**6.92.** (a) $-\frac{1}{2}z-\frac{3}{4}z^2-\frac{7}{8}z^3-\frac{15}{16}z^4-\cdots$ (b) $\cdots+\frac{1}{z^2}+\frac{1}{z}+1+\frac{1}{2}z+\frac{1}{4}z^2+\frac{1}{8}z^3+\cdots$

(c) $-\frac{1}{2}-\frac{3}{z^2}-\frac{7}{z^3}-\frac{15}{z^4}-\cdots$ (d) $-(z-1)^{-1}-2(z-1)^{-2}-2(z-1)^{-3}-\cdots$

(e) $1-2(z-2)^{-1}-(z-2)+(z-2)^2-(z-2)^3+(z-2)^4-\cdots$

**6.96.** (a) $\frac{z}{2!}-\frac{z^3}{4!}+\frac{z^5}{6!}-\cdots$; removable singularity (d) $z^2-z^6+\frac{z^{10}}{2!}-\frac{z^{14}}{3!}+\cdots$; ordinary point

(b) $\frac{1}{z^3}+\frac{1}{z}+\frac{z}{2!}+\frac{z^3}{3!}+\frac{z^5}{4!}+\frac{z^7}{5!}+\cdots$; pole of order 3 (e) $z^{3/2}+\frac{z^{5/2}}{3!}+\frac{z^{7/2}}{5!}+\frac{z^{9/2}}{7!}+\cdots$; branch point

(c) $\frac{1}{z}-\frac{1}{2!z^3}+\frac{1}{4!z^5}-\cdots$; essential singularity

**6.98.** (a) $\pi/6+2m\pi$, $(2m+1)\pi-\pi/6$, $m=0,\ \pm1,\ \pm2,\ldots$; poles of order 2

(b) $i/2m\pi$, $m=\pm1,\ \pm2,\ldots$; simple poles, $z=0$; essential singularity, $z=\infty$; pole of order 2

(c) $z=0,\ \infty$; essential singularities

(d) $z=-1\pm i$; branch points

(e) $z=2m\pi i$, $m=\pm1,\ \pm2,\ldots$; simple poles, $z=0$; removable singularity, $z=\infty$; essential singularity

**6.99.** (a) $e\left\{1+2(z-2)^{-1}+\frac{2^2(z-2)^{-2}}{2!}+\frac{2^3(z-2)^{-3}}{3!}+\cdots\right\}$

(b) $|z-2|>0$

(c) $z=2$; essential singularity, $z=\infty$; removable singularity

**6.104.** 2.62 to two decimal accuracy.

**6.108.** (b) $1/(1-z)$

**6.109.** (b) $-3-(9/4)i$

**6.112.** (a) div., (b) conv., (c) conv., (d) conv., (e) div., (f) conv.

**6.115.** $\sum_{n=1}^{\infty}\frac{(3-3^{2n-1})z^{2n-1}}{4(2n-1)!}$

**6.117.** $\cdots-\frac{1}{8}(z-1)^{-4}+\frac{1}{4}(z-1)^{-3}-\frac{1}{2}(z-1)^{-2}+(z-1)^{-1}-1-(z-1)-(z-1)^2-\cdots$

**6.118.** (a) $\frac{1}{z^3}-\frac{1}{3z}+\frac{z}{5}-\frac{z^3}{7}+\cdots$ (b) $|z|>0$ (c) $-1/3$

**6.119.** (a) $z+z^{-1}+\frac{z^{-3}}{2!}+\frac{z^{-5}}{3!}+\cdots$; $|z|>0$, $2z-\frac{2z^3}{3}+\frac{4z^5}{45}-\cdots$; $|z|\geq 0$, $\frac{z^{-1}}{4}+\frac{1}{16}+\frac{z}{64}+\frac{z^2}{256}+\cdots$; $0<|z|<4$

(b) essential singularity, removable singularity, pole, (c) $2\pi i,\ 0,\pi i/2$

**6.120.** (a) diverges.

**6.126.** Not uniformly convergent in any region that includes $z=0$; uniformly convergent in a region $|z|\geq\delta$ where $\delta$ is any positive number.

**6.137.** (a) entire, (b) meromorphic, (c) entire, (d) entire, (e) neither, (f) meromorphic, (g) entire, (h) neither

**6.139.** (a) $\frac{1}{z}+\frac{z}{2}-\frac{z}{12}+\frac{z^2}{24}+\frac{89z^3}{720}+\cdots$ (b) $0<|z|<1$

**6.144.** $e^{\cos\theta}\cos(\sin\theta)$

**6.163.** (b) $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_3=0$, $B_4=-\frac{1}{30}$, $B_5=0$, $B_6=\frac{1}{42}$

CHAPTER 7

# The Residue Theorem Evaluation of Integrals and Series

## 7.1 Residues

Let $f(z)$ be single-valued and analytic inside and on a circle $C$ except at the point $z = a$ chosen as the center of $C$. Then, as we have seen in Chapter 6, $f(z)$ has a Laurent series about $z = a$ given by

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} a_n(z-a)^n \\ &= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \end{aligned} \tag{7.1}$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad n = 0,\ \pm 1,\ \pm 2, \ldots \tag{7.2}$$

In the special case $n = -1$, we have from (7.2)

$$\oint_C f(z)\, dz = 2\pi i a_{-1} \tag{7.3}$$

Formally, we can obtain (7.3) from (7.1) by integrating term by term and using the results (Problems 4.21 and 4.22)

$$\oint_C \frac{dz}{(z-a)^p} = \begin{cases} 2\pi i & p = 1 \\ 0 & p = \text{integer} \neq 1 \end{cases} \tag{7.4}$$

Because of the fact that (7.3) involves only the coefficient $a_{-1}$ in (7.1), we call $a_{-1}$ the *residue* of $f(z)$ at $z = a$.

## 7.2 Calculation of Residues

To obtain the residue of a function $f(z)$ at $z = a$, it may appear from (7.1) that the Laurent expansion of $f(z)$ about $z = a$ must be obtained. However, in the case where $z = a$ is a pole of order $k$, there is a simple formula for $a_{-1}$ given by

$$a_{-1} = \lim_{z\to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \{(z-a)^k f(z)\} \tag{7.5}$$

If $k = 1$ (simple pole), then the result is especially simple and is given by

$$a_{-1} = \lim_{z\to a} (z-a) f(z) \tag{7.6}$$

which is a special case of (7.5) with $k = 1$ if we define $0! = 1$.

**EXAMPLE 7.1:** If $f(z) = z/(z-1)(z+1)^2$, then $z = 1$ and $z = -1$ are poles of orders one and two, respectively. We have, using (7.6) and (7.5) with $k = 2$,

$$\text{Residue at } z = 1 \text{ is } \lim_{z\to 1} (z-1)\left\{\frac{z}{(z-1)(z+1)^2}\right\} = \frac{1}{4}$$

$$\text{Residue at } z = -1 \text{ is } \lim_{z\to -1} \frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\left(\frac{z}{(z-1)(z+1)^2}\right)\right\} = -\frac{1}{4}$$

If $z = a$ is an essential singularity, the residue can sometimes be found by using known series expansions.

**EXAMPLE 7.2:** Let $f(z) = e^{-1/z}$. Then, $z = 0$ is an essential singularity and from the known expansion for $e^u$ with $u = -1/z$, we find

$$e^{-1/z} = 1 - \frac{1}{z} + \frac{1}{2!z^2} - \frac{1}{3!z^3} + \cdots$$

from which we see that the residue at $z = 0$ is the coefficient of $1/z$ and equals $-1$.

## 7.3 The Residue Theorem

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve $C$ except at the singularities $a, b, c, \ldots$ inside $C$, which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$ [see Fig. 7-1]. Then, the *residue theorem* states that

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) \tag{7.7}$$

i.e., the integral of $f(z)$ around $C$ is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by $C$. Note that (7.7) is a generalization of (7.3). Cauchy's theorem and integral formulas are special cases of this theorem (see Problem 7.75).

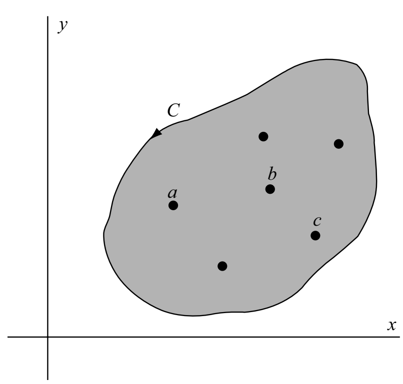


Fig. 7-1

## 7.4 Evaluation of Definite Integrals

The evaluation of definite integrals is often achieved by using the residue theorem together with a suitable function $f(z)$ and a suitable closed path or contour $C$, the choice of which may require great ingenuity. The following types are most common in practice.

1. $\int_{-\infty}^{\infty} F(x)\,dx$, where $F(x)$ is a rational function.

   Consider $\oint_C F(z)\,dz$ along a contour $C$ consisting of the line along the $x$ axis from $-R$ to $+R$ and the semicircle $\Gamma$ above the $x$ axis having this line as diameter [Fig. 7-2]. Then, let $R \to \infty$. If $F(x)$ is an even function, this can be used to evaluate $\int_0^{\infty} F(x)\,dx$. See Problems 7.7–7.10.

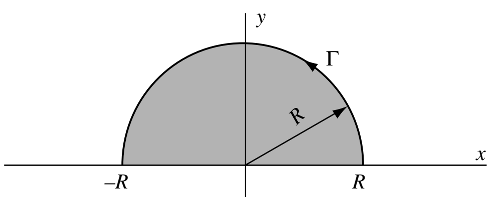


Fig. 7-2

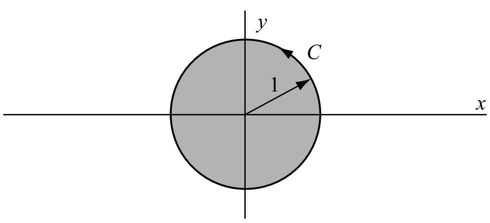


Fig. 7-3

2. $\int_0^{2\pi} G(\sin\theta,\ \cos\theta)\,d\theta$, where $G(\sin\theta,\ \cos\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$.

   Let $z = e^{i\theta}$. Then $\sin\theta = (z - z^{-1})/2i$, $\cos\theta = (z + z^{-1})/2$ and $dz = ie^{i\theta}\,d\theta$ or $d\theta = dz/iz$. The given integral is equivalent to $\oint_C F(z)\,dz$ where $C$ is the unit circle with center at the origin [Fig. 7-3]. See Problems 7.11–7.14.

3. $\displaystyle\int_{-\infty}^{\infty} F(x) \left\{ \begin{array}{c} \cos mx \\ \sin mx \end{array} \right\} dx$, where $F(x)$ is a rational function.

   Here, we consider $\oint_C F(z)e^{imz}\,dz$ where $C$ is the same contour as that in Type 1. See Problems 7.15–7.17 and 7.37.

4. Miscellaneous integrals involving particular contours. See Problems 7.18–7.23.

## 7.5 Special Theorems Used in Evaluating Integrals

In evaluating integrals such as those of Types 1 and 3 above, it is often necessary to show that $\int_\Gamma F(z)\,dz$ and $\int_\Gamma e^{imz}F(z)\,dz$ approach zero as $R \to \infty$. The following theorems are fundamental.

**THEOREM 7.1.** If $|F(z)| \le M/R^k$ for $z = Re^{i\theta}$, where $k > 1$ and $M$ are constants, then if $\Gamma$ is the semicircle of Fig. 7-2,

$$\lim_{R\to\infty} \int_\Gamma F(z)\,dz = 0$$

See Problem 7.7.

**THEOREM 7.2.** If $|F(z)| \le M/R^k$ for $z = Re^{i\theta}$, where $k > 0$ and $M$ are constants, then if $\Gamma$ is the semicircle of Fig. 7-2,

$$\lim_{R\to\infty} \int_{\Gamma} e^{imz} F(z)\, dz = 0$$

See Problem 7.15.

