left(\dfrac{n}{n + 1}\right)^n$

81. $a_n = \left(\dfrac{x^n}{2n + 1}\right)^{1/n}, \quad x > 0$

82. $a_n = \left(1 - \dfrac{1}{n^2}\right)^n$

83. $a_n = \dfrac{3^n \cdot 6^n}{2^{-n} \cdot n!}$

84. $a_n = \dfrac{(10/11)^n}{(9/10)^n + (11/12)^n}$

85. $a_n = \tanh n$

86. $a_n = \sinh(\ln n)$

87. $a_n = \dfrac{n^2}{2n - 1}\sin\dfrac{1}{n}$

**88.** $a_n = n\left(1 - \cos\frac{1}{n}\right)$

**89.** $a_n = \sqrt{n}\sin\frac{1}{\sqrt{n}}$

**90.** $a_n = (3^n + 5^n)^{1/n}$

**91.** $a_n = \tan^{-1} n$

**92.** $a_n = \frac{1}{\sqrt{n}}\tan^{-1} n$

**93.** $a_n = \left(\frac{1}{3}\right)^n + \frac{1}{\sqrt{2^n}}$

**94.** $a_n = \sqrt[n]{n^2 + n}$

**95.** $a_n = \frac{(\ln n)^{200}}{n}$

**96.** $a_n = \frac{(\ln n)^5}{\sqrt{n}}$

**97.** $a_n = n - \sqrt{n^2 - n}$

**98.** $a_n = \frac{1}{\sqrt{n^2 - 1} - \sqrt{n^2 + n}}$

**99.** $a_n = \frac{1}{n}\int_1^n \frac{1}{x}\,dx$

**100.** $a_n = \int_1^n \frac{1}{x^p}\,dx, \quad p > 1$

## Recursively Defined Sequences

In Exercises 101–108, assume that each sequence converges and find its limit.

**101.** $a_1 = 2, \quad a_{n+1} = \frac{72}{1 + a_n}$

**102.** $a_1 = -1, \quad a_{n+1} = \frac{a_n + 6}{a_n + 2}$

**103.** $a_1 = -4, \quad a_{n+1} = \sqrt{8 + 2a_n}$

**104.** $a_1 = 0, \quad a_{n+1} = \sqrt{8 + 2a_n}$

**105.** $a_1 = 5, \quad a_{n+1} = \sqrt{5a_n}$

**106.** $a_1 = 3, \quad a_{n+1} = 12 - \sqrt{a_n}$

**107.** $2, 2 + \frac{1}{2}, 2 + \cfrac{1}{2 + \cfrac{1}{2}}, 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}, \ldots$

**108.** $\sqrt{1}, \sqrt{1 + \sqrt{1}}, \sqrt{1 + \sqrt{1 + \sqrt{1}}},$
$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}, \ldots$

## Theory and Examples

**109.** The first term of a sequence is $x_1 = 1$. Each succeeding term is the sum of all those that come before it:

$$x_{n+1} = x_1 + x_2 + \cdots + x_n.$$

Write out enough early terms of the sequence to deduce a general formula for $x_n$ that holds for $n \ge 2$.

**110.** A sequence of rational numbers is described as follows:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots, \frac{a}{b}, \frac{a + 2b}{a + b}, \ldots.$$

Here the numerators form one sequence, the denominators form a second sequence, and their ratios form a third sequence. Let $x_n$ and $y_n$ be, respectively, the numerator and the denominator of the $n$th fraction $r_n = x_n/y_n$.

**a.** Verify that $x_1{}^2 - 2y_1{}^2 = -1, x_2{}^2 - 2y_2{}^2 = +1$ and, more generally, that if $a^2 - 2b^2 = -1$ or $+1$, then

$$(a + 2b)^2 - 2(a + b)^2 = +1 \quad \text{or} \quad -1,$$

respectively.

**b.** The fractions $r_n = x_n/y_n$ approach a limit as $n$ increases. What is that limit? (*Hint:* Use part (a) to show that $r_n{}^2 - 2 = \pm(1/y_n)^2$ and that $y_n$ is not less than $n$.)

**111. Newton's method** The following sequences come from the recursion formula for Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Do the sequences converge? If so, to what value? In each case, begin by identifying the function $f$ that generates the sequence.

**a.** $x_0 = 1, \quad x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n}{2} + \frac{1}{x_n}$

**b.** $x_0 = 1, \quad x_{n+1} = x_n - \frac{\tan x_n - 1}{\sec^2 x_n}$

**c.** $x_0 = 1, \quad x_{n+1} = x_n - 1$

**112. a.** Suppose that $f(x)$ is differentiable for all $x$ in $[0, 1]$ and that $f(0) = 0$. Define sequence $\{a_n\}$ by the rule $a_n = nf(1/n)$. Show that $\lim_{n\to\infty} a_n = f'(0)$. Use the result in part (a) to find the limits of the following sequences $\{a_n\}$.

**b.** $a_n = n\tan^{-1}\frac{1}{n}$

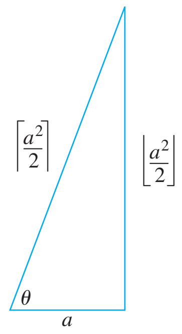
**c.** $a_n = n(e^{1/n} - 1)$

**d.** $a_n = n\ln\left(1 + \frac{2}{n}\right)$

**113. Pythagorean triples** A triple of positive integers $a$, $b$, and $c$ is called a **Pythagorean triple** if $a^2 + b^2 = c^2$. Let $a$ be an odd positive integer and let

$$b = \left\lfloor \frac{a^2}{2} \right\rfloor \quad \text{and} \quad c = \left\lceil \frac{a^2}{2} \right\rceil$$

be, respectively, the integer floor and ceiling for $a^2/2$.




**a.** Show that $a^2 + b^2 = c^2$. (*Hint:* Let $a = 2n + 1$ and express $b$ and $c$ in terms of $n$.)

**b.** By direct calculation, or by appealing to the accompanying figure, find

$$\lim_{a\to\infty} \frac{\left\lfloor \frac{a^2}{2} \right\rfloor}{\left\lceil \frac{a^2}{2} \right\rceil}.$$

**114. The $n$th root of $n!$**

**a.** Show that $\lim_{n\to\infty}(2n\pi)^{1/(2n)} = 1$ and hence, using Stirling's approximation (Chapter 8, Additional Exercise 52a), that

$$\sqrt[n]{n!} \approx \frac{n}{e} \quad \text{for large values of } n.$$

T **b.** Test the approximation in part (a) for $n = 40, 50, 60, \ldots,$ as far as your calculator will allow.

**115. a.** Assuming that $\lim_{n\to\infty}(1/n^c) = 0$ if $c$ is any positive constant, show that

$$\lim_{n\to\infty}\frac{\ln n}{n^c} = 0$$

if $c$ is any positive constant.

**b.** Prove that $\lim_{n\to\infty}(1/n^c) = 0$ if $c$ is any positive constant. (*Hint:* If $\varepsilon = 0.001$ and $c = 0.04$, how large should $N$ be to ensure that $|1/n^c - 0| < \varepsilon$ if $n > N$?)

**116. The zipper theorem** Prove the "zipper theorem" for sequences: If $\{a_n\}$ and $\{b_n\}$ both converge to $L$, then the sequence

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots$$

converges to $L$.

**117.** Prove that $\lim_{n\to\infty}\sqrt[n]{n} = 1$.

**118.** Prove that $\lim_{n\to\infty} x^{1/n} = 1, (x > 0)$.

**119.** Prove Theorem 2. **120.** Prove Theorem 3.

In Exercises 121–124, determine if the sequence is monotonic and if it is bounded.

**121.** $a_n = \dfrac{3n + 1}{n + 1}$ **122.** $a_n = \dfrac{(2n + 3)!}{(n + 1)!}$

**123.** $a_n = \dfrac{2^n 3^n}{n!}$ **124.** $a_n = 2 - \dfrac{2}{n} - \dfrac{1}{2^n}$

Which of the sequences in Exercises 125–134 converge, and which diverge? Give reasons for your answers.

**125.** $a_n = 1 - \dfrac{1}{n}$ **126.** $a_n = n - \dfrac{1}{n}$

**127.** $a_n = \dfrac{2^n - 1}{2^n}$ **128.** $a_n = \dfrac{2^n - 1}{3^n}$

**129.** $a_n = ((-1)^n + 1)\left(\dfrac{n + 1}{n}\right)$

**130.** The first term of a sequence is $x_1 = \cos(1)$. The next terms are $x_2 = x_1$ or $\cos(2)$, whichever is larger; and $x_3 = x_2$ or $\cos(3)$, whichever is larger (farther to the right). In general,

$$x_{n+1} = \max\{x_n, \cos(n + 1)\}.$$

**131.** $a_n = \dfrac{1 + \sqrt{2n}}{\sqrt{n}}$ **132.** $a_n = \dfrac{n + 1}{n}$

**133.** $a_n = \dfrac{4^{n+1} + 3^n}{4^n}$ **134.** $a_1 = 1, \quad a_{n+1} = 2a_n - 3$

In Exercises 135–136, use the definition of convergence to prove the given limit.

**135.** $\displaystyle\lim_{n\to\infty}\frac{\sin n}{n} = 0$ **136.** $\displaystyle\lim_{n\to\infty}\left(1 - \frac{1}{n^2}\right) = 1$

**137. The sequence $\{n/(n + 1)\}$ has a least upper bound of 1** Show that if $M$ is a number less than 1, then the terms of $\{n/(n + 1)\}$ eventually exceed $M$. That is, if $M < 1$ there is an integer $N$ such that $n/(n + 1) > M$ whenever $n > N$. Since $n/(n + 1) < 1$ for every $n$, this proves that 1 is a least upper bound for $\{n/(n + 1)\}$.

**138. Uniqueness of least upper bounds** Show that if $M_1$ and $M_2$ are least upper bounds for the sequence $\{a_n\}$, then $M_1 = M_2$. That is, a sequence cannot have two different least upper bounds.

**139.** Is it true that a sequence $\{a_n\}$ of positive numbers must converge if it is bounded from above? Give reasons for your answer.