## 7.6 The Cauchy Principal Value of Integrals

If $F(x)$ is continuous in $a \le x \le b$ except at a point $x_0$ such that $a < x_0 < b$, then if $\epsilon_1$ and $\epsilon_2$ are positive, we define

$$\int_a^b F(x)\, dx = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left\{ \int_a^{x_0-\epsilon_1} F(x)\, dx + \int_{x_0+\epsilon_2}^b F(x)\, dx \right\}$$

In some cases, the above limit does not exist for $\epsilon_1 \ne \epsilon_2$ but does exist if we take $\epsilon_1 = \epsilon_2 = \epsilon$. In such a case, we call

$$\int_a^b F(x)\, dx = \lim_{\epsilon\to 0} \left\{ \int_a^{x_0-\epsilon} F(x)\, dx + \int_{x_0+\epsilon}^b F(x)\, dx \right\}$$

the *Cauchy principal value* of the integral on the left.

**EXAMPLE 7.3:**

$$\int_{-1}^{1} \frac{dx}{x^3} = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left\{ \int_{-1}^{-\epsilon_1} \frac{dx}{x^3} + \int_{\epsilon_2}^{1} \frac{dx}{x^3} \right\} = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}} \left\{ \frac{1}{2\epsilon_2^2} - \frac{1}{2\epsilon_1^2} \right\}$$

does not exist. However, the Cauchy principal value with $\epsilon_1 = \epsilon_2 = \epsilon$ does exist and equals zero.

## 7.7 Differentiation Under the Integral Sign. Leibnitz's Rule

A useful method for evaluating integrals employs *Leibnitz's rule* for differentiation under the integral sign. This rule states that

$$\frac{d}{d\alpha} \int_a^b F(x, \alpha)\, dx = \int_a^b \frac{\partial F}{\partial \alpha}\, dx$$

The rule is valid if $a$ and $b$ are constants, $\alpha$ is a real parameter such that $\alpha_1 \le \alpha \le \alpha_2$ where $\alpha_1$ and $\alpha_2$ are constants, and $F(x, \alpha)$ is continuous and has a continuous partial derivative with respect to $\alpha$ for $a \le x \le b$, $\alpha_1 \le \alpha \le \alpha_2$. It can be extended to cases where the limits $a$ and $b$ are infinite or dependent on $\alpha$.

## 7.8 Summation of Series

The residue theorem can often be used to sum various types of series. The following results are valid under very mild restrictions on $f(z)$ that are generally satisfied whenever the series converge. See Problems 7.24, 7.32 and 7.38.

1. $\sum_{-\infty}^{\infty} f(n) = -\{\text{sum of residues of } \pi \cot \pi z f(z) \text{ at all the poles of } f(z)\}$
2. $\sum_{-\infty}^{\infty} (-1)^n f(n) = -\{\text{sum of residues of } \pi \csc \pi z f(z) \text{ at all the poles of } f(z)\}$
3. $\sum_{-\infty}^{\infty} f\left(\frac{2n+1}{2}\right) = \{\text{sum of residues of } \pi \tan \pi z f(z) \text{ at all the poles of } f(z)\}$
4. $\sum_{-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right) = \{\text{sum of residues of } \pi \sec \pi z f(z) \text{ at all the poles of } f(z)\}$

## 7.9 Mittag–Leffler's Expansion Theorem

1. Suppose that the only singularities of $f(z)$ in the finite $z$ plane are the simple poles $a_1, a_2, a_3, \ldots$ arranged in order of increasing absolute value.
2. Let the residues of $f(z)$ at $a_1, a_2, a_3, \ldots$ be $b_1, b_2, b_3, \ldots$.
3. Let $C_N$ be circles of radius $R_N$ that do not pass through any poles and on which $|f(z)| < M$, where $M$ is independent of $N$ and $R_N \to \infty$ as $N \to \infty$.

Then *Mittag–Leffler's expansion theorem* states that

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{z - a_n} + \frac{1}{a_n} \right\}$$

## 7.10 Some Special Expansions

1. $\csc z = \frac{1}{z} - 2z\left(\frac{1}{z^2 - \pi^2} - \frac{1}{z^2 - 4\pi^2} + \frac{1}{z^2 - 9\pi^2} - \cdots\right)$
2. $\sec z = \pi\left(\frac{1}{(\pi/2)^2 - z^2} - \frac{3}{(3\pi/2)^2 - z^2} + \frac{5}{(5\pi/2)^2 - z^2} - \cdots\right)$
3. $\tan z = 2z\left(\frac{1}{(\pi/2)^2 - z^2} + \frac{1}{(3\pi/2)^2 - z^2} + \frac{1}{(5\pi/2)^2 - z^2} + \cdots\right)$
4. $\cot z = \frac{1}{z} + 2z\left(\frac{1}{z^2 - \pi^2} + \frac{1}{z^2 - 4\pi^2} + \frac{1}{z^2 - 9\pi^2} + \cdots\right)$
5. $\operatorname{csch} z = \frac{1}{z} - 2z\left(\frac{1}{z^2 + \pi^2} - \frac{1}{z^2 + 4\pi^2} + \frac{1}{z^2 + 9\pi^2} - \cdots\right)$
6. $\operatorname{sech} z = \pi\left(\frac{1}{(\pi/2)^2 + z^2} - \frac{3}{(3\pi/2)^2 + z^2} + \frac{5}{(5\pi/2)^2 + z^2} - \cdots\right)$
7. $\tanh z = 2z\left(\frac{1}{z^2 + (\pi/2)^2} + \frac{1}{z^2 + (3\pi/2)^2} + \frac{1}{z^2 + (5\pi/2)^2} + \cdots\right)$
8. $\coth z = \frac{1}{z} + 2z\left(\frac{1}{z^2 + \pi^2} + \frac{1}{z^2 + 4\pi^2} + \frac{1}{z^2 + 9\pi^2} + \cdots\right)$

## SOLVED PROBLEMS

### Residues and the Residue Theorem

**7.1.** Let $f(z)$ be analytic inside and on a simple closed curve $C$ except at point $a$ inside $C$.

(a) Prove that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n \quad \text{where } a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz,\ n = 0,\ \pm 1,\ \pm 2, \ldots$$

i.e., $f(z)$ can be expanded into a converging Laurent series about $z = a$.

(b) Prove that

$$\oint_C f(z)\,dz = 2\pi i a_{-1}$$

***Solution***

(a) This follows from Problem 6.25 of Chapter 6.

(b) If we let $n = -1$ in the result of (a), we find

$$a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz, \quad \text{i.e., } \oint_C f(z)\,dz = 2\pi i a_{-1}$$

We call $a_{-1}$ the *residue* of $f(z)$ at $z = a$.

**7.2.** Prove the *residue theorem*. If $f(z)$ is analytic inside and on a simple closed curve $C$ except at a finite number of points $a, b, c, \ldots$ inside $C$ at which the residues are $a_{-1}, b_{-1}, c_{-1}, \ldots,$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

i.e., $2\pi i$ times the sum of the residues at all singularities enclosed by $C$.

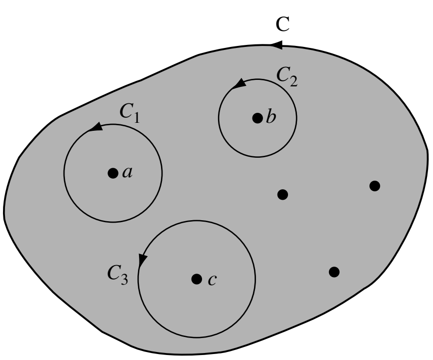

Fig. 7-4

***Solution***

With centers at $a, b, c, \ldots,$ respectively, construct circles $C_1, C_2, C_3, \ldots$ that lie entirely inside $C$ as shown in Fig. 7-4. This can be done since $a, b, c, \ldots$ are interior points. By Theorem 4.5, page 118, we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \cdots \tag{1}$$

But, by Problem 7.1,

$$\oint_{C_1} f(z)\,dz = 2\pi i a_{-1}, \quad \oint_{C_2} f(z)\,dz = 2\pi i b_{-1}, \quad \oint_{C_3} f(z)\,dz = 2\pi i c_{-1}, \ldots \tag{2}$$

Then, from (1) and (2), we have, as required,

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) = 2\pi i \text{ (sum of residues)}$$

The proof given here establishes the residue theorem for simply-connected regions containing a finite number of singularities of $f(z)$. It can be extended to regions with infinitely many isolated singularities and to multiply-connected regions (see Problems 7.96 and 7.97).

**7.3.** Let $f(z)$ be analytic inside and on a simple closed curve $C$ except at a pole $a$ of order $m$ inside $C$. Prove that the residue of $f(z)$ at $a$ is given by

$$a_{-1} = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}$$

### Solution

***Method 1.*** Suppose $f(z)$ has a pole $a$ of order $m$. Then the Laurent series of $f(z)$ is

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \tag{1}$$

Then multiplying both sides by $(z-a)^m$, we have

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \cdots + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + \cdots \tag{2}$$

This represents the Taylor series about $z = a$ of the analytic function on the left. Differentiating both sides $m-1$ times with respect to $z$, we have

$$\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = (m-1)!a_{-1} + m(m-1)\cdots 2a_0(z-a) + \cdots$$

Thus, on letting $z \to a$,

$$\lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = (m-1)!\,a_{-1}$$

from which the required result follows.

***Method 2.*** The required result also follows directly from Taylor's theorem on noting that the coefficient of $(z-a)^{m-1}$ in the expansion (2) is

$$a_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\bigg|_{z=a}$$

***Method 3.*** See Problem 5.28, page 161.

**7.4.** Find the residues of (a) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ and (b) $f(z) = e^z \csc^2 z$ at all its poles in the finite plane.

### Solution

(a) $f(z)$ has a double pole at $z = -1$ and simple poles at $z = \pm 2i$.

***Method 1.*** Residue at $z = -1$ is

$$\lim_{z\to -1} \frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2 \cdot \frac{z^2-2z}{(z+1)^2(z^2+4)}\right\} = \lim_{z\to -1} \frac{(z^2+4)(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2} = -\frac{14}{25}$$

Residue at $z = 2i$ is

$$\lim_{z\to 2i}\left\{(z-2i)\cdot \frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4-4i}{(2i+1)^2(4i)} = \frac{7+i}{25}$$

Residue at $z = -2i$ is

$$\lim_{z\to -2i}\left\{(z+2i)\cdot\frac{z^2-2z}{(z+1)^2(z-2i)(z+2i)}\right\} = \frac{-4+4i}{(-2i+1)^2(-4i)} = \frac{7-i}{25}$$

***Method 2.*** Residue at $z = 2i$ is

$$\lim_{z\to 2i}\left\{\frac{(z-2i)(z^2-2z)}{(z+1)^2(z^2+4)}\right\} = \left\{\lim_{z\to 2i}\frac{z^2-2z}{(z+1)^2}\right\}\left\{\lim_{z\to 2i}\frac{z-2i}{z^2+4}\right\}$$
$$= \frac{-4-4i}{(2i+1)^2}\cdot\lim_{z\to 2i}\frac{1}{2z} = \frac{-4-4i}{(2i+1)^2}\cdot\frac{1}{4i} = \frac{7+i}{25}$$

using L'Hospital's rule. In a similar manner, or by replacing $i$ by $-i$ in the result, we can obtain the residue at $z = -2i$.

(b) $f(z) = e^z \csc^2 z = e^z/\sin^2 z$ has double poles at $z = 0, \pm\pi, \pm 2\pi, \ldots$, i.e., $z = m\pi$ where $m = 0, \pm 1, \pm 2, \ldots$.

***Method 1.*** Residue at $z = m\pi$ is

$$\lim_{z\to m\pi}\frac{1}{1!}\frac{d}{dz}\left\{(z-m\pi)^2\frac{e^z}{\sin^2 z}\right\} = \lim_{z\to m\pi}\frac{e^z[(z-m\pi)^2\sin z + 2(z-m\pi)\sin z - 2(z-m\pi)^2\cos z]}{\sin^3 z}$$

Letting $z - m\pi = u$ or $z = u + m\pi$, this limit can be written

$$\lim_{u\to 0} e^{u+m\pi}\left\{\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{\sin^3 u}\right\} = e^{m\pi}\left\{\lim_{u\to 0}\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{\sin^3 u}\right\}$$

The limit in braces can be obtained using L'Hospital's rule. However, it is easier to first note that

$$\lim_{u\to 0}\frac{u^3}{\sin^3 u} = \lim_{u\to 0}\left(\frac{u}{\sin u}\right)^3 = 1$$

and thus write the limit as

$$e^{m\pi}\lim_{u\to 0}\left(\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{u^3}\cdot\frac{u^3}{\sin^3 u}\right) = e^{m\pi}\lim_{u\to 0}\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{u^3} = e^{m\pi}$$

using L'Hospital's rule several times. In evaluating this limit, we can instead use the series expansions $\sin u = u - u^3/3! + \cdots$, $\cos u = 1 - u^2/2! + \cdots$.