**140.** Prove that if $\{a_n\}$ is a convergent sequence, then to every positive number $\varepsilon$ there corresponds an integer $N$ such that

$$|a_m - a_n| < \varepsilon \quad \text{whenever} \quad m > N \quad \text{and} \quad n > N.$$

**141. Uniqueness of limits** Prove that limits of sequences are unique. That is, show that if $L_1$ and $L_2$ are numbers such that $a_n \to L_1$ and $a_n \to L_2$, then $L_1 = L_2$.

**142. Limits and subsequences** If the terms of one sequence appear in another sequence in their given order, we call the first sequence a **subsequence** of the second. Prove that if two sub-sequences of a sequence $\{a_n\}$ have different limits $L_1 \neq L_2$, then $\{a_n\}$ diverges.

**143.** For a sequence $\{a_n\}$ the terms of even index are denoted by $a_{2k}$ and the terms of odd index by $a_{2k+1}$. Prove that if $a_{2k} \to L$ and $a_{2k+1} \to L$, then $a_n \to L$.

**144.** Prove that a sequence $\{a_n\}$ converges to 0 if and only if the sequence of absolute values $\{|a_n|\}$ converges to 0.

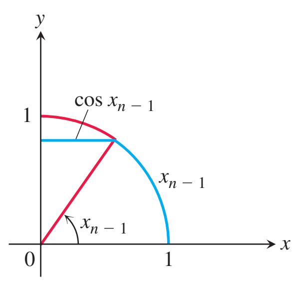
**145. Sequences generated by Newton's method** Newton's method, applied to a differentiable function $f(x)$, begins with a starting value $x_0$ and constructs from it a sequence of numbers $\{x_n\}$ that under favorable circumstances converges to a zero of $f$. The recursion formula for the sequence is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

**a.** Show that the recursion formula for $f(x) = x^2 - a, a > 0$, can be written as $x_{n+1} = (x_n + a/x_n)/2$.

T **b.** Starting with $x_0 = 1$ and $a = 3$, calculate successive terms of the sequence until the display begins to repeat. What number is being approximated? Explain.

T **146. A recursive definition of $\pi/2$** If you start with $x_1 = 1$ and define the subsequent terms of $\{x_n\}$ by the rule $x_n = x_{n-1} + \cos x_{n-1}$, you generate a sequence that converges rapidly to $\pi/2$. **(a)** Try it. **(b)** Use the accompanying figure to explain why the convergence is so rapid.



**COMPUTER EXPLORATIONS**

Use a CAS to perform the following steps for the sequences in Exercises 147–158.

**a.** Calculate and then plot the first 25 terms of the sequence. Does the sequence appear to be bounded from above or below? Does it appear to converge or diverge? If it does converge, what is the limit $L$?

**b.** If the sequence converges, find an integer $N$ such that $|a_n - L| \le 0.01$ for $n \ge N$. How far in the sequence do you have to get for the terms to lie within 0.0001 of $L$?

**147.** $a_n = \sqrt[n]{n}$

**148.** $a_n = \left(1 + \frac{0.5}{n}\right)^n$

**149.** $a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{5^n}$

**150.** $a_1 = 1, \quad a_{n+1} = a_n + (-2)^n$

**151.** $a_n = \sin n$

**152.** $a_n = n \sin \frac{1}{n}$

**153.** $a_n = \frac{\sin n}{n}$

**154.** $a_n = \frac{\ln n}{n}$

**155.** $a_n = (0.9999)^n$

**156.** $a_n = (123456)^{1/n}$

**157.** $a_n = \frac{8^n}{n!}$

**158.** $a_n = \frac{n^{41}}{19^n}$

## 10.2 Infinite Series

An *infinite series* is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The goal of this section is to understand the meaning of such an infinite sum and to develop methods to calculate it. Since there are infinitely many terms to add in an infinite series, we cannot just keep adding to see what comes out. Instead we look at the result of summing just the first $n$ terms of the sequence. The sum of the first $n$ terms

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

is an ordinary finite sum and can be calculated by normal addition. It is called the *nth partial sum*. As $n$ gets larger, we expect the partial sums to get closer and closer to a limiting value in the same sense that the terms of a sequence approach a limit, as discussed in Section 10.1.

For example, to assign meaning to an expression like

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

we add the terms one at a time from the beginning and look for a pattern in how these partial sums grow.

| Partial sum | | Value | Suggestive expression for partial sum |
|---|---|---|---|
| First: | $s_1 = 1$ | $1$ | $2 - 1$ |
| Second: | $s_2 = 1 + \frac{1}{2}$ | $\frac{3}{2}$ | $2 - \frac{1}{2}$ |
| Third: | $s_3 = 1 + \frac{1}{2} + \frac{1}{4}$ | $\frac{7}{4}$ | $2 - \frac{1}{4}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n$th: | $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$ | $\frac{2^n - 1}{2^{n-1}}$ | $2 - \frac{1}{2^{n-1}}$ |

Indeed there is a pattern. The partial sums form a sequence whose $n$th term is

$$s_n = 2 - \frac{1}{2^{n-1}}.$$

This sequence of partial sums converges to 2 because $\lim_{n\to\infty}(1/2^{n-1}) = 0$. We say

$$\text{“the sum of the infinite series } 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots \text{ is 2.”}$$

Is the sum of any finite number of terms in this series equal to 2? No. Can we actually add an infinite number of terms one by one? No. But we can still define their sum by defining it to be the limit of the sequence of partial sums as $n \to \infty$, in this case 2 (Figure 10.8). Our knowledge of sequences and limits enables us to break away from the confines of finite sums.

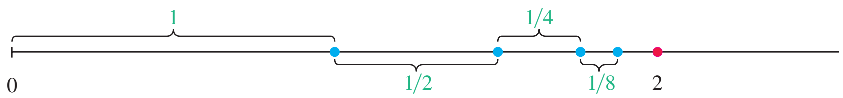



**FIGURE 10.8** As the lengths 1, 1/2, 1/4, 1/8, . . . are added one by one, the sum approaches 2.

**HISTORICAL BIOGRAPHY**
**Blaise Pascal**
**(1623–1662)**
www.goo.gl/9NNLtv

**DEFINITIONS** Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number $a_n$ is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number $s_n$ being the ***n*th partial sum**. If the sequence of partial sums converges to a limit $L$, we say that the series **converges** and that its **sum** is $L$. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

We can represent each term in an infinite series by the area of a rectangle. If all the terms $a_n$ in the series are positive, then the series converges if the total area is finite, and diverges otherwise. Figure 10.9a shows an example where the series converges and Figure 10.9b shows an example where it diverges. The convergence of the total area is related to the convergence or divergence of improper integrals, as we found in Section 8.8. We make this connection explicit in the next section, where we develop an important test for convergence of series, the Integral Test.

When we begin to study a given series $a_1 + a_2 + \cdots + a_n + \cdots$, we might not know whether it converges or diverges. In either case, it is convenient to use sigma notation to write the series as

$$\sum_{n=1}^{\infty} a_n, \qquad \sum_{k=1}^{\infty} a_k, \qquad \text{or} \qquad \sum a_n$$

A useful shorthand when summation from 1 to $\infty$ is understood

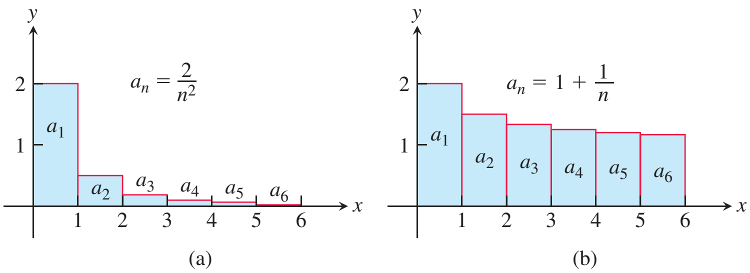



**FIGURE 10.9** The sum of a series with positive terms can be interpreted as a total area of an infinite collection of rectangles. The series converges when the total area of the rectangles is finite (a) and diverges when the total area is unbounded (b). Note that the total area can be infinite even if the area of the rectangles is decreasing.

## Geometric Series

**Geometric series** are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which $a$ and $r$ are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** $r$ can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots, \qquad r = 1/2,\ a = 1$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots. \qquad r = -1/3,\ a = 1$$

If $r = 1$, the $n$th partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

and the series diverges because $\lim_{n\to\infty} s_n = \pm\infty$, depending on the sign of $a$. If $r = -1$, the series diverges because the $n$th partial sums alternate between $a$ and 0 and never approach a single limit. If $|r| \neq 1$, we can determine the convergence or divergence of the series in the following way:

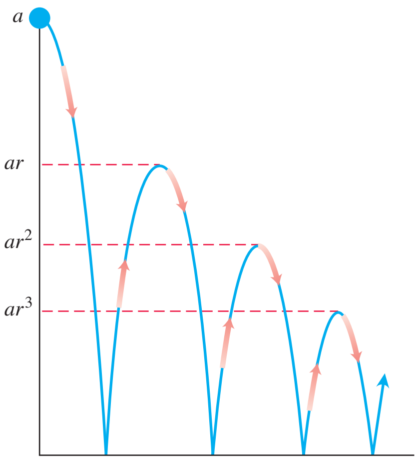
$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} && \text{Write the } n\text{th partial sum.} \\ rs_n &= ar + ar^2 + \cdots + ar^{n-1} + ar^n && \text{Multiply } s_n \text{ by } r. \\ s_n - rs_n &= a - ar^n && \text{Subtract } rs_n \text{ from } s_n.\text{ Most of the terms on the right cancel.} \\ s_n(1 - r) &= a(1 - r^n) && \text{Factor.} \\ s_n &= \frac{a(1 - r^n)}{1 - r}, \quad (r \neq 1). && \text{We can solve for } s_n \text{ if } r \neq 1. \end{aligned}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ (as in Section 10.1), so $s_n \to a/(1 - r)$ in this case. On the other hand, if $|r| > 1$, then $|r^n| \to \infty$ and the series diverges.

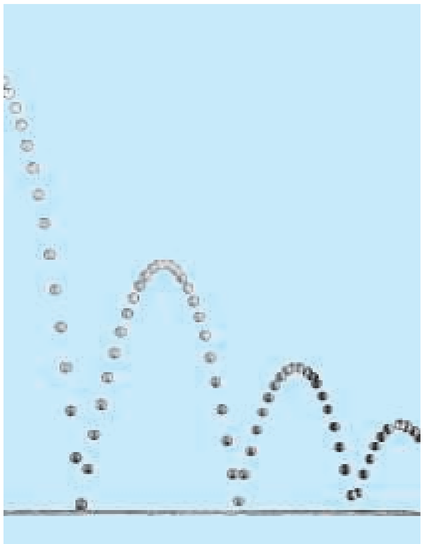
If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.