***Method 2*** (using Laurent's series).

In this method, we expand $f(z) = e^z \csc^2 z$ in a Laurent series about $z = m\pi$ and obtain the coefficient of $1/(z - m\pi)$ as the required residue. To make the calculation easier, let $z = u + m\pi$. Then, the function to be expanded in a Laurent series about $u = 0$ is $e^{m\pi+u}\csc^2(m\pi + u) = e^{m\pi}e^u\csc^2 u$. Using the Maclaurin expansions for $e^u$ and $\sin u$, we find using long division

$$e^{m\pi}e^u\csc^2 u = \frac{e^{m\pi}\left(1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right)}{\left(u-\frac{u^3}{3!}+\frac{u^5}{5!}-\cdots\right)^2} = \frac{e^{m\pi}\left(1+u+\frac{u^2}{2}+\cdots\right)}{u^2\left(1-\frac{u^2}{6}+\frac{u^4}{120}-\cdots\right)^2}$$
$$= \frac{e^{m\pi}\left(1+u+\frac{u^2}{2!}+\cdots\right)}{u^2\left(1-\frac{u^2}{3}+\frac{2u^4}{45}+\cdots\right)} = e^{m\pi}\left(\frac{1}{u^2}+\frac{1}{u}+\frac{5}{6}+\frac{u}{3}+\cdots\right)$$

and so the residue is $e^{m\pi}$.

**7.5.** Find the residue of $F(z) = \dfrac{\cot z \coth z}{z^3}$ at $z = 0$.

***Solution***

We have, as in Method 2 of Problem 7.4(b),

$$F(z) = \frac{\cos z \cosh z}{z^3 \sin z \sinh z} = \frac{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)}{z^3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)}$$

$$= \frac{\left(1 - \frac{z^4}{6} + \cdots\right)}{z^5\left(1 - \frac{z^4}{90} + \cdots\right)} = \frac{1}{z^5}\left(1 - \frac{7z^4}{45} + \cdots\right)$$

and so the residue (coefficient of $1/z$) is $-7/45$.

***Another Method.*** The result can also be obtained by finding

$$\lim_{z \to 0} \frac{1}{4!} \frac{d^4}{dz^4}\left\{z^5 \frac{\cos z \cosh z}{z^3 \sin z \sinh z}\right\}$$

but this method is much more laborious than that given above.

**7.6.** Evaluate $\dfrac{1}{2\pi i} \displaystyle\oint_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)}\, dz$ around the circle $C$ with equation $|z| = 3$.

***Solution***

The integrand $e^{zt}/\{z^2(z^2 + 2z + 2)\}$ has a double pole at $z = 0$ and two simple poles at $z = -1 \pm i$ [roots of $z^2 + 2z + 2 = 0$]. All these poles are inside $C$.

Residue at $z = 0$ is

$$\lim_{z \to 0} \frac{1}{1!} \frac{d}{dz}\left\{z^2 \frac{e^{zt}}{z^2(z^2 + 2z + 2)}\right\} = \lim_{z \to 0} \frac{(z^2 + 2z + 2)(te^{zt}) - (e^{zt})(2z + 2)}{(z^2 + 2z + 2)^2} = \frac{t - 1}{2}$$

Residue at $z = -1 + i$ is

$$\lim_{z \to -1+i}\left\{[z - (-1 + i)]\frac{e^{zt}}{z^2(z^2 + 2z + 2)}\right\} = \lim_{z \to -1+i}\left\{\frac{e^{zt}}{z^2}\right\} \lim_{z \to -1+i}\left\{\frac{z + 1 - i}{z^2 + 2z + 2}\right\}$$

$$= \frac{e^{(-1+i)t}}{(-1 + i)^2} \cdot \frac{1}{2i} = \frac{e^{(-1+i)t}}{4}$$

Residue at $z = -1 - i$ is

$$\lim_{z \to -1-i}\left\{[z - (-1 - i)]\frac{e^{zt}}{z^2(z^2 + 2z + 2)}\right\} = \frac{e^{(-1-i)t}}{4}$$

Then, by the residue theorem

$$\oint_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)}\, dz = 2\pi i \text{ (sum of residues)} = 2\pi i\left\{\frac{t - 1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}\right\}$$

$$= 2\pi i\left\{\frac{t - 1}{2} + \frac{1}{2}e^{-t}\cos t\right\}$$

that is,

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{t-1}{2}+\frac{1}{2}e^{-t}\cos t$$

## Definite Integrals of the Type $\int_{-\infty}^{\infty} F(x)\,dx$

**7.7.** Let $|F(z)| \le M/R^k$ for $z = Re^{i\theta}$ where $k > 1$ and $M$ are constants. Prove that $\lim_{R\to\infty}\int_\Gamma F(z)\,dz = 0$ where $\Gamma$ is the semi-circular arc of radius $R$ shown in Fig. 7-5.

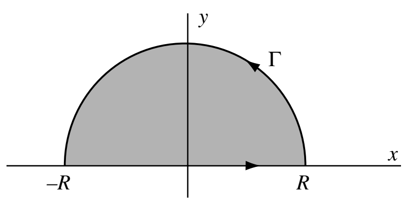

Fig. 7-5

***Solution***

By Property (e), page 112, we have

$$\left|\int_\Gamma F(z)\,dz\right| \le \frac{M}{R^k}\cdot \pi R = \frac{\pi M}{R^{k-1}}$$

since the length of arc $L = \pi R$. Then

$$\lim_{R\to\infty}\left|\int_\Gamma F(z)\,dz\right| = 0 \quad\text{and so}\quad \lim_{R\to\infty}\int_\Gamma F(z)\,dz = 0$$

**7.8.** Show that for $z = Re^{i\theta}$, $|f(z)| \le M/R^k$, $k > 1$ if $f(z) = 1/(z^6+1)$.

***Solution***

Suppose $z = Re^{i\theta}$. Then

$$|f(z)| = \left|\frac{1}{R^6e^{6i\theta}+1}\right| \le \frac{1}{|R^6e^{6i\theta}|-1} = \frac{1}{R^6-1} \le \frac{2}{R^6}$$

where $R$ is large enough (say $R > 2$, for example), so that $M = 2$, $k = 6$.

Note that we have made use of the inequality $|z_1+z_2| \ge |z_1| - |z_2|$ with $z_1 = R^6e^{6i\theta}$ and $z_2 = 1$.

**7.9.** Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^6+1}$.

***Solution***

Consider $\oint_C dz/(z^6+1)$, where $C$ is the closed contour of Fig. 7-5 consisting of the line from $-R$ to $R$ and the semicircle $\Gamma$, traversed in the positive (counterclockwise) sense.

Since $z^6+1=0$ when $z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$, these are simple poles of $1/(z^6+1)$. Only the poles $e^{\pi i/6}$, $e^{3\pi i/6}$, and $e^{5\pi i/6}$ lie within $C$. Then, using L'Hospital's rule,

$$\text{Residue at } e^{\pi i/6} = \lim_{z\to e^{\pi i/6}}\left\{(z-e^{\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{\pi i/6}}\frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/6}$$

$$\text{Residue at } e^{3\pi i/6} = \lim_{z\to e^{3\pi i/6}}\left\{(z-e^{3\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{3\pi i/6}}\frac{1}{6z^5} = \frac{1}{6}e^{-5\pi i/2}$$

$$\text{Residue at } e^{5\pi i/6} = \lim_{z\to e^{5\pi i/6}}\left\{(z-e^{5\pi i/6})\frac{1}{z^6+1}\right\} = \lim_{z\to e^{5\pi i/6}}\frac{1}{6z^5} = \frac{1}{6}e^{-25\pi i/6}$$

Thus

$$\oint_C \frac{dz}{z^6+1} = 2\pi i\left\{\frac{1}{6}e^{-5\pi i/6} + \frac{1}{6}e^{-5\pi i/2} + \frac{1}{6}e^{-25\pi i/6}\right\} = \frac{2\pi}{3}$$

that is,

$$\int_{-R}^{R} \frac{dx}{x^6+1} + \int_\Gamma \frac{dz}{z^6+1} = \frac{2\pi}{3} \tag{1}$$

Taking the limit of both sides of (1) as $R \to \infty$ and using Problems 7.7 and 7.8, we have

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3} \tag{2}$$

Since

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_0^{\infty} \frac{dx}{x^6+1}$$

the required integral has the value $\pi/3$.

**7.10.** Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}.$

***Solution***

The poles of $z^2/(z^2+1)^2(z^2+2z+2)$ enclosed by the contour $C$ of Fig. 7-5 are $z = i$ of order 2 and $z = -1+i$ of order 1.

Residue at $z = i$ is

$$\lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2 \frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\} = \frac{9i-12}{100}$$

Residue at $z = -1+i$ is

$$\lim_{z\to -1+i} (z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$$

Then

$$\oint_C \frac{z^2\, dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\left\{\frac{9i-12}{100} + \frac{3-4i}{25}\right\} = \frac{7\pi}{50}$$

or

$$\int_{-R}^{R} \frac{x^2\, dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma \frac{z^2\, dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero by Problem 7.7, we obtain the required result.

### Definite Integrals of the Type $\int_0^{2\pi} G(\sin\theta, \cos\theta)\, d\theta$

**7.11.** Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

***Solution***

Let $z = e^{i\theta}$. Then $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i = (z - z^{-1})/2i$, $\cos\theta = (e^{i\theta} + e^{-i\theta})/2 = (z + z^{-1})/2$, $dz = iz\, d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = \oint_C \frac{dz/iz}{3 - 2(z + z^{-1})/2 + (z - z^{-1})/2i} = \oint_C \frac{2\, dz}{(1 - 2i)z^2 + 6iz - 1 - 2i}$$

where $C$ is the circle of unit radius with center at the origin (Fig. 7-6).

The poles of $2/\{(1 - 2i)z^2 + 6iz - 1 - 2i\}$ are the simple poles

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1 - 2i)(-1 - 2i)}}{2(1 - 2i)}$$

$$= \frac{-6i \pm 4i}{2(1 - 2i)} = 2 - i,\ (2 - i)/5$$

Only $(2 - i)/5$ lies inside $C$.

Residue at

$$(2 - i)/5 = \lim_{z \to (2-i)/5} \{z - (2 - i)/5\} \left\{ \frac{2}{(1 - 2i)z^2 + 6iz - 1 - 2i} \right\}$$

$$= \lim_{z \to (2-i)/5} \frac{2}{2(1 - 2i)z + 6i} = \frac{1}{2i}$$

by L'Hospital's rule.

Then

$$\oint_C \frac{2\, dz}{(1 - 2i)z^2 + 6iz - 1 - 2i} = 2\pi i \left( \frac{1}{2i} \right) = \pi,$$

the required value.

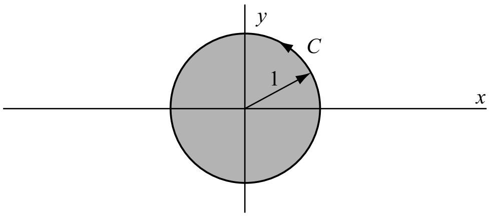


Fig. 7-6

**7.12.** Given $a > |b|$, show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$.

***Solution***

Let $z = e^{i\theta}$. Then, $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i = (z - z^{-1})/2i$, $dz = ie^{i\theta}\, d\theta = iz\, d\theta$ so that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \oint_C \frac{dz/iz}{a + b(z - z^{-1})/2i} = \oint_C \frac{2\, dz}{bz^2 + 2aiz - b}$$

where $C$ is the circle of unit radius with center at the origin, as shown in Fig. 7-6.

The poles of $2/(bz^2 + 2aiz - b)$ are obtained by solving $bz^2 + 2aiz - b = 0$ and are given by

$$z = \frac{-2ai \pm \sqrt{-4a^2 + 4b^2}}{2b} = \frac{-ai \pm \sqrt{a^2 - b^2}\, i}{b}$$

$$= \left\{ \frac{-a + \sqrt{a^2 - b^2}}{b} \right\} i,\ \left\{ \frac{-a - \sqrt{a^2 - b^2}}{b} \right\} i$$

Only $\left\{\left(-a+\sqrt{a^2-b^2}\right)/b\right\}i$ lies inside $C$, since

$$\left|\frac{-a+\sqrt{a^2-b^2}}{b}i\right| = \left|\frac{\sqrt{a^2-b^2}-a}{b}\cdot\frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2-b^2}+a}\right| = \left|\frac{b}{(\sqrt{a^2-b^2}+a)}\right| < 1$$

when $a > |b|$.

Residue at

$$z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}i = \lim_{z\to z_1}(z-z_1)\frac{2}{bz^2+2aiz-b}$$
$$= \lim_{z\to z_1}\frac{2}{2bz+2ai} = \frac{1}{bz_1+ai} = \frac{1}{\sqrt{a^2-b^2}i}$$

by L'Hospital's rule.

Then

$$\oint_C \frac{2\,dz}{bz^2+2aiz-b} = 2\pi i\left(\frac{1}{\sqrt{a^2-b^2}i}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

the required value.

**7.13.** Show that $\displaystyle\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \frac{\pi}{12}$.

***Solution***

Let $z=e^{i\theta}$. Then $\cos\theta = (z+z^{-1})/2$, $\cos 3\theta = (e^{3i\theta}+e^{-3i\theta})/2 = (z^3+z^{-3})/2$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta = \oint_C \frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2}\frac{dz}{iz} = -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}dz$$

where $C$ is the contour of Fig. 7-6.

The integrand has a pole of order 3 at $z=0$ and a simple pole $z=\frac{1}{2}$ inside $C$.

Residue at $z=0$ is

$$\lim_{z\to 0}\frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = \frac{21}{8}$$

Residue at $z=\frac{1}{2}$ is

$$\lim_{z\to 1/2}\left\{\left(z-\frac{1}{2}\right)\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = -\frac{65}{24}$$

Then

$$-\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}dz = -\frac{1}{2i}(2\pi i)\left\{\frac{21}{8}-\frac{65}{24}\right\} = \frac{\pi}{12} \text{ as required.}$$

**7.14.** Show that $\displaystyle\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$.

***Solution***

Letting $z=e^{i\theta}$, we have $\sin\theta = (z-z^{-1})/2i$, $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ and so

$$\int_0^{2\pi}\frac{d\theta}{(5-3\sin\theta)^2} = \oint_C \frac{dz/iz}{\{5-3(z-z^{-1})/2i\}^2} = -\frac{4}{i}\oint_C \frac{z\,dz}{(3z^2-10iz-3)^2}$$

where $C$ is the contour of Fig. 7-6.

The integrand has poles of order 2 at $z = \left(10i \pm \sqrt{-100+36}\right)/6 = (10i \pm 8i)/6 = 3i,\ i/3$. Only the pole $i/3$ lies inside $C$.

Residue at

$$z = i/3 = \lim_{z\to i/3} \frac{d}{dz}\left\{(z-i/3)^2 \cdot \frac{z}{(3z^2-10iz-3)^2}\right\}$$
$$= \lim_{z\to i/3} \frac{d}{dz}\left\{(z-i/3)^2 \cdot \frac{z}{(3z-i)^2(z-3i)^2}\right\} = -\frac{5}{256}$$

Then

$$-\frac{4}{i}\oint_C \frac{z\,dz}{(3z^2-10iz-3)^2} = -\frac{4}{i}(2\pi i)\left(\frac{-5}{256}\right) = \frac{5\pi}{32}$$

***Another Method.*** From Problem 7.12, we have for $a > |b|$,

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Then, by differentiating both sides with respect to $a$ (considering $b$ as constant) using Leibnitz's rule, we have

$$\frac{d}{da}\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \int_0^{2\pi} \frac{\partial}{\partial a}\left(\frac{1}{a+b\sin\theta}\right)d\theta = -\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2}$$
$$= \frac{d}{da}\left(\frac{2\pi}{\sqrt{a^2-b^2}}\right) = \frac{-2\pi a}{(a^2-b^2)^{3/2}}$$

that is,

$$\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Letting $a = 5$ and $b = -3$, we have

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{2\pi(5)}{(5^2-3^2)^{3/2}} = \frac{5\pi}{32}$$

## Definite Integrals of the Type $\int_{-\infty}^{\infty} F(x)\begin{Bmatrix}\cos mx\\ \sin mx\end{Bmatrix}dx$

**7.15.** Let $|F(z)| \le M/R^k$ for $z = Re^{i\theta}$ where $k > 0$ and $M$ are constants. Prove that

$$\lim_{R\to\infty}\int_\Gamma e^{imz}F(z)\,dz = 0$$

where $\Gamma$ is the semicircular arc of Fig. 7-5 and $m$ is a positive constant.

***Solution***

Let $z = Re^{i\theta}$. Then $\int_\Gamma e^{imz}F(z)\,dz = \int_0^\pi e^{imRe^{i\theta}}F(Re^{i\theta})iRe^{i\theta}\,d\theta$. Then

$$\left|\int_0^\pi e^{imRe^{i\theta}}F(Re^{i\theta})iRe^{i\theta}\,d\theta\right| \le \int_0^\pi |e^{imRe^{i\theta}}F(Re^{i\theta})iRe^{i\theta}|\,d\theta$$
$$= \int_0^\pi |e^{imR\cos\theta - mR\sin\theta}F(Re^{i\theta})iRe^{i\theta}|\,d\theta$$
$$= \int_0^\pi e^{-mR\sin\theta}|F(Re^{i\theta})|R\,d\theta$$
$$\le \frac{M}{R^{k-1}}\int_0^\pi e^{-mR\sin\theta}\,d\theta = \frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta$$

Now $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi/2$, as can be seen geometrically from Fig. 7-7 or analytically from Problem 7.99.

Then, the last integral is less than or equal to

$$\frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{-2mR\theta/\pi}d\theta = \frac{\pi M}{mR^k}\left(1 - e^{-mR}\right)$$

As $R \to \infty$, this approaches zero, since $m$ and $k$ are positive, and the required result is proved.

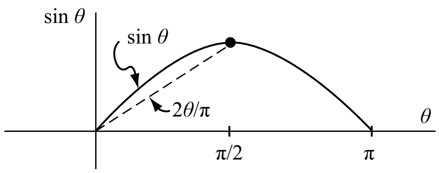

Fig. 7-7

**7.16.** Show that $\displaystyle\int_0^\infty \frac{\cos mx}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}$, $m > 0$.

***Solution***

Consider $\oint_C \{e^{imz}/(z^2+1)\}\,dz$ where $C$ is the contour of Fig. 7-5. The integrand has simple poles at $z = \pm i$, but only $z = i$ lies inside $C$.

Residue at $z = i$ is

$$\lim_{z\to i}\left\{(z-i)\frac{e^{imz}}{(z-i)(z+i)}\right\} = \frac{e^{-m}}{2i}$$

Then

$$\oint_C \frac{e^{imz}}{z^2+1}\,dz = 2\pi i\left(\frac{e^{-m}}{2i}\right) = \pi e^{-m}$$

or

$$\int_{-R}^{R} \frac{e^{imz}}{x^2+1}\,dx + \int_\Gamma \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

that is,

$$\int_{-R}^{R} \frac{\cos mx}{x^2+1}\,dx + i\int_{-R}^{R} \frac{\sin mx}{x^2+1}\,dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

and so

$$2\int_{0}^{R} \frac{\cos mx}{x^2+1}\,dx + \int_{\Gamma} \frac{e^{imz}}{z^2+1}\,dz = \pi e^{-m}$$

Taking the limit as $R \to \infty$ and using Problem 7.15 to show that the integral around $\Gamma$ approaches zero, we obtain the required result.

**7.17.** Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5}\,dx$.

### *Solution*

Consider $\oint_C \{ze^{i\pi z}/(z^2+2z+5)\}\,dz$ where $C$ is the contour of Fig. 7-5. The integrand has simple poles at $z = -1 \pm 2i$, but only $z = -1+2i$ lies inside $C$.

Residue at $z = -1+2i$ is

$$\lim_{z\to -1+2i}\left\{(z+1-2i)\cdot\frac{ze^{i\pi z}}{z^2+2z+5}\right\} = (-1+2i)\frac{e^{-i\pi-2\pi}}{4i}$$

Then

$$\oint_C \frac{ze^{i\pi z}}{z^2+2z+5}\,dz = 2\pi i(-1+2i)\left(\frac{e^{-i\pi-2\pi}}{4i}\right) = \frac{\pi}{2}(1-2i)e^{-2\pi}$$

or

$$\int_{-R}^{R} \frac{xe^{i\pi x}}{x^2+2x+5}\,dx + \int_{\Gamma} \frac{ze^{i\pi z}}{z^2+2z+5}\,dz = \frac{\pi}{2}(1-2i)e^{-2\pi}$$

that is,

$$\int_{-R}^{R} \frac{x\cos \pi x}{x^2+2x+5}\,dx + i\int_{-R}^{R} \frac{x\sin \pi x}{x^2+2x+5}\,dx + \int_{\Gamma} \frac{ze^{i\pi z}}{z^2+2z+5}\,dz = \frac{\pi}{2}(1-2i)e^{-2\pi}$$

Taking the limit as $R \to \infty$ and using Problem 7.15 to show that the integral around $\Gamma$ approaches zero, this becomes

$$\int_{-\infty}^{\infty} \frac{x\cos \pi x}{x^2+2x+5}\,dx + i\int_{-\infty}^{\infty} \frac{x\sin \pi x}{x^2+2x+5}\,dx = \frac{\pi}{2}e^{-2\pi} - i\pi e^{-2\pi}$$

Equating real and imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x\cos \pi x}{x^2+2x+5}\,dx = \frac{\pi}{2}e^{-2\pi}, \qquad \int_{-\infty}^{\infty} \frac{x\sin \pi x}{x^2+2x+5}\,dx = -\pi e^{-2\pi}$$

Thus, we have obtained the value of another integral in addition to the required one.

### Miscellaneous Definite Integrals

**7.18.** Show that $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

#### *Solution*

The method of Problem 7.16 leads us to consider the integral of $e^{iz}/z$ around the contour of Fig. 7-5. However, since $z = 0$ lies on this path of integration and since we cannot integrate through a singularity, we modify that contour by indenting the path at $z = 0$, as shown in Fig. 7-8, which we call contour $C'$ or $ABDEFGHJA$.

Since $z = 0$ is outside $C'$, we have

$$\oint_{C'} \frac{e^{iz}}{z}\,dz = 0$$

or

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\,dx + \int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0$$

Replacing $x$ by $-x$ in the first integral and combining with the third integral, we find

$$\int_{\epsilon}^{R} \frac{e^{ix} - e^{-ix}}{x}\,dx + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0$$

or

$$2i\int_{\epsilon}^{R} \frac{\sin x}{x}\,dx = -\int_{HJA} \frac{e^{iz}}{z}\,dz - \int_{BDEFG} \frac{e^{iz}}{z}\,dz$$

Let $\epsilon \to 0$ and $R \to \infty$. By Problem 7.15, the second integral on the right approaches zero. Letting $z = \epsilon e^{i\theta}$ in the first integral on the right, we see that it approaches

$$-\lim_{\epsilon\to 0}\int_{\pi}^{0} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}}\,i\epsilon e^{i\theta}\,d\theta = -\lim_{\epsilon\to 0}\int_{\pi}^{0} i e^{i\epsilon e^{i\theta}}\,d\theta = \pi i$$

since the limit can be taken under the integral sign.