(a)



(b)

**FIGURE 10.10** (a) Example 3 shows how to use a geometric series to calculate the total vertical distance traveled by a bouncing ball if the height of each rebound is reduced by the factor $r$. (b) A stroboscopic photo of a bouncing ball. (*Source: PSSC Physics*, 2nd ed., Reprinted by permission of Educational Development Center, Inc.)

The formula $a/(1 - r)$ for the sum of a geometric series applies *only* when the summation index begins with $n = 1$ in the expression $\sum_{n=1}^{\infty} ar^{n-1}$ (or with the index $n = 0$ if we write the series as $\sum_{n=0}^{\infty} ar^n$).

**EXAMPLE 1** The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

**EXAMPLE 2** The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$

**EXAMPLE 3** You drop a ball from $a$ meters above a flat surface. Each time the ball hits the surface after falling a distance $h$, it rebounds a distance $rh$, where $r$ is positive but less than 1. Find the total distance the ball travels up and down (Figure 10.10).

**Solution** The total distance is

$$s = a + \underbrace{2ar + 2ar^2 + 2ar^3 + \cdots}_{\text{This sum is } 2ar/(1 - r).} = a + \frac{2ar}{1 - r} = a\frac{1 + r}{1 - r}.$$

If $a = 6$ m and $r = 2/3$, for instance, the distance is

$$s = 6 \cdot \frac{1 + (2/3)}{1 - (2/3)} = 6\left(\frac{5/3}{1/3}\right) = 30 \text{ m}.$$

**EXAMPLE 4** Express the repeating decimal 5.232323 . . . as the ratio of two integers.

**Solution** From the definition of a decimal number, we get a geometric series

$$\begin{aligned} 5.232323\ldots &= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \cdots \\ &= 5 + \frac{23}{100}\underbrace{\left(1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \cdots\right)}_{1/(1 - 0.01)} \qquad a = 1,\ r = 1/100 \\ &= 5 + \frac{23}{100}\left(\frac{1}{0.99}\right) = 5 + \frac{23}{99} = \frac{518}{99} \end{aligned}$$

Unfortunately, formulas like the one for the sum of a convergent geometric series are rare and we usually have to settle for an estimate of a series' sum (more about this later). The next example, however, is another case in which we can find the sum exactly.

**EXAMPLE 5** Find the sum of the "telescoping" series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n + 1)}$.

**Solution** We look for a pattern in the sequence of partial sums that might lead to a formula for $s_k$. The key observation is the partial fraction decomposition

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

so

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

and

$$s_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

Removing parentheses and canceling adjacent terms of opposite sign collapses the sum to

$$s_k = 1 - \frac{1}{k+1}.$$

We now see that $s_k \to 1$ as $k \to \infty$. The series converges, and its sum is 1:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

## The *n*th-Term Test for a Divergent Series

One reason that a series may fail to converge is that its terms don't become small.

**EXAMPLE 6** The series

$$\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} + \cdots$$

diverges because the partial sums eventually outgrow every preassigned number. Each term is greater than 1, so the sum of $n$ terms is greater than $n$.

We now show that $\lim_{n\to\infty} a_n$ must equal zero if the series $\sum_{n=1}^{\infty} a_n$ converges. To see why, let $S$ represent the series' sum and $s_n = a_1 + a_2 + \cdots + a_n$ the $n$th partial sum. When $n$ is large, both $s_n$ and $s_{n-1}$ are close to $S$, so their difference, $a_n$, is close to zero. More formally,

$$a_n = s_n - s_{n-1} \quad \longrightarrow \quad S - S = 0. \qquad \text{Difference Rule for sequences}$$

This establishes the following theorem.

**Caution**
Theorem 7 *does not say* that $\sum_{n=1}^{\infty} a_n$ converges if $a_n \to 0$. It is possible for a series to diverge when $a_n \to 0$. (See Example 8.)

**THEOREM 7** If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Theorem 7 leads to a test for detecting the kind of divergence that occurred in Example 6.

**The *n*th-Term Test for Divergence**

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

**EXAMPLE 7** The following are all examples of divergent series.

**(a)** $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$.

**(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\displaystyle\frac{n+1}{n} \rightarrow 1.$ $\lim_{n\rightarrow\infty} a_n \neq 0$

**(c)** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\rightarrow\infty}(-1)^{n+1}$ does not exist.

**(d)** $\displaystyle\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\displaystyle\lim_{n\rightarrow\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0.$ ■

**EXAMPLE 8** The series

$$1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{\text{2 terms}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{\text{4 terms}} + \cdots + \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{2^n \text{ terms}} + \cdots$$

diverges because the terms can be grouped into infinitely many clusters each of which adds to 1, so the partial sums increase without bound. However, the terms of the series form a sequence that converges to 0. Example 1 of Section 10.3 shows that the harmonic series $\sum 1/n$ also behaves in this manner. ■

## Combining Series

Whenever we have two convergent series, we can add them term by term, subtract them term by term, or multiply them by constants to make new convergent series.

**THEOREM 8** If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (any number $k$).

**Proof** The three rules for series follow from the analogous rules for sequences in Theorem 1, Section 10.1. To prove the Sum Rule for series, let

$$A_n = a_1 + a_2 + \cdots + a_n, \quad B_n = b_1 + b_2 + \cdots + b_n.$$

Then the partial sums of $\sum(a_n + b_n)$ are

$$\begin{aligned} s_n &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) \\ &= (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) \\ &= A_n + B_n. \end{aligned}$$

Since $A_n \rightarrow A$ and $B_n \rightarrow B$, we have $s_n \rightarrow A + B$ by the Sum Rule for sequences. The proof of the Difference Rule is similar.

To prove the Constant Multiple Rule for series, observe that the partial sums of $\sum ka_n$ form the sequence

$$s_n = ka_1 + ka_2 + \cdots + ka_n = k(a_1 + a_2 + \cdots + a_n) = kA_n,$$

which converges to $kA$ by the Constant Multiple Rule for sequences. ■

As corollaries of Theorem 8, we have the following results. We omit the proofs.

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

**Caution** Remember that $\sum(a_n + b_n)$ can converge *even if* both $\sum a_n$ and $\sum b_n$ diverge. For example, $\sum a_n = 1 + 1 + 1 + \cdots$ and $\sum b_n = (-1) + (-1) + (-1) + \cdots$ diverge, whereas $\sum(a_n + b_n) = 0 + 0 + 0 + \cdots$ converges to 0.

**EXAMPLE 9** Find the sums of the following series.

**(a)**
$$\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \qquad \text{Difference Rule}$$

$$= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)} \qquad \text{Geometric series with } a = 1 \text{ and } r = 1/2, 1/6$$

$$= 2 - \frac{6}{5} = \frac{4}{5}$$

**(b)**
$$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4 \sum_{n=0}^{\infty} \frac{1}{2^n} \qquad \text{Constant Multiple Rule}$$

$$= 4\left(\frac{1}{1 - (1/2)}\right) \qquad \text{Geometric series with } a = 1, r = 1/2$$

$$= 8$$

## Adding or Deleting Terms

We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence this will usually change the sum. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$ and

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{k-1} + \sum_{n=k}^{\infty} a_n.$$

Conversely, if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$, then $\sum_{n=1}^{\infty} a_n$ converges. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

and

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n}\right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}.$$

The convergence or divergence of a series is not affected by its first few terms. Only the "tail" of the series, the part that remains when we sum beyond some finite number of initial terms, influences whether it converges or diverges.

**HISTORICAL BIOGRAPHY**
**Richard Dedekind**
**(1831–1916)**
www.goo.gl/aPN8sH

## Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index $h$ units, replace the $n$ in the formula for $a_n$ by $n - h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \cdots.$$

To lower the starting value of the index $h$ units, replace the $n$ in the formula for $a_n$ by $n + h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + a_3 + \cdots.$$

We saw this reindexing in starting a geometric series with the index $n = 0$ instead of the index $n = 1$, but we can use any other starting index value as well. We usually give preference to indexings that lead to simple expressions.

**EXAMPLE 10** We can write the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$$

as

$$\sum_{n=0}^{\infty} \frac{1}{2^n}, \qquad \sum_{n=5}^{\infty} \frac{1}{2^{n-5}}, \qquad \text{or even} \qquad \sum_{n=-4}^{\infty} \frac{1}{2^{n+4}}.$$

The partial sums remain the same no matter what indexing we choose to use. ■

## EXERCISES 10.2

### Finding *n*th Partial Sums

In Exercises 1–6, find a formula for the $n$th partial sum of each series and use it to find the series' sum if the series converges.

1. $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots + \frac{2}{3^{n-1}} + \cdots$
2. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$
3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}} + \cdots$
4. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1}\, 2^{n-1} + \cdots$
5. $\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$
6. $\frac{5}{1 \cdot 2} + \frac{5}{2 \cdot 3} + \frac{5}{3 \cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

### Series with Geometric Terms

In Exercises 7–14, write out the first eight terms of each series to show how the series starts. Then find the sum of the series or show that it diverges.

7. $\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}$
8. $\sum_{n=2}^{\infty} \frac{1}{4^n}$
9. $\sum_{n=1}^{\infty} \left(1 - \frac{7}{4^n}\right)$
10. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$
11. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$
12. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right)$
13. $\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$
14. $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n}\right)$

In Exercises 15–22, determine if the geometric series converges or diverges. If a series converges, find its sum.

15. $1 + \left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \cdots$
16. $1 + (-3) + (-3)^2 + (-3)^3 + (-3)^4 + \cdots$
17. $\left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \left(\frac{1}{8}\right)^4 + \left(\frac{1}{8}\right)^5 + \cdots$
18. $\left(\frac{-2}{3}\right)^2 + \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^4 + \left(\frac{-2}{3}\right)^5 + \left(\frac{-2}{3}\right)^6 + \cdots$
19. $1 - \left(\frac{2}{e}\right) + \left(\frac{2}{e}\right)^2 - \left(\frac{2}{e}\right)^3 + \left(\frac{2}{e}\right)^4 - \cdots$
20. $\left(\frac{1}{3}\right)^{-2} - \left(\frac{1}{3}\right)^{-1} + 1 - \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 - \cdots$
21. $1 + \left(\frac{10}{9}\right)^2 + \left(\frac{10}{9}\right)^4 + \left(\frac{10}{9}\right)^6 + \left(\frac{10}{9}\right)^8 + \cdots$
22. $\frac{9}{4} - \frac{27}{8} + \frac{81}{16} - \frac{243}{32} + \frac{729}{64} - \cdots$

### Repeating Decimals

Express each of the numbers in Exercises 23–30 as the ratio of two integers.

23. $0.\overline{23} = 0.23\ 23\ 23 \ldots$
24. $0.\overline{234} = 0.234\ 234\ 234 \ldots$
25. $0.\overline{7} = 0.7777 \ldots$
26. $0.\overline{d} = 0.dddd \ldots$, where $d$ is a digit
27. $0.0\overline{6} = 0.06666 \ldots$
28. $1.\overline{414} = 1.414\ 414\ 414 \ldots$
29. $1.24\overline{123} = 1.24\ 123\ 123\ 123 \ldots$
30. $3.\overline{142857} = 3.142857\ 142857 \ldots$

### Using the *n*th-Term Test

In Exercises 31–38, use the $n$th-Term Test for divergence to show that the series is divergent, or state that the test is inconclusive.

31. $\sum_{n=1}^{\infty} \frac{n}{n+10}$
32. $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$
33. $\sum_{n=0}^{\infty} \frac{1}{n+4}$
34. $\sum_{n=1}^{\infty} \frac{n}{n^2+3}$

**35.** $\sum_{n=1}^{\infty} \cos \frac{1}{n}$

**36.** $\sum_{n=0}^{\infty} \frac{e^n}{e^n + n}$

**37.** $\sum_{n=1}^{\infty} \ln \frac{1}{n}$

**38.** $\sum_{n=0}^{\infty} \cos n\pi$

## Telescoping Series

In Exercises 39–44, find a formula for the $n$th partial sum of the series and use it to determine if the series converges or diverges. If a series converges, find its sum.

**39.** $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

**40.** $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$

**41.** $\sum_{n=1}^{\infty} \left(\ln \sqrt{n+1} - \ln \sqrt{n}\right)$

**42.** $\sum_{n=1}^{\infty} \left(\tan (n) - \tan (n-1)\right)$

**43.** $\sum_{n=1}^{\infty} \left(\cos^{-1}\left(\frac{1}{n+1}\right) - \cos^{-1}\left(\frac{1}{n+2}\right)\right)$

**44.** $\sum_{n=1}^{\infty} \left(\sqrt{n+4} - \sqrt{n+3}\right)$

Find the sum of each series in Exercises 45–52.

**45.** $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$

**46.** $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

**47.** $\sum_{n=1}^{\infty} \frac{40n}{(2n-1)^2(2n+1)^2}$

**48.** $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

**49.** $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

**50.** $\sum_{n=1}^{\infty} \left(\frac{1}{2^{1/n}} - \frac{1}{2^{1/(n+1)}}\right)$

**51.** $\sum_{n=1}^{\infty} \left(\frac{1}{\ln (n+2)} - \frac{1}{\ln (n+1)}\right)$

**52.** $\sum_{n=1}^{\infty} \left(\tan^{-1}(n) - \tan^{-1}(n+1)\right)$

## Convergence or Divergence

Which series in Exercises 53–76 converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

**53.** $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$

**54.** $\sum_{n=0}^{\infty} (\sqrt{2})^n$

**55.** $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{2^n}$

**56.** $\sum_{n=1}^{\infty} (-1)^{n+1} n$

**57.** $\sum_{n=0}^{\infty} \cos\left(\frac{n\pi}{2}\right)$

**58.** $\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n}$

**59.** $\sum_{n=0}^{\infty} e^{-2n}$

**60.** $\sum_{n=1}^{\infty} \ln \frac{1}{3^n}$

**61.** $\sum_{n=1}^{\infty} \frac{2}{10^n}$

**62.** $\sum_{n=0}^{\infty} \frac{1}{x^n}, \quad |x| > 1$

**63.** $\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$

**64.** $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

**65.** $\sum_{n=0}^{\infty} \frac{n!}{1000^n}$

**66.** $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

**67.** $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$

**68.** $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$

**69.** $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

**70.** $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

**71.** $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

**72.** $\sum_{n=0}^{\infty} \frac{e^{n\pi}}{\pi^{ne}}$

**73.** $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} - \frac{n+2}{n+3}\right)$

**74.** $\sum_{n=2}^{\infty} \left(\sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n-1}\right)\right)$

**75.** $\sum_{n=1}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right)\right)$

**76.** $\sum_{n=0}^{\infty} \left(\ln (4e^n - 1) - \ln (2e^n + 1)\right)$

## Geometric Series with a Variable $x$

In each of the geometric series in Exercises 77–80, write out the first few terms of the series to find $a$ and $r$, and find the sum of the series. Then express the inequality $|r| < 1$ in terms of $x$ and find the values of $x$ for which the inequality holds and the series converges.

**77.** $\sum_{n=0}^{\infty} (-1)^n x^n$

**78.** $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

**79.** $\sum_{n=0}^{\infty} 3\left(\frac{x-1}{2}\right)^n$

**80.** $\sum_{n=0}^{\infty} \frac{(-1)^n}{2}\left(\frac{1}{3 + \sin x}\right)^n$

In Exercises 81–86, find the values of $x$ for which the given geometric series converges. Also, find the sum of the series (as a function of $x$) for those values of $x$.

**81.** $\sum_{n=0}^{\infty} 2^n x^n$

**82.** $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$

**83.** $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

**84.** $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$

**85.** $\sum_{n=0}^{\infty} \sin^n x$

**86.** $\sum_{n=0}^{\infty} (\ln x)^n$

## Theory and Examples

**87.** The series in Exercise 5 can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} \quad \text{and} \quad \sum_{n=-1}^{\infty} \frac{1}{(n+3)(n+4)}.$$

Write it as a sum beginning with **(a)** $n = -2$, **(b)** $n = 0$, **(c)** $n = 5$.

**88.** The series in Exercise 6 can also be written as

$$\sum_{n=1}^{\infty} \frac{5}{n(n+1)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{5}{(n+1)(n+2)}.$$

Write it as a sum beginning with **(a)** $n = -1$, **(b)** $n = 3$, **(c)** $n = 20$.

**89.** Make up an infinite series of nonzero terms whose sum is

**a.** 1 **b.** $-3$ **c.** 0.

**90.** (*Continuation of Exercise 89.*) Can you make an infinite series of nonzero terms that converges to any number you want? Explain.

**91.** Show by example that $\sum(a_n/b_n)$ may diverge even though $\sum a_n$ and $\sum b_n$ converge and no $b_n$ equals 0.

**92.** Find convergent geometric series $A = \sum a_n$ and $B = \sum b_n$ that illustrate the fact that $\sum a_n b_n$ may converge without being equal to $AB$.

**93.** Show by example that $\sum(a_n/b_n)$ may converge to something other than $A/B$ even when $A = \sum a_n$, $B = \sum b_n \neq 0$, and no $b_n$ equals 0.

**94.** If $\sum a_n$ converges and $a_n > 0$ for all $n$, can anything be said about $\sum(1/a_n)$? Give reasons for your answer.

**95.** What happens if you add a finite number of terms to a divergent series or delete a finite number of terms from a divergent series? Give reasons for your answer.

**96.** If $\sum a_n$ converges and $\sum b_n$ diverges, can anything be said about their term-by-term sum $\sum(a_n + b_n)$? Give reasons for your answer.

**97.** Make up a geometric series $\sum ar^{n-1}$ that converges to the number 5 if

**a.** $a = 2$ **b.** $a = 13/2$.

**98.** Find the value of $b$ for which

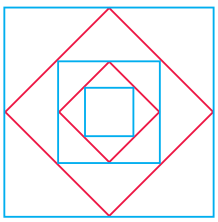
$$1 + e^b + e^{2b} + e^{3b} + \cdots = 9.$$

**99.** For what values of $r$ does the infinite series

$$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + \cdots$$

converge? Find the sum of the series when it converges.

**100.** The accompanying figure shows the first five of a sequence of squares. The outermost square has an area of 4 m$^2$. Each of the other squares is obtained by joining the midpoints of the sides of the squares before it. Find the sum of the areas of all the squares.



**101. Drug dosage** A patient takes a 300 mg tablet for the control of high blood pressure every morning at the same time. The concentration of the drug in the patient's system decays exponentially at a constant hourly rate of $k = 0.12$.

**a.** How many milligrams of the drug are in the patient's system just before the second tablet is taken? Just before the third tablet is taken?

**b.** In the long run, after taking the medication for at least six months, what quantity of drug is in the patient's body just before taking the next regularly scheduled morning tablet?

**102.** Show that the error $(L - s_n)$ obtained by replacing a convergent geometric series with one of its partial sums $s_n$ is $ar^n/(1 - r)$.

**103. The Cantor set** To construct this set, we begin with the closed interval $[0, 1]$. From that interval, remove the middle open interval $(1/3, 2/3)$, leaving the two closed intervals $[0, 1/3]$ and $[2/3, 1]$. At the second step we remove the open middle third interval from each of those remaining. From $[0, 1/3]$ we remove the open interval $(1/9, 2/9)$, and from $[2/3, 1]$ we remove $(7/9, 8/9)$, leaving behind the four closed intervals $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$, and $[8/9, 1]$. At the next step, we remove the middle open third interval from each closed interval left behind, so $(1/27, 2/27)$ is removed from $[0, 1/9]$, leaving the closed intervals $[0, 1/27]$ and $[2/27, 1/9]$; $(7/27, 8/27)$ is removed from $[2/9, 1/3]$, leaving behind $[2/9, 7/27]$ and $[8/27, 1/3]$, and so forth. We continue this process repeatedly without stopping, at each step removing the open third interval from every closed interval remaining behind from the preceding step. The numbers remaining in the interval $[0, 1]$, after all open middle third intervals have been removed, are the points in the Cantor set (named after Georg Cantor, 1845–1918). The set has some interesting properties.