Then we have

$$\lim_{\substack{R\to\infty \\ \epsilon\to 0}} 2i\int_{\epsilon}^{R} \frac{\sin x}{x}\,dx = \pi i \qquad \text{or} \qquad \int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

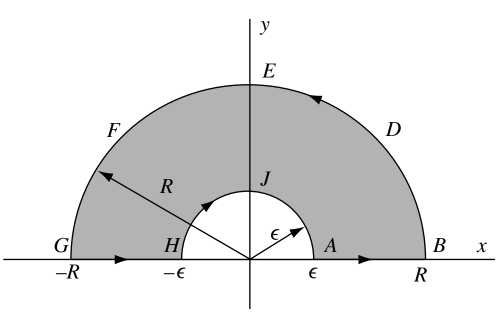


Fig. 7-8

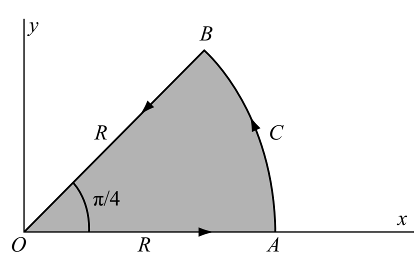


Fig. 7-9

**7.19.** Prove that

$$\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

### *Solution*

Let $C$ be the contour indicated in Fig. 7-9, where $AB$ is the arc of a circle with center at $O$ and radius $R$. By Cauchy's theorem,

$$\oint_C e^{iz^2}\,dz = 0$$

or

$$\int_{OA} e^{iz^2}\,dz + \int_{AB} e^{iz^2}\,dz + \int_{BO} e^{iz^2}\,dz = 0 \qquad (1)$$

Now on $OA$, $z = x$ (from $x = 0$ to $x = R$); on $AB$, $z = Re^{i\theta}$ (from $\theta = 0$ to $\theta = \pi/4$); on $BO$, $z = re^{\pi i/4}$ (from $r = R$ to $r = 0$). Hence from (1),

$$\int_0^R e^{ix^2}\,dx + \int_0^{\pi/4} e^{iR^2e^{2i\theta}} iRe^{i\theta}\,d\theta + \int_R^0 e^{ir^2e^{\pi i/2}} e^{\pi i/4}\,dr = 0 \qquad (2)$$

that is,

$$\int_0^R (\cos x^2 + i\sin x^2)\,dx = e^{\pi i/4}\int_0^R e^{-r^2}\,dr - \int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2\sin 2\theta} iRe^{i\theta}\,d\theta \qquad (3)$$

We consider the limit of (3) as $R \to \infty$. The first integral on the right becomes [see Problem 10.14]

$$e^{\pi i/4}\int_0^\infty e^{-r^2}\,dr = \frac{\sqrt{\pi}}{2}e^{\pi i/4} = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}} \qquad (4)$$

The absolute value of the second integral on the right of (3) is

$$\left|\int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2\sin 2\theta} iRe^{i\theta}\,d\theta\right| \le \int_0^{\pi/4} e^{-R^2\sin 2\theta} R\,d\theta = \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi$$

$$\le \frac{R}{2}\int_0^{\pi/2} e^{-2R^2\phi/\pi}\,d\phi = \frac{\pi}{4R}(1 - e^{-R^2})$$

where we have used the transformation $2\theta = \phi$ and the inequality $\sin\phi \ge 2\phi/\pi$, $0 \le \phi \le \pi/2$ (see Problem 7.15). This shows that as $R \to \infty$, the second integral on the right of (3) approaches zero. Then (3) becomes

$$\int_0^\infty (\cos x^2 + i\sin x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}}$$

and so, equating real and imaginary parts, we have as required,

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

**7.20.** Show that $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi},\ 0 < p < 1.$

### *Solution*

Consider $\oint_C (z^{p-1}/1+z)\,dz$. Since $z=0$ is a branch point, choose $C$ as the contour of Fig. 7-10 where the positive real axis is the branch line and where $AB$ and $GH$ are actually coincident with the $x$ axis but are shown separated for visual purposes.

The integrand has the simple pole $z=-1$ inside $C$.

Residue at $z=-1=e^{\pi i}$ is

$$\lim_{z\to -1}(z+1)\frac{z^{p-1}}{1+z} = (e^{\pi i})^{p-1} = e^{(p-1)\pi i}$$

Then

$$\oint_C \frac{z^{p-1}}{1+z}dz = 2\pi i e^{(p-i)\pi i}$$

or, omitting the integrand,

$$\int_{AB} + \int_{BDEFG} + \int_{GH} + \int_{HJA} = 2\pi i e^{(p-1)\pi i}$$

We thus have

$$\int_\epsilon^R \frac{x^{p-1}}{1+x}dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1} iRe^{i\theta}\,d\theta}{1+Re^{i\theta}} + \int_R^\epsilon \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}}dx + \int_{2\pi}^0 \frac{(\epsilon e^{i\theta})^{p-1} i\epsilon e^{i\theta}\,d\theta}{1+\epsilon e^{i\theta}} = 2\pi i e^{(p-1)\pi i}$$

where we have used $z = xe^{2\pi i}$ for the integral along $GH$, since the argument of $z$ is increased by $2\pi$ in going around the circle $BDEFG$.

Taking the limit as $\epsilon \to 0$ and $R\to\infty$ and nothing that the second and fourth integrals approach zero, we find

$$\int_0^\infty \frac{x^{p-1}}{1+x}dx + \int_\infty^0 \frac{e^{2\pi i(p-1)}x^{p-1}}{1+x}dx = 2\pi e^{(p-1)\pi i}$$

or

$$(1-e^{2\pi i(p-1)})\int_0^\infty \frac{x^{p-1}}{1+x}dx = 2\pi i e^{(p-1)\pi i}$$

so that

$$\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{2\pi i e^{(p-1)\pi i}}{1-e^{2\pi i(p-1)}} = \frac{2\pi i}{e^{p\pi i}-e^{-p\pi i}} = \frac{\pi}{\sin p\pi}$$

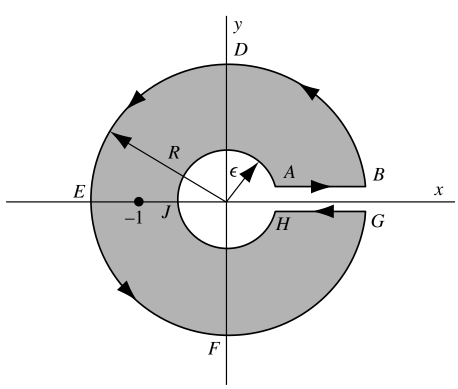


Fig. 7-10

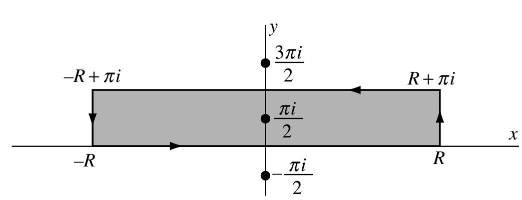


Fig. 7-11

**7.21.** Prove that $\displaystyle\int_0^\infty \frac{\cosh ax}{\cosh x}\,dx = \frac{\pi}{2\cos(\pi a/2)}$ where $|a| < 1$.

### *Solution*

Consider $\oint_C (e^{az}/\cosh z)\,dz$ where $C$ is a rectangle having vertices at $-R$, $R$, $R + \pi i$, $-R + \pi i$ (see Fig. 7-11).

The poles of $e^{az}/\cosh z$ are simple and occur where $\cosh z = 0$, i.e., $z = \left(n + \frac{1}{2}\right)\pi i$, $n = 0, \pm 1, \pm 2, \ldots$. The only pole enclosed by $C$ is $\pi i/2$.

Residue of $e^{az}/\cosh z$ at $z = \pi i/2$ is

$$\lim_{z\to\pi i/2} (z - \pi i/2)\frac{e^{az}}{\cosh z} = \frac{e^{a\pi i/2}}{\sinh(\pi i/2)} = \frac{e^{a\pi i/2}}{i\sin(\pi/2)} = -ie^{a\pi i/2}$$

Then, by the residue theorem,

$$\oint_C \frac{e^{az}}{\cosh z}\,dz = 2\pi i(-ie^{a\pi i/2}) = 2\pi e^{a\pi i/2}$$

This can be written

$$\int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx + \int_0^{\pi} \frac{e^{a(R+iy)}}{\cosh(R+iy)}\,i\,dy + \int_R^{-R} \frac{e^{a(x+\pi i)}}{\cosh(x+\pi i)}\,dx + \int_\pi^0 \frac{e^{a(-R+iy)}}{\cosh(-R+iy)}\,i\,dy = 2\pi e^{a\pi i/2} \qquad (1)$$

As $R \to \infty$, the second and fourth integrals on the left approach zero. To show this, let us consider the second integral. Since

$$|\cosh(R+iy)| = \left|\frac{e^{R+iy} + e^{-R-iy}}{2}\right| \geq \frac{1}{2}\{|e^{R+iy}| - |e^{-R-iy}|\} = \frac{1}{2}(e^R - e^{-R}) \geq \frac{1}{4}e^R$$

we have

$$\left|\int_0^\pi \frac{e^{a(R+iy)}}{\cosh(R+iy)}\,i\,dy\right| \leq \int_0^\pi \frac{e^{aR}}{e^{R/4}}\,dy = 4\pi e^{(a-1)R}$$

and the result follows on noting that the right side approaches zero as $R \to \infty$ since $|a| < 1$. In a similar manner, we can show that the fourth integral on the left of (1) approaches zero as $R \to \infty$. Hence, (1) becomes

$$\lim_{R\to\infty}\left\{\int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx + e^{a\pi i}\int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx\right\} = 2\pi e^{a\pi i/2}$$

since $\cosh(x + \pi i) = -\cosh x$. Thus

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx = \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}\,dx = \frac{2\pi e^{a\pi i/2}}{1 + e^{a\pi i}} = \frac{2\pi}{e^{a\pi i/2} + e^{-a\pi i/2}} = \frac{\pi}{\cos(\pi a/2)}$$

Now

$$\int_{-\infty}^{0} \frac{e^{ax}}{\cosh x} dx + \int_{0}^{\infty} \frac{e^{ax}}{\cosh x} dx = \frac{\pi}{\cos(\pi a/2)}$$

Then, replacing $x$ by $-x$ in the first integral, we have

$$\int_{0}^{\infty} \frac{e^{-ax}}{\cosh x} dx + \int_{0}^{\infty} \frac{e^{ax}}{\cosh x} dx = 2\int_{0}^{\infty} \frac{\cosh ax}{\cosh x} dx = \frac{\pi}{\cos(\pi a/2)}$$

from which the required result follows.

**7.22.** Prove that $\displaystyle\int_{0}^{\infty} \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2.$

### *Solution*

Consider $\oint_C \{\ln(z+i)/z^2+1\}\, dz$ around the contour $C$ consisting of the real axis from $-R$ to $R$ and the semicircle $\Gamma$ of radius $R$ (see Fig. 7-12).

The only pole of $\ln(z+i)/(z^2+1)$ inside $C$ is the simple pole $z = i$, and the residue is

$$\lim_{z \to i} (z-i) \frac{\ln(z+i)}{(z-i)(z+i)} = \frac{\ln(2i)}{2i}$$

Hence, by the residue theorem,

$$\oint_C \frac{\ln(z+i)}{z^2+1} dz = 2\pi i \left\{ \frac{\ln(2i)}{2i} \right\} = \pi \ln(2i) = \pi \ln 2 + \frac{1}{2}\pi^2 i \tag{1}$$

on writing $\ln(2i) = \ln 2 + \ln i = \ln 2 + \ln e^{\pi i/2} = \ln 2 + \pi i/2$ using principal values of the logarithm. The result can be written

$$\int_{-R}^{R} \frac{\ln(x+i)}{x^2+1} dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1} dz = \pi \ln 2 + \frac{1}{2}\pi^2 i$$

or

$$\int_{-R}^{0} \frac{\ln(x+i)}{x^2+1} dx + \int_{0}^{R} \frac{\ln(x+i)}{x^2+1} dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1} dz = \pi \ln 2 + \frac{1}{2}\pi^2 i$$

Replacing $x$ by $-x$ in the first integral, this can be written

$$\int_{0}^{R} \frac{\ln(i-x)}{x^2+1} dx + \int_{0}^{R} \frac{\ln(i+x)}{x^2+1} dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1} dz = \pi \ln 2 + \frac{1}{2}\pi^2 i$$

or, since $\ln(i-x) + \ln(i+x) = \ln(i^2 - x^2) = \ln(x^2+1) + \pi i$,

$$\int_{0}^{R} \frac{\ln(x^2+1)}{x^2+1} dx + \int_{0}^{R} \frac{\pi i}{x^2+1} dx + \int_{\Gamma} \frac{\ln(z+i)}{z^2+1} dz = \pi \ln 2 + \frac{1}{2}\pi^2 i \tag{2}$$