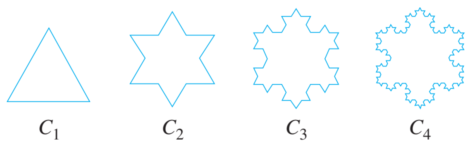
**a.** The Cantor set contains infinitely many numbers in $[0, 1]$. List 12 numbers that belong to the Cantor set.

**b.** Show, by summing an appropriate geometric series, that the total length of all the open middle third intervals that have been removed from $[0, 1]$ is equal to 1.

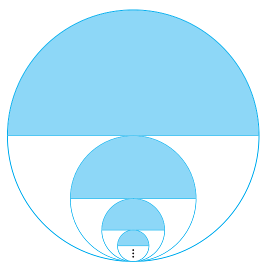
**104. Helga von Koch's snowflake curve** Helga von Koch's snowflake is a curve of infinite length that encloses a region of finite area. To see why this is so, suppose the curve is generated by starting with an equilateral triangle whose sides have length 1.

**a.** Find the length $L_n$ of the $n$th curve $C_n$ and show that $\lim_{n\to\infty} L_n = \infty$.

**b.** Find the area $A_n$ of the region enclosed by $C_n$ and show that $\lim_{n\to\infty} A_n = (8/5)A_1$.




**105.** The largest circle in the accompanying figure has radius 1. Consider the sequence of circles of maximum area inscribed in semicircles of diminishing size. What is the sum of the areas of all of the circles?



# 10.3 The Integral Test

The most basic question we can ask about a series is whether it converges. In this section we begin to study this question, starting with series that have nonnegative terms. Such a series converges if its sequence of partial sums is bounded. If we establish that a given series does converge, we generally do not have a formula available for its sum. So to get an estimate for the sum of a convergent series, we investigate the error involved when using a partial sum to approximate the total sum.

## Nondecreasing Partial Sums

Suppose that $\sum_{n=1}^{\infty} a_n$ is an infinite series with $a_n \geq 0$ for all $n$. Then each partial sum is greater than or equal to its predecessor because $s_{n+1} = s_n + a_n$, so

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_n \leq s_{n+1} \leq \cdots.$$

Since the partial sums form a nondecreasing sequence, the Monotonic Sequence Theorem (Theorem 6, Section 10.1) gives the following result.

**Corollary of Theorem 6**

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

**EXAMPLE 1** As an application of the above corollary, consider the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

Although the $n$th term $1/n$ does go to zero, the series diverges because there is no upper bound for its partial sums. To see why, group the terms of the series in the following way:

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{> \frac{8}{16} = \frac{1}{2}} + \cdots.$$

The sum of the first two terms is 1.5. The sum of the next two terms is $1/3 + 1/4$, which is greater than $1/4 + 1/4 = 1/2$. The sum of the next four terms is $1/5 + 1/6 + 1/7 + 1/8$, which is greater than $1/8 + 1/8 + 1/8 + 1/8 = 1/2$. The sum of the next eight terms is $1/9 + 1/10 + 1/11 + 1/12 + 1/13 + 1/14 + 1/15 + 1/16$, which is greater than $8/16 = 1/2$. The sum of the next 16 terms is greater than $16/32 = 1/2$, and so on. In general, the sum of $2^n$ terms ending with $1/2^{n+1}$ is greater than $2^n/2^{n+1} = 1/2$. If $n = 2^k$, the partial sum $s_n$ is greater than $k/2$, so the sequence of partial sums is not bounded from above. The harmonic series diverges. ■

## The Integral Test

We now introduce the Integral Test with a series that is related to the harmonic series, but whose $n$th term is $1/n^2$ instead of $1/n$.

**EXAMPLE 2** Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

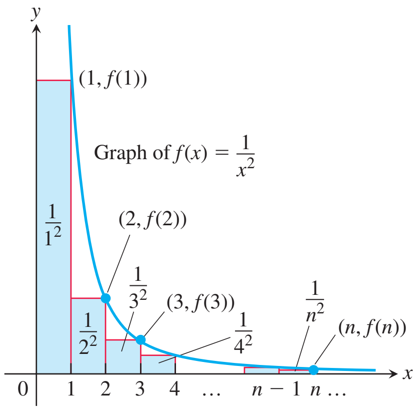



**FIGURE 10.11** The sum of the areas of the rectangles under the graph of $f(x) = 1/x^2$ is less than the area under the graph (Example 2).

**Solution** We determine the convergence of $\sum_{n=1}^{\infty}(1/n^2)$ by comparing it with $\int_1^{\infty}(1/x^2)\,dx$. To carry out the comparison, we think of the terms of the series as values of the function $f(x) = 1/x^2$ and interpret these values as the areas of rectangles under the curve $y = 1/x^2$.

As Figure 10.11 shows,

$$\begin{aligned} s_n &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &= f(1) + f(2) + f(3) + \cdots + f(n) \\ &< f(1) + \int_1^n \frac{1}{x^2}\,dx && \text{Rectangle areas sum to less than area under graph.} \\ &< 1 + \int_1^{\infty} \frac{1}{x^2}\,dx && \textstyle\int_1^n (1/x^2)\,dx < \int_1^{\infty}(1/x^2)\,dx \\ &< 1 + 1 = 2. && \text{As in Section 8.8, Example 3, } \textstyle\int_1^{\infty}(1/x^2)\,dx = 1. \end{aligned}$$

Thus the partial sums of $\sum_{n=1}^{\infty}(1/n^2)$ are bounded from above (by 2) and the series converges. ■

**Caution**
The series and integral need not have the same value in the convergent case. You will see in Example 6 that

$$\textstyle\sum_{n=1}^{\infty}(1/n^2) \neq \int_1^{\infty}(1/x^2)\,dx = 1.$$

### THEOREM 9—The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where $f$ is a continuous, positive, decreasing function of $x$ for all $x \geq N$ ($N$ a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

**Proof** We establish the test for the case $N = 1$. The proof for general $N$ is similar.

We start with the assumption that $f$ is a decreasing function with $f(n) = a_n$ for every $n$. This leads us to observe that the rectangles in Figure 10.12a, which have areas $a_1, a_2, \ldots, a_n$, collectively enclose more area than that under the curve $y = f(x)$ from $x = 1$ to $x = n + 1$. That is,

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n.$$

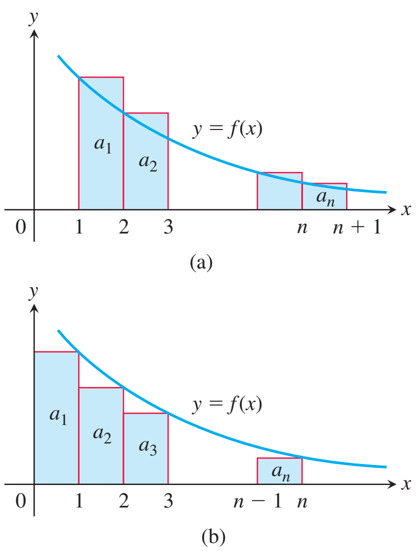



**FIGURE 10.12** Subject to the conditions of the Integral Test, the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

In Figure 10.12b the rectangles have been faced to the left instead of to the right. If we momentarily disregard the first rectangle of area $a_1$, we see that

$$a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx.$$

If we include $a_1$, we have

$$a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

Combining these results gives

$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^n f(x)\,dx.$$

These inequalities hold for each $n$, and continue to hold as $n \to \infty$.

If $\int_1^{\infty} f(x)\,dx$ is finite, the right-hand inequality shows that $\sum a_n$ is finite. If $\int_1^{\infty} f(x)\,dx$ is infinite, the left-hand inequality shows that $\sum a_n$ is infinite. Hence the series and the integral are either both finite or both infinite. ■

**The *p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \le 1$.

**EXAMPLE 3** Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

($p$ a real constant) converges if $p > 1$, and diverges if $p \le 1$.

**Solution** If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of $x$. Since

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b \quad \text{Evaluate the improper integral}$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1}, \quad b^{p-1}\to\infty \text{ as } b\to\infty \text{ because } p-1>0.$$

the series converges by the Integral Test. We emphasize that the sum of the $p$-series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p \le 0$, the series diverges by the $n$th-term test. If $0 < p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}(b^{1-p} - 1) = \infty.$$

Therefore, the series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

In summary, we have convergence for $p > 1$ but divergence for all other values of $p$. ■

The $p$-series with $p = 1$ is the **harmonic series** (Example 1). The $p$-Series Test shows that the harmonic series is just *barely* divergent; if we increase $p$ to 1.000000001, for instance, the series converges!

The slowness with which the partial sums of the harmonic series approach infinity is impressive. For instance, it takes more than 178 million terms of the harmonic series to move the partial sums beyond 20. (See also Exercise 49b.)

**EXAMPLE 4** The series $\sum_{n=1}^{\infty}(1/(n^2+1))$ is not a $p$-series, but it converges by the Integral Test. The function $f(x) = 1/(x^2+1)$ is positive, continuous, and decreasing for $x \ge 1$, and

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\left[\arctan x\right]_1^b$$

$$= \lim_{b\to\infty}\left[\arctan b - \arctan 1\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

The Integral Test tells us that the series converges, but it does *not* say that $\pi/4$ or any other number is the sum of the series. ■

**EXAMPLE 5** Determine the convergence or divergence of the series.

**(a)** $\sum_{n=1}^{\infty} ne^{-n^2}$ **(b)** $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

## Solutions

**(a)** We apply the Integral Test and find that

$$\int_1^\infty \frac{x}{e^{x^2}}\,dx = \frac{1}{2}\int_1^\infty \frac{du}{e^u} \qquad u = x^2,\ du = 2x\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}e^{-u}\right]_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{2e^b} + \frac{1}{2e}\right) = \frac{1}{2e}.$$

Since the integral converges, the series also converges.

**(b)** Again applying the Integral Test,

$$\int_1^\infty \frac{dx}{2^{\ln x}} = \int_0^\infty \frac{e^u\,du}{2^u} \qquad u = \ln x,\ x = e^u,\ dx = e^u\,du$$

$$= \int_0^\infty \left(\frac{e}{2}\right)^u du$$

$$= \lim_{b\to\infty} \frac{1}{\ln\left(\frac{e}{2}\right)}\left(\left(\frac{e}{2}\right)^b - 1\right) = \infty. \qquad (e/2) > 1$$

The improper integral diverges, so the series diverges also. ■

## Error Estimation

For some convergent series, such as the geometric series or the telescoping series in Example 5 of Section 10.2, we can actually find the total sum of the series. That is, we can find the limiting value $S$ of the sequence of partial sums. For most convergent series, however, we cannot easily find the total sum. Nevertheless, we can *estimate* the sum by adding the first $n$ terms to get $s_n$, but we need to know how far off $s_n$ is from the total sum $S$. An approximation to a function or to a number is more useful when it is accompanied by a bound on the size of the worst possible error that could occur. With such an error bound we can try to make an estimate or approximation that is close enough for the problem at hand. Without a bound on the error size, we are just guessing and hoping that we are close to the actual answer. We now show a way to bound the error size using integrals.

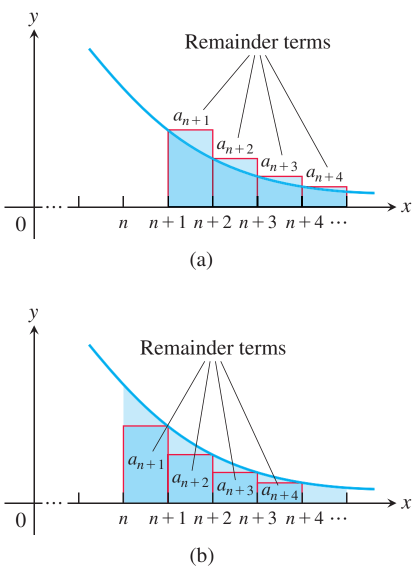



**FIGURE 10.13** The remainder when using $n$ terms is (a) larger than the integral of $f$ over $[n+1, \infty)$. (b) smaller than the integral of $f$ over $[n, \infty)$.

Suppose that a series $\Sigma a_n$ with positive terms is shown to be convergent by the Integral Test, and we want to estimate the size of the **remainder** $R_n$ measuring the difference between the total sum $S$ of the series and its $n$th partial sum $s_n$. That is, we wish to estimate

$$R_n = S - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots.$$

To get a lower bound for the remainder, we compare the sum of the areas of the rectangles with the area under the curve $y = f(x)$ for $x \geq n$ (see Figure 10.13a). We see that

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^\infty f(x)\,dx.$$

Similarly, from Figure 10.13b, we find an upper bound with

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^\infty f(x)\,dx.$$

These comparisons prove the following result, giving bounds on the size of the remainder.

**Bounds for the Remainder in the Integral Test**

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where $f$ is a continuous positive decreasing function of $x$ for all $x \geq n$, and that $\Sigma a_n$ converges to $S$. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx. \qquad (1)$$

If we add the partial sum $s_n$ to each side of the inequalities in (1), we get

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \leq S \leq s_n + \int_{n}^{\infty} f(x)\,dx \qquad (2)$$

since $s_n + R_n = S$. The inequalities in (2) are useful for estimating the error in approximating the sum of a series known to converge by the Integral Test. The error can be no larger than the length of the interval containing $S$, with endpoints given by (2).

**EXAMPLE 6** Estimate the sum of the series $\Sigma(1/n^2)$ using the inequalities in (2) and $n = 10$.

**Solution** We have that

$$\int_{n}^{\infty} \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_n^b = \lim_{b\to\infty}\left(-\frac{1}{b} + \frac{1}{n}\right) = \frac{1}{n}.$$

Using this result with the inequalities in (2), we get

$$s_{10} + \frac{1}{11} \leq S \leq s_{10} + \frac{1}{10}.$$

Taking $s_{10} = 1 + (1/4) + (1/9) + (1/16) + \cdots + (1/100) \approx 1.54977$, these last inequalities give

$$1.64068 \leq S \leq 1.64977.$$

If we approximate the sum $S$ by the midpoint of this interval, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \approx 1.6452.$$

The error in this approximation is then less than half the length of the interval, so the error is less than 0.005. Using a trigonometric *Fourier series* (studied in advanced calculus), it can be shown that $S$ is equal to $\pi^2/6 \approx 1.64493$. ■

**The $p$-series for $p = 2$**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.64493$$

## EXERCISES 10.3

### Applying the Integral Test

Use the Integral Test to determine if the series in Exercises 1–12 converge or diverge. Be sure to check that the conditions of the Integral Test are satisfied.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$
2. $\sum_{n=1}^{\infty} \frac{1}{n^{0.2}}$
3. $\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$
4. $\sum_{n=1}^{\infty} \frac{1}{n + 4}$
5. $\sum_{n=1}^{\infty} e^{-2n}$
6. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
7. $\sum_{n=1}^{\infty} \frac{n}{n^2 + 4}$
8. $\sum_{n=2}^{\infty} \frac{\ln (n^2)}{n}$
9. $\sum_{n=1}^{\infty} \frac{n^2}{e^{n/3}}$
10. $\sum_{n=2}^{\infty} \frac{n - 4}{n^2 - 2n + 1}$
11. $\sum_{n=1}^{\infty} \frac{7}{\sqrt{n} + 4}$
12. $\sum_{n=2}^{\infty} \frac{1}{5n + 10\sqrt{n}}$

## Determining Convergence or Divergence

Which of the series in Exercises 13–46 converge, and which diverge? Give reasons for your answers. (When you check an answer, remember that there may be more than one way to determine the series' convergence or divergence.)

13. $\sum_{n=1}^{\infty} \frac{1}{10^n}$ 14. $\sum_{n=1}^{\infty} e^{-n}$ 15. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

16. $\sum_{n=1}^{\infty} \frac{5}{n+1}$ 17. $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ 18. $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$

19. $\sum_{n=1}^{\infty} -\frac{1}{8^n}$ 20. $\sum_{n=1}^{\infty} \frac{-8}{n}$ 21. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

22. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ 23. $\sum_{n=1}^{\infty} \frac{2^n}{3^n}$ 24. $\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

25. $\sum_{n=0}^{\infty} \frac{-2}{n+1}$ 26. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ 27. $\sum_{n=1}^{\infty} \frac{2^n}{n+1}$

28. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ 29. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$ 30. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\left(\sqrt{n}+1\right)}$

31. $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$ 32. $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$

33. $\sum_{n=3}^{\infty} \frac{(1/n)}{(\ln n)\sqrt{\ln^2 n - 1}}$ 34. $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$

35. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$ 36. $\sum_{n=1}^{\infty} n \tan \frac{1}{n}$

37. $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$ 38. $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$

39. $\sum_{n=1}^{\infty} \frac{e^n}{10+e^n}$ 40. $\sum_{n=1}^{\infty} \frac{e^n}{(10+e^n)^2}$

41. $\sum_{n=2}^{\infty} \frac{\sqrt{n+2}-\sqrt{n+1}}{\sqrt{n+1}\sqrt{n+2}}$ 42. $\sum_{n=3}^{\infty} \frac{7}{\sqrt{n+1}\,\ln\sqrt{n+1}}$

43. $\sum_{n=1}^{\infty} \frac{8\tan^{-1} n}{1+n^2}$ 44. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

45. $\sum_{n=1}^{\infty} \operatorname{sech} n$ 46. $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$

## Theory and Examples

For what values of $a$, if any, do the series in Exercises 47 and 48 converge?

47. $\sum_{n=1}^{\infty} \left(\frac{a}{n+2} - \frac{1}{n+4}\right)$ 48. $\sum_{n=3}^{\infty} \left(\frac{1}{n-1} - \frac{2a}{n+1}\right)$

49. a. Draw illustrations like those in Figures 10.12a and 10.12b to show that the partial sums of the harmonic series satisfy the inequalities

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,dx \le 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$\le 1 + \int_1^{n} \frac{1}{x}\,dx = 1 + \ln n.$$

T b. There is absolutely no empirical evidence for the divergence of the harmonic series even though we know it diverges. The partial sums just grow too slowly. To see what we mean, suppose you had started with $s_1 = 1$ the day the universe was formed, 13 billion years ago, and added a new term every *second*. About how large would the partial sum $s_n$ be today, assuming a 365-day year?

50. Are there any values of $x$ for which $\sum_{n=1}^{\infty} (1/nx)$ converges? Give reasons for your answer.

51. Is it true that if $\sum_{n=1}^{\infty} a_n$ is a divergent series of positive numbers, then there is also a divergent series $\sum_{n=1}^{\infty} b_n$ of positive numbers with $b_n < a_n$ for every $n$? Is there a "smallest" divergent series of positive numbers? Give reasons for your answers.

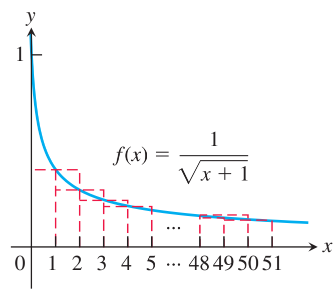
52. (*Continuation of Exercise 51.*) Is there a "largest" convergent series of positive numbers? Explain.

**53.** $\sum_{n=1}^{\infty} \left(1/\sqrt{n+1}\right)$ **diverges**

a. Use the accompanying graph to show that the partial sum $s_{50} = \sum_{n=1}^{50} \left(1/\sqrt{n+1}\right)$ satisfies

$$\int_1^{51} \frac{1}{\sqrt{x+1}}\,dx < s_{50} < \int_0^{50} \frac{1}{\sqrt{x+1}}\,dx.$$

Conclude that $11.5 < s_{50} < 12.3$.

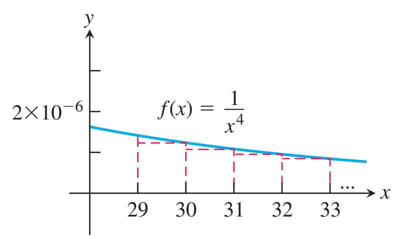



b. What should $n$ be in order that the partial sum

$$s_n = \sum_{i=1}^{n} \left(1/\sqrt{i+1}\right) \text{ satisfy } s_n > 1000?$$

**54.** $\sum_{n=1}^{\infty} (1/n^4)$ **converges**

a. Use the accompanying graph to find an upper bound for the error if $s_{30} = \sum_{n=1}^{30} (1/n^4)$ is used to estimate the value of $\sum_{n=1}^{\infty} (1/n^4)$.




b. Find $n$ so that the partial sum $s_n = \sum_{i=1}^{n} (1/i^4)$ estimates the value of $\sum_{n=1}^{\infty} (1/n^4)$ with an error of at most 0.000001.