As $R \to \infty$, we can show that the integral around $\Gamma$ approaches zero (see Problem 7.101). Hence, on taking real parts, we find as required,

$$\lim_{R\to\infty}\int_0^R \frac{\ln(x^2+1)}{x^2+1}\,dx = \int_0^\infty \frac{\ln(x^2+1)}{x^2+1}\,dx = \pi \ln 2$$

**7.23.** Prove that $\displaystyle\int_0^{\pi/2} \ln \sin x\,dx = \int_0^{\pi/2} \ln\cos x\,dx = -\frac{1}{2}\pi \ln 2.$

***Solution***

Letting $x = \tan\theta$ in the result of Problem 7.22, we find

$$\int_0^{\pi/2} \frac{\ln(\tan^2\theta + 1)}{\tan^2\theta + 1}\sec^2\theta\,d\theta = -2\int_0^{\pi/2} \ln\cos\theta\,d\theta = \pi\ln 2$$

from which

$$\int_0^{\pi/2} \ln\cos\theta\,d\theta = -\frac{1}{2}\pi\ln 2 \tag{1}$$

which establishes part of the required result. Letting $\theta = \pi/2 - \phi$ in (1), we find

$$\int_0^{\pi/2} \ln\sin\phi\,d\phi = -\frac{1}{2}\pi\ln 2$$

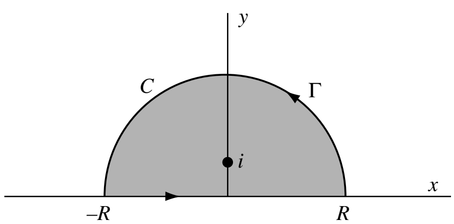

Fig. 7-12

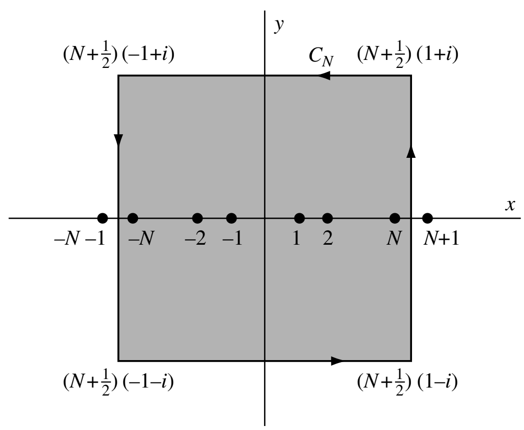

Fig. 7-13

### Summation of Series

**7.24.** Let $C_N$ be a square with vertices at

$$\left(N+\frac{1}{2}\right)(1+i), \quad \left(N+\frac{1}{2}\right)(-1+i), \quad \left(N+\frac{1}{2}\right)(-1-i), \quad \left(N+\frac{1}{2}\right)(1-i)$$

as shown in Fig. 7-13. Prove that on $C_N$, $|\cot \pi z| < A$ where $A$ is a constant.

### Solution

We consider the parts of $C_N$ which lie in the regions $y > \frac{1}{2}$, $-\frac{1}{2} \le y \le \frac{1}{2}$ and $y < -\frac{1}{2}$.

**Case 1:** $y > \frac{1}{2}$. In this case, if $z = x + iy$,

$$|\cot \pi z| = \left|\frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}}\right| = \left|\frac{e^{\pi i x - \pi y} + e^{-\pi i x + \pi y}}{e^{\pi i x - \pi y} - e^{-\pi i x + \pi y}}\right|$$
$$\le \frac{|e^{\pi i x - \pi y}| + |e^{-\pi i x + \pi y}|}{|e^{-\pi i x + \pi y}| - |e^{\pi i x - \pi y}|} = \frac{e^{-\pi y} + e^{\pi y}}{e^{\pi y} - e^{-\pi y}} = \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \le \frac{1 + e^{-\pi}}{1 - e^{-\pi}} = A_1$$

**Case 2:** $y < -\frac{1}{2}$. Here, as in Case 1,

$$|\cot \pi z| \le \frac{|e^{\pi i x - \pi y}| + |e^{-\pi i x + \pi y}|}{|e^{\pi i x - \pi y}| - |e^{-\pi i x + \pi y}|} = \frac{e^{-\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}} = \frac{1 + e^{2\pi y}}{1 - e^{2\pi y}} \le \frac{1 + e^{-\pi}}{1 - e^{-\pi}} = A_1$$

**Case 3:** $-\frac{1}{2} \le y \le \frac{1}{2}$. Consider $z = N + \frac{1}{2} + iy$. Then

$$|\cot \pi z| = |\cot \pi(N + \tfrac{1}{2} + iy)| = |\cot(\pi/2 + \pi i y)| = |\tanh \pi y| \le \tanh(\pi/2) = A_2$$

If $z = -N - \frac{1}{2} + iy$, we have similarly

$$|\cot \pi z| = |\cot \pi(-N - \tfrac{1}{2} + iy)| = |\tanh \pi y| \le \tanh(\pi/2) = A_2$$

Thus, if we choose $A$ as a number greater than the larger of $A_1$ and $A_2$, we have $|\cot \pi z| < A$ on $C_N$ where $A$ is independent of $N$. It is of interest to note that we actually have $|\cot \pi z| \le A_1 = \coth(\pi/2)$ since $A_2 < A_1$.

**7.25.** Let $f(z)$ be such that along the path $C_N$ of Fig. 7-13, $|f(z)| \le M/|z|^k$ where $k > 1$ and $M$ are constants independent of $N$. Prove that

$$\sum_{-\infty}^{\infty} f(n) = -\{\text{sum of residues of } \pi \cot \pi z f(z) \text{ at the poles of } f(z)\}$$

### Solution

**Case 1:** $f(z)$ has a finite number of poles.

In this case, we can choose $N$ so large that the path $C_N$ of Fig. 7-13 encloses all poles of $f(z)$. The poles of $\cot \pi z$ are simple and occur at $z = 0, \pm 1, \pm 2, \ldots$.

Residue of $\pi \cot \pi z f(z)$ at $z = n$, $n = 0, \pm 1, \pm 2, \ldots$, is

$$\lim_{z \to n} (z - n)\pi \cot \pi z f(z) = \lim_{z \to n} \pi\left(\frac{z - n}{\sin \pi z}\right) \cos \pi z f(z) = f(n)$$

using L'Hospital's rule. We have assumed here that $f(z)$ has no poles at $z = n$, since otherwise the given series diverges.

By the residue theorem,

$$\oint_{C_N} \pi \cot \pi z f(z)\, dz = \sum_{n=-N}^{N} f(n) + S \tag{1}$$

where $S$ is the sum of the residues of $\pi \cot \pi z f(z)$ at the poles of $f(z)$. By Problem 7.24 and our assumption on $f(z)$, we have

$$\left|\oint_{C_N} \pi \cot \pi z f(z)\, dz\right| \le \frac{\pi A M}{N^k}(8N + 4)$$

since the length of path $C_N$ is $8N + 4$. Then, taking the limit as $N \to \infty$, we see that

$$\lim_{N\to\infty} \oint_{C_N} \pi \cot \pi z f(z)\, dz = 0 \tag{2}$$

Thus, from (1) we have as required,

$$\sum_{-\infty}^{\infty} f(n) = -S \tag{3}$$

**Case 2:** $f(z)$ has infinitely many poles.

If $f(z)$ has an infinite number of poles, we can obtain the required result by an appropriate limiting procedure. See Problem 7.103.

**7.26.** Prove that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a$ where $a > 0$.

***Solution***

Let $f(z) = 1/(z^2 + a^2)$, which has simple poles at $z = \pm ai$.

Residue of $\pi \cot \pi z/(z^2 + a^2)$ at $z = ai$ is

$$\lim_{z\to ai} (z - ai) \frac{\pi \cot \pi z}{(z - ai)(z + ai)} = \frac{\pi \cot \pi ai}{2ai} = -\frac{\pi}{2a} \coth \pi a$$

Similarly, the residue at $z = -ai$ is $(-\pi/2a)$ coth $\pi a$, and the sum of the residues is $-(\pi/a)$ coth $\pi a$. Then, by Problem 7.25,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = -(\text{sum of residues}) = \frac{\pi}{a} \coth \pi a$$

**7.27.** Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth \pi a - \frac{1}{2a^2}$ where $a > 0$.

***Solution***

The result of Problem 7.26 can be written in the form

$$\sum_{n=-\infty}^{-1} \frac{1}{n^2 + a^2} + \frac{1}{a^2} + \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a$$

or

$$2\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} + \frac{1}{a^2} = \frac{\pi}{a} \coth \pi a$$

which gives the required result.

**7.28.** Prove that $\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots=\dfrac{\pi^2}{6}.$

***Solution***

We have

$$F(z)=\frac{\pi\cot\pi z}{z^2}=\frac{\pi\cos\pi z}{z^2\sin\pi z}=\frac{\left(1-\dfrac{\pi^2z^2}{2!}+\dfrac{\pi^4z^4}{4!}-\cdots\right)}{z^3\left(1-\dfrac{\pi^2z^2}{3!}+\dfrac{\pi^4z^4}{5!}-\cdots\right)}$$

$$=\frac{1}{z^3}\left(1-\frac{\pi^2z^2}{2!}+\cdots\right)\left(1+\frac{\pi^2z^2}{3!}+\cdots\right)=\frac{1}{z^3}\left(1-\frac{\pi^2z^2}{3}+\cdots\right)$$

so that the residue at $z=0$ is $-\pi^{2/3}$.

Then, as in Problems 7.26 and 7.27,

$$\oint_{C_N}\frac{\pi\cot\pi z}{z^2}\,dz=\sum_{n=-N}^{-1}\frac{1}{n^2}+\sum_{n=1}^{N}\frac{1}{n^2}-\frac{\pi^2}{3}=2\sum_{n=1}^{N}\frac{1}{n^2}-\frac{\pi^2}{3}$$

Taking the limit as $N\to\infty$, we have, since the left side approaches zero,

$$2\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{\pi^2}{3}=0\quad\text{or}\quad\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

***Another Method.*** Take the limit as $a\to 0$ in the result of Problem 7.27. Then, using L'Hospital's rule,

$$\lim_{a\to 0}\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}=\lim_{a\to 0}\frac{\pi a\coth\pi a-1}{2a^2}=\frac{\pi^2}{6}$$

**7.29.** Suppose $f(z)$ satisfies the same conditions given in Problem 7.25. Prove that

$$\sum_{-\infty}^{\infty}(-1)^n f(n)=-\{\text{sum of residues of }\pi\csc\pi z f(z)\text{ at the poles of }f(z)\}$$

***Solution***

We proceed in a manner similar to that in Problem 7.25. The poles of $\csc\pi z$ are simple and occur at $z=0,\ \pm1,\ \pm2,\ldots.$

Residue of $\pi\csc\pi z\,f(z)$ at $z=n$, $n=0,\ \pm1,\ \pm2,\ldots$, is

$$\lim_{z\to n}(z-n)\pi\csc\pi z\,f(z)=\lim_{z\to n}\pi\left(\frac{z-n}{\sin\pi z}\right)f(z)=(-1)^n f(n)$$

By the residue theorem,

$$\oint_{C_N}\pi\csc\pi z\,f(z)\,dz=\sum_{n=-N}^{N}(-1)^n f(n)+S \tag{1}$$

where $S$ is the sum of the residues of $\pi\csc\pi z f(z)$ at the poles of $f(z)$.

Letting $N \to \infty$, the integral on the left of (1) approaches zero (Problem 7.106) so that, as required, (1) becomes

$$\sum_{-\infty}^{\infty} (-1)^n f(n) = -S \tag{2}$$

**7.30.** Prove that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$ where $a$ is real and different from $0, \ \pm 1, \ \pm 2, \ldots$.

***Solution***

Let $f(z) = 1/(z+a)^2$ which has a double pole at $z = -a$.