55. Estimate the value of $\sum_{n=1}^{\infty} (1/n^3)$ to within 0.01 of its exact value.

56. Estimate the value of $\sum_{n=2}^{\infty} \left(1/(n^2+4)\right)$ to within 0.1 of its exact value.

57. How many terms of the convergent series $\sum_{n=1}^{\infty} \left(1/n^{1.1}\right)$ should be used to estimate its value with error at most 0.00001?

**58.** How many terms of the convergent series $\sum_{n=4}^{\infty} 1/(n(\ln n)^3)$ should be used to estimate its value with error at most 0.01?

**59. The Cauchy condensation test** The Cauchy condensation test says: Let $\{a_n\}$ be a nonincreasing sequence ($a_n \geq a_{n+1}$ for all $n$) of positive terms that converges to 0. Then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. For example, $\sum(1/n)$ diverges because $\sum 2^n \cdot (1/2^n) = \sum 1$ diverges. Show why the test works.

**60.** Use the Cauchy condensation test from Exercise 59 to show that

**a.** $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges;

**b.** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

**61. Logarithmic *p*-series**

**a.** Show that the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} \quad (p \text{ a positive constant})$$

converges if and only if $p > 1$.

**b.** What implications does the fact in part (a) have for the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}?$$

Give reasons for your answer.

**62.** (*Continuation of Exercise 61.*) Use the result in Exercise 61 to determine which of the following series converge and which diverge. Support your answer in each case.

**a.** $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ **b.** $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$

**c.** $\sum_{n=2}^{\infty} \frac{1}{n \ln(n^3)}$ **d.** $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$

**63. Euler's constant** Graphs like those in Figure 10.12 suggest that as $n$ increases there is little change in the difference between the sum

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and the integral

$$\ln n = \int_1^n \frac{1}{x}\,dx.$$

To explore this idea, carry out the following steps.

**a.** By taking $f(x) = 1/x$ in the proof of Theorem 9, show that

$$\ln(n+1) \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq 1 + \ln n$$

or

$$0 < \ln(n+1) - \ln n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \leq 1.$$

Thus, the sequence

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

is bounded from below and from above.

**b.** Show that

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx = \ln(n+1) - \ln n,$$

and use this result to show that the sequence $\{a_n\}$ in part (a) is decreasing.

Since a decreasing sequence that is bounded from below converges, the numbers $a_n$ defined in part (a) converge:

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n \to \gamma.$$

The number $\gamma$, whose value is 0.5772 . . . , is called *Euler's constant*.

**64.** Use the Integral Test to show that the series

$$\sum_{n=0}^{\infty} e^{-n^2}$$

converges.

**65. a.** For the series $\sum(1/n^3)$, use the inequalities in Equation (2) with $n = 10$ to find an interval containing the sum $S$.

**b.** As in Example 5, use the midpoint of the interval found in part (a) to approximate the sum of the series. What is the maximum error for your approximation?

**66.** Repeat Exercise 65 using the series $\sum(1/n^4)$.

**67. Area** Consider the sequence $\{1/n\}_{n=1}^{\infty}$. On each subinterval $(1/(n+1), 1/n)$ within the interval $[0, 1]$, erect the rectangle with area $a_n$ having height $1/n$ and width equal to the length of the subinterval. Find the total area $\sum a_n$ of all the rectangles. (*Hint:* Use the result of Example 5 in Section 10.2.)

**68. Area** Repeat Exercise 67, using trapezoids instead of rectangles. That is, on the subinterval $(1/(n+1), 1/n)$, let $a_n$ denote the area of the trapezoid having heights $y = 1/(n+1)$ at $x = 1/(n+1)$ and $y = 1/n$ at $x = 1/n$.

## 10.4 Comparison Tests

We have seen how to determine the convergence of geometric series, $p$-series, and a few others. We can test the convergence of many more series by comparing their terms to those of a series whose convergence is already known.

**THEOREM 10—Direct Comparison Test**
Let $\sum a_n$ and $\sum b_n$ be two series with $0 \le a_n \le b_n$ for all $n$. Then

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

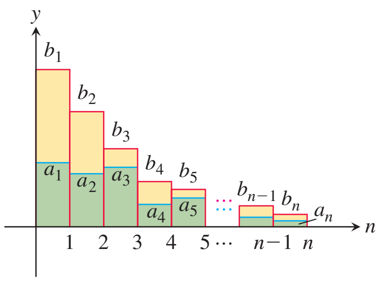



**FIGURE 10.14** If the total area $\sum b_n$ of the taller $b_n$ rectangles is finite, then so is the total area $\sum a_n$ of the shorter $a_n$ rectangles.

**Proof** The series $\sum a_n$ and $\sum b_n$ have nonnegative terms. The Corollary of Theorem 6 stated in Section 10.3 tells us that the series $\sum a_n$ and $\sum b_n$ converge if and only if their partial sums are bounded from above.

In Part (1) we assume that $\sum b_n$ converges to some number $M$. The partial sums $\sum_{n=1}^{N} a_n$ are all bounded from above by $M = \sum b_n$, since

$$s_N = a_1 + a_2 + \cdots + a_N \le b_1 + b_2 + \cdots + b_N \le \sum_{n=1}^{\infty} b_n = M.$$

Since the partial sums of $\sum a_n$ are bounded from above, the Corollary of Theorem 6 implies that $\sum a_n$ converges. We conclude that when $\sum b_n$ converges, then so does $\sum a_n$. Figure 10.12 illustrates this result, with each term of each series interpreted as the area of a rectangle.

In Part (2), where we assume that $\sum a_n$ diverges, the partial sums of $\sum_{n=1}^{\infty} b_n$ are not bounded from above. If they were, the partial sums for $\sum a_n$ would also be bounded from above, since

$$a_1 + a_2 + \cdots + a_N \le b_1 + b_2 + \cdots + b_N,$$

and this would mean that $\sum a_n$ converges. We conclude that if $\sum a_n$ diverges, then so does $\sum b_n$. ■

**EXAMPLE 1** We apply Theorem 10 to several series.

**(a)** The series

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}$$

diverges because its $n$th term

$$\frac{5}{5n-1} = \frac{1}{n - \frac{1}{5}} > \frac{1}{n}$$

is greater than the $n$th term of the divergent harmonic series.

**(b)** The series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

converges because its terms are all positive and less than or equal to the corresponding terms of

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots.$$

The geometric series on the left converges and we have

$$1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1 - (1/2)} = 3.$$

The fact that 3 is an upper bound for the partial sums of $\sum_{n=0}^{\infty} (1/n!)$ does not mean that the series converges to 3. As we will see in Section 10.9, the series converges to $e$.

**HISTORICAL BIOGRAPHY**
**Albert of Saxony**
**(ca. 1316–1390)**
www.goo.gl/Q2dO0w

**(c)** The series

$$5 + \frac{2}{3} + \frac{1}{7} + 1 + \frac{1}{2 + \sqrt{1}} + \frac{1}{4 + \sqrt{2}} + \frac{1}{8 + \sqrt{3}} + \cdots + \frac{1}{2^n + \sqrt{n}} + \cdots$$

converges. To see this, we ignore the first three terms and compare the remaining terms with those of the convergent geometric series $\sum_{n=0}^{\infty}(1/2^n)$. The term $1/(2^n + \sqrt{n})$ of the truncated sequence is less than the corresponding term $1/2^n$ of the geometric series. We see that term by term we have the comparison

$$1 + \frac{1}{2 + \sqrt{1}} + \frac{1}{4 + \sqrt{2}} + \frac{1}{8 + \sqrt{3}} + \cdots \leq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

So the truncated series and the original series converge by an application of the Direct Comparison Test. ■

## The Limit Comparison Test

We now introduce a comparison test that is particularly useful for series in which $a_n$ is a rational function of $n$.

**THEOREM 11—Limit Comparison Test**
Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ ($N$ an integer).

**1.** If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ and $c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

**2.** If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

**3.** If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

**Proof** We will prove Part 1. Parts 2 and 3 are left as Exercises 57a and b.

Since $c/2 > 0$, there exists an integer $N$ such that

$$\left|\frac{a_n}{b_n} - c\right| < \frac{c}{2} \qquad \text{whenever} \qquad n > N.$$

Limit definition with $\varepsilon = c/2$, $L = c$, and $a_n$ replaced by $a_n/b_n$

Thus, for $n > N$,

$$-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2},$$

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2},$$

$$\left(\frac{c}{2}\right)b_n < a_n < \left(\frac{3c}{2}\right)b_n.$$

If $\sum b_n$ converges, then $\sum(3c/2)b_n$ converges and $\sum a_n$ converges by the Direct Comparison Test. If $\sum b_n$ diverges, then $\sum(c/2)b_n$ diverges and $\sum a_n$ diverges by the Direct Comparison Test. ■

**EXAMPLE 2** Which of the following series converge, and which diverge?

**(a)** $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{n^2 + 2n + 1}$

**(b)** $\dfrac{1}{1} + \dfrac{1}{3} + \dfrac{1}{7} + \dfrac{1}{15} + \cdots = \sum_{n=1}^{\infty} \dfrac{1}{2^n - 1}$

**(c)** $\dfrac{1 + 2\ln 2}{9} + \dfrac{1 + 3\ln 3}{14} + \dfrac{1 + 4\ln 4}{21} + \cdots = \sum_{n=2}^{\infty} \dfrac{1 + n\ln n}{n^2 + 5}$

**Solution** We apply the Limit Comparison Test to each series.

**(a)** Let $a_n = (2n + 1)/(n^2 + 2n + 1)$. For large $n$, we expect $a_n$ to behave like $2n/n^2 = 2/n$ since the leading terms dominate for large $n$, so we let $b_n = 1/n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 + n}{n^2 + 2n + 1} = 2,$$

$\sum a_n$ diverges by Part 1 of the Limit Comparison Test. We could just as well have taken $b_n = 2/n$, but $1/n$ is simpler.

**(b)** Let $a_n = 1/(2^n - 1)$. For large $n$, we expect $a_n$ to behave like $1/2^n$, so we let $b_n = 1/2^n$. Since

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - (1/2^n)} = 1,$$

$\sum a_n$ converges by Part 1 of the Limit Comparison Test.