Residue of $\pi \csc \pi z/(z+a)^2$ at $z = -a$ is

$$\lim_{z \to -a} \frac{d}{dz} \left\{ (z+a)^2 \cdot \frac{\pi \csc \pi z}{(z+a)^2} \right\} = -\pi^2 \csc \pi a \cot \pi a$$

Then, by Problem 7.29,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = -(\text{sum of residues}) = \pi^2 \csc \pi a \cot \pi a = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

**7.31.** Suppose $a \neq 0, \ \pm 1, \ \pm 2, \ldots$. Prove that

$$\frac{a^2+1}{(a^2-1)^2} - \frac{a^2+4}{(a^2-4)^2} + \frac{a^2+9}{(a^2-9)^2} - \cdots = \frac{1}{2a^2} - \frac{\pi^2 \cos \pi a}{2 \sin^2 \pi a}$$

***Solution***

The result of Problem 7.30 can be written in the form

$$\frac{1}{a^2} - \left\{ \frac{1}{(a+1)^2} + \frac{1}{(a-1)^2} \right\} + \left\{ \frac{1}{(a+2)^2} + \frac{1}{(a-2)^2} \right\} + \cdots = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

or

$$\frac{1}{a^2} - \frac{2(a^2+1)}{(a^2-1)^2} + \frac{2(a^2+4)}{(a^2-4)^2} - \frac{2(a^2+9)}{(a^2-9)^2} + \cdots = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$$

from which the required result follows. Note that the grouping of terms in the infinite series is permissible since the series is absolutely convergent.

**7.32.** Prove that $\displaystyle\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$.

***Solution***

We have

$$F(z) = \frac{\pi \sec \pi z}{z^3} = \frac{\pi}{z^3 \cos \pi z} = \frac{\pi}{z^3(1 - \pi^2 z^2/2! + \cdots)}$$
$$= \frac{\pi}{z^3}\left(1 + \frac{\pi^2 z^2}{2} + \cdots\right) = \frac{\pi}{z^3} + \frac{\pi^3}{2z} + \cdots$$

so that the residue at $z = 0$ is $\pi^3/2$.

The residue of $F(z)$ at $z = n + \frac{1}{2}$, $n = 0, \pm 1, \pm 2, \ldots$ [which are the simple poles of $\sec \pi z$], is

$$\lim_{z \to n+1/2} \{z - (n + \tfrac{1}{2})\} \frac{\pi}{z^3 \cos \pi z} = \frac{\pi}{(n+\frac{1}{2})^3} \lim_{z \to n+1/2} \frac{z - (n+\frac{1}{2})}{\cos \pi z} = \frac{-(-1)^n}{(n+\frac{1}{2})^3}$$

If $C_N$ is a square with vertices at $N(1+i)$, $N(1-i)$, $N(-1+i)$, $N(-1-i)$, then

$$\oint_{C_N} \frac{\pi \sec \pi z}{z^3} dz = -\sum_{n=-N}^{N} \frac{(-1)^n}{(n+\frac{1}{2})^3} + \frac{\pi^3}{2} = -8 \sum_{n=-N}^{N} \frac{(-1)^n}{(2n+1)^3} + \frac{\pi^3}{2}$$

and since the integral on the left approaches zero as $N \to \infty$, we have

$$\sum_{-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^3} = 2\left\{\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots\right\} = \frac{\pi^3}{16}$$

from which the required result follows.

## Mittag–Leffler's Expansion Theorem

**7.33.** Prove Mittag–Leffler's expansion theorem (see page 209).

***Solution***

Let $f(z)$ have poles at $z = a_n$, $n = 1, 2, \ldots$, and suppose that $z = \zeta$ is not a pole of $f(z)$. Then, the function $f(z)/z - \zeta$ has poles at $z = a_n$, $n = 1, 2, 3, \ldots$ and $\zeta$.

Residue of $f(z)/z - \zeta$ at $z = a_n$, $n = 1, 2, 3, \ldots$, is

$$\lim_{z \to a_n} (z - a_n) \frac{f(z)}{z - \zeta} = \frac{b_n}{a_n - \zeta}$$

Residue of $f(z)/z - \zeta$ at $z = \zeta$ is

$$\lim_{z \to \zeta} (z - \zeta) \frac{f(z)}{z - \zeta} = f(\zeta)$$

Then, by the residue theorem,

$$\frac{1}{2\pi i} \oint_{C_N} \frac{f(z)}{z - \zeta} dz = f(\zeta) + \sum_n \frac{b_n}{a_n - \zeta} \qquad (1)$$

where the last summation is taken over all poles inside circle $C_N$ of radius $R_N$ (Fig. 7-14).

Suppose that $f(z)$ is analytic at $z = 0$. Then, putting $\zeta = 0$ in (1), we have

$$\frac{1}{2\pi i} \oint_{C_N} \frac{f(z)}{z} dz = f(0) + \sum_n \frac{b_n}{a_n} \qquad (2)$$

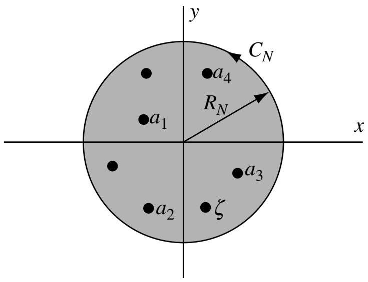

Fig. 7-14

Subtraction of (2) from (1) yields

$$f(\zeta) - f(0) + \sum_n b_n \left(\frac{1}{a_n - \zeta} - \frac{1}{a_n}\right) = \frac{1}{2\pi i} \oint_{C_N} f(z) \left\{\frac{1}{z - \zeta} - \frac{1}{z}\right\} dz$$

$$= \frac{\zeta}{2\pi i} \oint_{C_N} \frac{f(z)}{z(z - \zeta)} dz \qquad (3)$$

Now since $|z - \zeta| \geq |z| - |\zeta| = R_N - |\zeta|$ for $z$ on $C_N$, we have, if $|f(z)| \leq M$,

$$\left| \oint_{C_N} \frac{f(z)}{z(z-\zeta)} dz \right| \leq \frac{M \cdot 2\pi R_N}{R_N(R_N - |\zeta|)}$$

As $N \to \infty$ and therefore $R_N \to \infty$, it follows that the integral on the left approaches zero, i.e.,

$$\lim_{N\to\infty} \oint_{C_N} \frac{f(z)}{z(z-\zeta)} dz = 0$$

Hence from (3), letting $N \to \infty$, we have as required

$$f(\zeta) = f(0) + \sum_n b_n \left( \frac{1}{\zeta - a_n} + \frac{1}{a_n} \right)$$

the result on page 209 being obtained on replacing $\zeta$ by $z$.

**7.34.** Prove that $\cot z = \dfrac{1}{z} + \sum_n \left( \dfrac{1}{z - n\pi} + \dfrac{1}{n\pi} \right)$ where the summation extends over $n = \pm 1, \pm 2, \ldots$.

***Solution***

Consider the function

$$f(z) = \cot z - \frac{1}{z} = \frac{z\cos z - \sin z}{z \sin z}$$

Then $f(z)$ has simple poles at $z = n\pi$, $n = \pm 1, \pm 2, \pm 3, \ldots$, and the residue at these poles is

$$\lim_{z\to n\pi} (z - n\pi)\left( \frac{z\cos z - \sin z}{z\sin z} \right) = \lim_{z\to n\pi} \left( \frac{z - n\pi}{\sin z} \right) \lim_{z\to n\pi} \left( \frac{z\cos z - \sin z}{z} \right) = 1$$

At $z = 0$, $f(z)$ has a removable singularity since

$$\lim_{z\to 0} \left( \cot z - \frac{1}{z} \right) = \lim_{z\to 0} \left( \frac{z\cos z - \sin z}{z\sin z} \right) = 0$$

by L'Hospital's rule. Hence, we can define $f(0) = 0$.

By Problem 7.110, it follows that $f(z)$ is bounded on circles $C_N$ having center at the origin and radius $R_N = (N + \frac{1}{2})\pi$. Hence, by Problem 7.33,

$$\cot z - \frac{1}{z} = \sum_n \left( \frac{1}{z - n\pi} + \frac{1}{n\pi} \right)$$

from which the required result follows.

**7.35.** Prove that $\cot z = \dfrac{1}{z} + 2z\left\{\dfrac{1}{z^2 - \pi^2} + \dfrac{1}{z^2 - 4\pi^2} + \cdots\right\}$.

***Solution***

We can write the result of Problem 7.34 in the form

$$\begin{aligned}
\cot z &= \frac{1}{z} + \lim_{N\to\infty}\left\{\sum_{n=-N}^{-1}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right) + \sum_{n=1}^{N}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)\right\} \\
&= \frac{1}{z} + \lim_{N\to\infty}\left\{\left(\frac{1}{z+\pi} + \frac{1}{z-\pi}\right) + \left(\frac{1}{z+2\pi} + \frac{1}{z-2\pi}\right) + \cdots + \left(\frac{1}{z+N\pi} + \frac{1}{z-N\pi}\right)\right\} \\
&= \frac{1}{z} + \lim_{N\to\infty}\left\{\frac{2z}{z^2-\pi^2} + \frac{2z}{z^2-4\pi^2} + \cdots + \frac{2z}{z^2-N^2\pi^2}\right\} \\
&= \frac{1}{z} + 2z\left\{\frac{1}{z^2-\pi^2} + \frac{1}{z^2-4\pi^2} + \cdots\right\}
\end{aligned}$$

## Miscellaneous Problems

**7.36.** Evaluate $\dfrac{1}{2\pi i}\displaystyle\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z+1}}\, dz$ where $a$ and $t$ are any positive constants.

***Solution***

The integrand has a branch point at $z = -1$. We shall take as a branch line that part of the real axis to the left of $z = -1$. Since we cannot cross this branch line, let us consider

$$\oint_C \frac{e^{zt}}{\sqrt{z+1}}\, dz$$

where $C$ is the contour $ABDEFGHJKA$ shown in Fig. 7-15. In this figure, $EF$ and $HJ$ actually lie on the real axis but have been shown separated for visual purposes. Also, $FGH$ is a circle of radius $\epsilon$ while $BDE$ and $JKA$ represent arcs of a circle of radius $R$.

Since $e^{zt}/\sqrt{z+1}$ is analytic inside and on $C$, we have by Cauchy's theorem

$$\oint_C \frac{e^{zt}}{\sqrt{z+1}}dz = 0 \tag{1}$$

Omitting the integrand, this can be written

$$\int_{AB} + \int_{BDE} + \int_{EF} + \int_{FGH} + \int_{HJ} + \int_{JKA} = 0 \tag{2}$$

Now, on $BDE$ and $JKA$, $z = Re^{i\theta}$ where $\theta$ goes from $\theta_0$ to $\pi$ and $\pi$ to $2\pi - \theta_0$, respectively.

On $EF$, $z + 1 = ue^{\pi i}$, $\sqrt{z+1} = \sqrt{u}e^{\pi i/2} = i\sqrt{u}$; whereas on $HJ$, $z+1 = ue^{-\pi i}$, $\sqrt{z+1} = \sqrt{u}e^{-\pi i/2} = -i\sqrt{u}$. In both cases, $z = -u - 1$, $dz = -du$, where $u$ varies from $R - 1$ to $\epsilon$ along $EF$ and $\epsilon$ to $R - 1$ along $HJ$.