**(c)** Let $a_n = (1 + n\ln n)/(n^2 + 5)$. For large $n$, we expect $a_n$ to behave like $(n\ln n)/n^2 = (\ln n)/n$, which is greater than $1/n$ for $n \geq 3$, so we let $b_n = 1/n$. Since

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n} \text{ diverges}$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + n^2\ln n}{n^2 + 5} = \infty,$$

$\sum a_n$ diverges by Part 3 of the Limit Comparison Test. ■

**EXAMPLE 3** Does $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^{3/2}}$ converge?

**Solution** Because $\ln n$ grows more slowly than $n^c$ for any positive constant $c$ (Section 10.1, Exercise 115), we can compare the series to a convergent $p$-series. To get the $p$-series, we see that

$$\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

for $n$ sufficiently large. Then taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\ln n}{n^{1/4}}$$

$$= \lim_{n\to\infty}\frac{1/n}{(1/4)n^{-3/4}} \qquad \text{l'Hôpital's Rule}$$

$$= \lim_{n\to\infty}\frac{4}{n^{1/4}} = 0.$$

Since $\sum b_n = \sum(1/n^{5/4})$ is a $p$-series with $p > 1$, it converges. Therefore $\sum a_n$ converges by Part 2 of the Limit Comparison Test. ■

## EXERCISES 10.4

### Direct Comparison Test

In Exercises 1–8, use the Direct Comparison Test to determine if each series converges or diverges.

**1.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+30}$ **2.** $\displaystyle\sum_{n=1}^{\infty}\frac{n-1}{n^4+2}$ **3.** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}-1}$

**4.** $\displaystyle\sum_{n=2}^{\infty}\frac{n+2}{n^2-n}$ **5.** $\displaystyle\sum_{n=1}^{\infty}\frac{\cos^2 n}{n^{3/2}}$ **6.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n3^n}$

**7.** $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n+4}{n^4+4}}$ **8.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}+1}{\sqrt{n^2+3}}$

### Limit Comparison Test

In Exercises 9–16, use the Limit Comparison Test to determine if each series converges or diverges.

**9.** $\displaystyle\sum_{n=1}^{\infty}\frac{n-2}{n^3-n^2+3}$

(*Hint:* Limit Comparison with $\sum_{n=1}^{\infty}(1/n^2)$)

**10.** $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n+1}{n^2+2}}$

(*Hint:* Limit Comparison with $\sum_{n=1}^{\infty}(1/\sqrt{n})$)

**11.** $\displaystyle\sum_{n=2}^{\infty}\frac{n(n+1)}{(n^2+1)(n-1)}$ **12.** $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{3+4^n}$

**13.** $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{\sqrt{n}\,4^n}$ **14.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{5n+4}\right)^n$

**15.** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\ln n}$

(*Hint:* Limit Comparison with $\sum_{n=2}^{\infty}(1/n)$)

**16.** $\displaystyle\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n^2}\right)$

(*Hint:* Limit Comparison with $\sum_{n=1}^{\infty}(1/n^2)$)

### Determining Convergence or Divergence

Which of the series in Exercises 17–56 converge, and which diverge? Use any method, and give reasons for your answers.

**17.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}+\sqrt[3]{n}}$ **18.** $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n+\sqrt{n}}$ **19.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sin^2 n}{2^n}$

**20.** $\displaystyle\sum_{n=1}^{\infty}\frac{1+\cos n}{n^2}$ **21.** $\displaystyle\sum_{n=1}^{\infty}\frac{2n}{3n-1}$ **22.** $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^2\sqrt{n}}$

**23.** $\displaystyle\sum_{n=1}^{\infty}\frac{10n+1}{n(n+1)(n+2)}$ **24.** $\displaystyle\sum_{n=3}^{\infty}\frac{5n^3-3n}{n^2(n-2)(n^2+5)}$

**25.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{3n+1}\right)^n$ **26.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+2}}$ **27.** $\displaystyle\sum_{n=3}^{\infty}\frac{1}{\ln(\ln n)}$

**28.** $\displaystyle\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^3}$ **29.** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\ln n}$ **30.** $\displaystyle\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{3/2}}$

**31.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+\ln n}$ **32.** $\displaystyle\sum_{n=2}^{\infty}\frac{\ln(n+1)}{n+1}$ **33.** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{n^2-1}}$

**34.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$ **35.** $\displaystyle\sum_{n=1}^{\infty}\frac{1-n}{n2^n}$ **36.** $\displaystyle\sum_{n=1}^{\infty}\frac{n+2^n}{n^2 2^n}$

**37.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^{n-1}+1}$ **38.** $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n-1}+1}{3^n}$ **39.** $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n^2+3n}\cdot\frac{1}{5n}$

**40.** $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+3^n}{3^n+4^n}$ **41.** $\displaystyle\sum_{n=1}^{\infty}\frac{2^n-n}{n2^n}$ **42.** $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{n}\,e^n}$

**43.** $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n!}$

(*Hint:* First show that $(1/n!) \le (1/n(n-1))$ for $n \ge 2$.)

**44.** $\displaystyle\sum_{n=1}^{\infty}\frac{(n-1)!}{(n+2)!}$ **45.** $\displaystyle\sum_{n=1}^{\infty}\sin\frac{1}{n}$ **46.** $\displaystyle\sum_{n=1}^{\infty}\tan\frac{1}{n}$

**47.** $\displaystyle\sum_{n=1}^{\infty}\frac{\tan^{-1} n}{n^{1.1}}$ **48.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sec^{-1} n}{n^{1.3}}$ **49.** $\displaystyle\sum_{n=1}^{\infty}\frac{\coth n}{n^2}$

**50.** $\displaystyle\sum_{n=1}^{\infty}\frac{\tanh n}{n^2}$ **51.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt[n]{n}}$ **52.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt[n]{n}}{n^2}$

**53.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+2+3+\cdots+n}$ **54.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+2^2+3^2+\cdots+n^2}$

**55.** $\displaystyle\sum_{n=2}^{\infty}\frac{n}{(\ln n)^2}$ **56.** $\displaystyle\sum_{n=2}^{\infty}\frac{(\ln n)^2}{n}$

## Theory and Examples

**57.** Prove **(a)** Part 2 and **(b)** Part 3 of the Limit Comparison Test.

**58.** If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers, can anything be said about $\sum_{n=1}^{\infty}(a_n/n)$? Explain.

**59.** Suppose that $a_n > 0$ and $b_n > 0$ for $n \geq N$ ($N$ an integer). If $\lim_{n\to\infty}(a_n/b_n) = \infty$ and $\sum a_n$ converges, can anything be said about $\sum b_n$? Give reasons for your answer.

**60.** Prove that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ converges.

**61.** Suppose that $a_n > 0$ and $\lim\limits_{n\to\infty} a_n = \infty$. Prove that $\sum a_n$ diverges.

**62.** Suppose that $a_n > 0$ and $\lim\limits_{n\to\infty} n^2 a_n = 0$. Prove that $\sum a_n$ converges.

**63.** Show that $\sum_{n=2}^{\infty}\left((\ln n)^q / n^p\right)$ converges for $-\infty < q < \infty$ and $p > 1$.

(*Hint:* Limit Comparison with $\sum_{n=2}^{\infty} 1/n^r$ for $1 < r < p$.)

**64.** (*Continuation of Exercise 63.*) Show that $\sum_{n=2}^{\infty}((\ln n)^q/n^p)$ diverges for $-\infty < q < \infty$ and $0 < p < 1$.

(*Hint:* Limit Comparison with an appropriate $p$-series.)

**65. Decimal numbers** Any real number in the interval $[0, 1]$ can be represented by a decimal (not necessarily unique) as

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots,$$

where $d_i$ is one of the integers 0, 1, 2, 3, . . . , 9. Prove that the series on the right-hand side always converges.

**66.** If $\sum a_n$ is a convergent series of positive terms, prove that $\sum \sin(a_n)$ converges.

In Exercises 67–72, use the results of Exercises 63 and 64 to determine if each series converges or diverges.

**67.** $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^3}{n^4}$

**68.** $\displaystyle\sum_{n=2}^{\infty} \sqrt{\frac{\ln n}{n}}$

**69.** $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^{1000}}{n^{1.001}}$

**70.** $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^{1/5}}{n^{0.99}}$

**71.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.1}(\ln n)^3}$

**72.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n \cdot \ln n}}$

## COMPUTER EXPLORATIONS

**73.** It is not yet known whether the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2 n}$$

converges or diverges. Use a CAS to explore the behavior of the series by performing the following steps.

**a.** Define the sequence of partial sums

$$s_k = \sum_{n=1}^{k} \frac{1}{n^3 \sin^2 n}.$$

What happens when you try to find the limit of $s_k$ as $k \to \infty$? Does your CAS find a closed form answer for this limit?

**b.** Plot the first 100 points $(k, s_k)$ for the sequence of partial sums. Do they appear to converge? What would you estimate the limit to be?

**c.** Next plot the first 200 points $(k, s_k)$. Discuss the behavior in your own words.

**d.** Plot the first 400 points $(k, s_k)$. What happens when $k = 355$? Calculate the number $355/113$. Explain from you calculation what happened at $k = 355$. For what values of $k$ would you guess this behavior might occur again?

**74. a.** Use Theorem 8 to show that

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{n(n+1)}\right)$$

where $S = \sum_{n=1}^{\infty}(1/n^2)$, the sum of a convergent $p$-series.

**b.** From Example 5, Section 10.2, show that

$$S = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}.$$

**c.** Explain why taking the first $M$ terms in the series in part (b) gives a better approximation to $S$ than taking the first $M$ terms in the original series $\sum_{n=1}^{\infty}(1/n^2)$.

**d.** We know the exact value of $S$ is $\pi^2/6$. Which of the sums

$$\sum_{n=1}^{1000000} \frac{1}{n^2} \quad \text{or} \quad 1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$$

gives a better approximation to $S$?

# 10.5 Absolute Convergence; The Ratio and Root Tests

When some of the terms of a series are positive and others are negative, the series may or may not converge. For example, the geometric series

$$5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots = \sum_{n=0}^{\infty} 5\left(\frac{-1}{4}\right)^n \tag{1}$$

converges (since $|r| = \frac{1}{4} < 1$), whereas the different geometric series

$$1 - \frac{5}{4} + \frac{25}{16} - \frac{125}{64} + \cdots = \sum_{n=0}^{\infty}\left(\frac{-5}{4}\right)^n \tag{2}$$