On $FGH$, $z+1=\epsilon e^{i\phi}$ where $\phi$ goes from $-\pi$ to $\pi$. Thus, (2) can be written

$$\int_{a-iT}^{a+iT}\frac{e^{zt}}{\sqrt{z+1}}dz+\int_{\theta_0}^{\pi}\frac{e^{Re^{i\theta}t}}{\sqrt{Re^{i\theta}+1}}iRe^{i\theta}d\theta+\int_{R-1}^{\epsilon}\frac{e^{-(u+1)t}(-du)}{i\sqrt{u}}$$
$$+\int_{\pi}^{-\pi}\frac{e^{(\epsilon e^{i\phi}-1)t}}{\sqrt{\epsilon e^{i\phi}+1}}i\epsilon e^{i\phi}\,d\phi+\int_{\epsilon}^{R-1}\frac{e^{-(u+1)t}(-du)}{-i\sqrt{u}}$$
$$+\int_{\pi}^{2\pi-\theta_0}\frac{e^{Re^{i\theta}t}}{\sqrt{Re^{i\theta}+1}}iRe^{i\theta}d\theta=0 \tag{3}$$

Let us now take the limit as $R\to\infty$ (and $T=\sqrt{R^2-a^2}\to\infty$) and $\epsilon\to 0$. We can show (see Problem 7.111) that the second, fourth, and sixth integrals approach zero. Hence, we have

$$\int_{a-i\infty}^{a+i\infty}\frac{e^{zt}}{\sqrt{z+1}}dz=\lim_{\substack{\epsilon\to 0\\ R\to\infty}}2i\int_{\epsilon}^{R-1}\frac{e^{-(n+1)t}}{\sqrt{u}}du=2i\int_0^{\infty}\frac{e^{-(u+1)t}}{\sqrt{u}}du$$

or letting $u=v^2$,

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{zt}}{\sqrt{z+1}}dz=\frac{1}{\pi}\int_0^{\infty}\frac{e^{-(u+1)t}}{\sqrt{u}}du=\frac{2e^{-t}}{\pi}\int_0^{\infty}e^{-v^2t}\,dv=\frac{e^{-t}}{\sqrt{\pi t}}$$

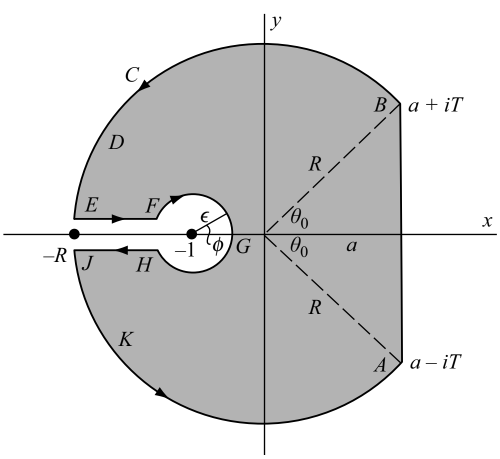


Fig. 7-15

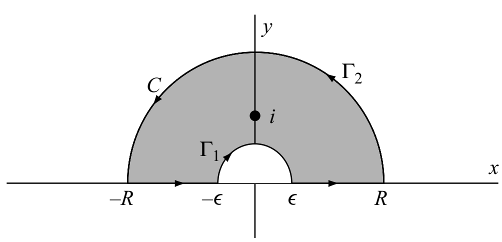


Fig. 7-16

**7.37.** Prove that $\displaystyle\int_0^{\infty}\frac{(\ln u)^2}{u^2+1}du=\frac{\pi^3}{8}$.

***Solution***

Let $C$ be the closed curve of Fig. 7-16 where $\Gamma_1$ and $\Gamma_2$ are semicircles of radii $\epsilon$ and $R$, respectively, and center at the origin. Consider

$$\oint_C\frac{(\ln z)^2}{z^2+1}\,dz$$

Since the integrand has a simple pole $z = i$ inside $C$ and since the residue at this pole is

$$\lim_{z\to i}(z-i)\frac{(\ln z)^2}{(z-i)(z+i)} = \frac{(\ln i)^2}{2i} = \frac{(\pi i/2)^2}{2i} = \frac{-\pi^2}{8i}$$

we have by the residue theorem

$$\oint_C \frac{(\ln z)^2}{z^2+1}\,dz = 2\pi i\left(\frac{-\pi^2}{8i}\right) = \frac{-\pi^3}{4} \tag{1}$$

Now

$$\oint_C \frac{(\ln z)^2}{z^2+1}\,dz = \int_{-R}^{-\epsilon} \frac{(\ln z)^2}{z^2+1}\,dz + \int_{\Gamma_1} \frac{(\ln z)^2}{z^2+1}\,dz + \int_{\epsilon}^{R} \frac{(\ln z)^2}{z^2+1}\,dz + \int_{\Gamma_2} \frac{(\ln z)^2}{z^2+1}\,dz \tag{2}$$

Let $z = -u$ in the first integral on the right so that $\ln z = \ln(-u) = \ln u + \ln(-1) = \ln u + \pi i$ and $dz = -du$. Also, let $z = u$ (so that $dz = du$ and $\ln z = \ln u$) in the third integral on the right. Then, using (1), we have

$$\int_{\epsilon}^{R} \frac{(\ln u + \pi i)^2}{u^2+1}\,du + \int_{\Gamma_1} \frac{(\ln z)^2}{z^2+1}\,dz + \int_{\epsilon}^{R} \frac{(\ln u)^2}{u^2+1}\,du + \int_{\Gamma_2} \frac{(\ln z)^2}{z^2+1}\,dz = \frac{-\pi^3}{4}$$

Now, let $\epsilon \to 0$ and $R \to \infty$. Since the integrals around $\Gamma_1$ and $\Gamma_2$ approach zero, we have

$$\int_0^{\infty} \frac{(\ln u + \pi i)^2}{u^2+1}\,du + \int_0^{\infty} \frac{(\ln u)^2}{u^2+1}\,du = \frac{-\pi^3}{4}$$

or

$$2\int_0^{\infty} \frac{(\ln u)^2}{u^2+1}\,du + 2\pi i\int_0^{\infty} \frac{\ln u}{u^2+1}\,du - \pi^2\int_0^{\infty} \frac{du}{u^2+1} = \frac{-\pi^3}{4}$$

Using the fact that $\displaystyle\int_0^{\infty} \frac{du}{u^2+1} = \tan^{-1} u\Big|_0^{\infty} = \frac{\pi}{2}$,

$$2\int_0^{\infty} \frac{(\ln u)^2}{u^2+1}\,du + 2\pi i\int_0^{\infty} \frac{\ln u}{u^2+1}\,du = \frac{\pi^3}{4}$$

Equating real and imaginary parts, we find

$$\int_0^{\infty} \frac{(\ln u)^2}{u^2+1}\,du = \frac{\pi^3}{8}, \qquad \int_0^{\infty} \frac{\ln u}{u^2+1}\,du = 0$$

the second integral being a by-product of the evaluation.

**7.38.** Prove that

$$\frac{\coth \pi}{1^3} + \frac{\coth 2\pi}{2^3} + \frac{\coth 3\pi}{3^3} + \cdots = \frac{7\pi^3}{180}$$

***Solution***

Consider

$$\oint_{C_N} \frac{\pi \cot \pi z \coth \pi z}{z^3} dz$$

taken around the square $C_N$ shown in Fig. 7-17. The poles of the integrand are located at: $z = 0$ (pole of order 5); $z = \pm 1, \pm 2, \ldots$ (simple poles); $z = \pm i, \pm 2i, \ldots$ (simple poles).

By Problem 7.5 (replacing $z$ by $\pi z$), we see that:

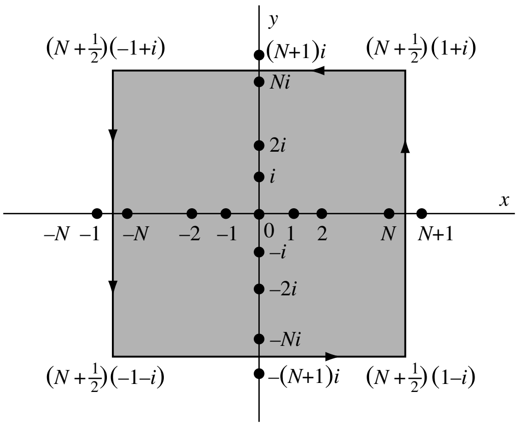


Fig. 7-17

Residue at $z = 0$ is $\dfrac{-7\pi^3}{45}$.

Residue at $z = n\,(n = \pm 1, \pm 2, \ldots)$ is

$$\lim_{z \to n} \left\{ \frac{(z-n)}{\sin \pi z} \cdot \frac{\pi \cos \pi z \coth \pi z}{z^3} \right\} = \frac{\coth n\pi}{n^3}$$

Residue at $z = ni\ (n = \pm 1, \pm 2, \ldots)$ is

$$\lim_{z \to ni} \left\{ \frac{(z-ni)}{\sinh \pi z} \cdot \frac{\pi \cot \pi z \cosh \pi z}{z^3} \right\} = \frac{\coth n\pi}{n^3}$$

Hence, by the residue theorem,

$$\oint_{C_N} \frac{\pi \cot \pi z \coth \pi z}{z^3} dz = \frac{-7\pi^3}{45} + 4 \sum_{n=1}^{N} \frac{\coth n\pi}{n^3}$$

Taking the limit as $N \to \infty$, we find as in Problem 7.25 that the integral on the left approaches zero and the required result follows.

## SUPPLEMENTARY PROBLEMS

### Residues and the Residue Theorem

**7.39.** For each of the following functions, determine the poles and the residues at the poles:

(a) $\dfrac{2z+1}{z^2 - z - 2}$, (b) $\left(\dfrac{z+1}{z-1}\right)^2$, (c) $\dfrac{\sin z}{z^2}$, (d) $\operatorname{sech} z$, (e) $\cot z$.

**7.40.** Prove that $\displaystyle\oint_C \frac{\cosh z}{z^3} dz = \pi i$ if $C$ is the square with vertices at $\pm 2 \pm 2i$.

**7.41.** Show that the residue of $(\csc z \operatorname{csch} z)/z^3$ at $z = 0$ is $-1/60$.

**7.42.** Evaluate $\displaystyle\oint_C \frac{e^z\, dz}{\cosh z}$ around the circle $C$ defined by $|z| = 5$.

**7.43.** Find the zeros and poles of $f(z) = \dfrac{z^2 + 4}{z^3 + 2z^2 + 2z}$ and determine the residues at the poles.

**7.44.** Evaluate $\displaystyle\oint_C e^{-1/z} \sin(1/z)\, dz$ where $C$ is the circle $|z| = 1$.

**7.45.** Let $C$ be a square bounded by $x = \pm 2$, $y = \pm 2$. Evaluate $\oint_C \frac{\sinh 3z}{(z - \pi i/4)^3} dz$.

**7.46.** Evaluate $\oint_C \frac{2z^2 + 5}{(z+2)^3(z^2+4)z^2} dz$ where $C$ is (a) $|z - 2i| = 6$, (b) the square with vertices at $1 + i$, $2 + i$, $2 + 2i$, $1 + 2i$.

**7.47.** Evaluate $\oint_C \frac{2 + 3\sin \pi z}{z(z-1)^2} dz$ where $C$ is a square having vertices at $3 + 3i$, $3 - 3i$, $-3 + 3i$, $-3 - 3i$.

**7.48.** Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z(z^2+1)} dz$, $t > 0$ around the square with vertices at $2 + 2i$, $-2 + 2i$, $-2 - 2i$, $2 - 2i$.

## Definite Integrals

**7.49.** Prove that $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

**7.50.** Evaluate $\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2}$.

**7.51.** Evaluate $\int_0^{2\pi} \frac{\sin 3\theta}{5 - 3\cos\theta} d\theta$.

**7.52.** Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta} d\theta$.

**7.53.** Prove that $\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta} d\theta = \frac{3\pi}{8}$.

**7.54.** Prove that if $m > 0$, $\int_0^\infty \frac{\cos mx}{(x^2+1)^2} dx = \frac{\pi e^{-m}(1+m)}{4}$.

**7.55.** (a) Find the residue of $\frac{e^{iz}}{(z^2+1)^5}$ at $z = i$. (b) Evaluate $\int_0^\infty \frac{\cos x}{(x^2+1)^5} dx$.

**7.56.** Given $a^2 > b^2 + c^2$. Prove that $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2 - c^2}}$.

**7.57.** Prove that $\int_0^{2\pi} \frac{\cos 3\theta}{(5 - 3\cos\theta)^4} d\theta = \frac{135\pi}{16{,}384}$.

**7.58.** Evaluate $\int_0^\infty \frac{dx}{x^4 + x^2 + 1}$.

**7.59.** Evaluate $\int_{-\infty}^\infty \frac{dx}{(x^2 + 4x + 5)^2}$.

**7.60.** Prove that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$.

**7.61.** Discuss the validity of the following solution to Problem 7.19. Let $u = (1+i)x/\sqrt{2}$ in the result $\int_0^\infty e^{-u^2} du = \frac{1}{2}\sqrt{\pi}$ to obtain $\int_0^\infty e^{-ix^2} dx = \frac{1}{2}(1-i)\sqrt{\pi/2}$ from which $\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\pi/2}$ on equating real and imaginary parts.

**7.62.** Show that $\int_0^\infty \frac{\cos 2\pi x}{x^4 + x^2 + 1} dx = \frac{-\pi}{2\sqrt{3}} e^{-\pi/\sqrt{3}}$.

## Summation of Series

**7.63.** Prove that $\sum_{n=1}^\infty \frac{1}{(n^2+1)^2} = \frac{\pi}{4}\coth\pi + \frac{\pi^2}{4}\operatorname{csch}^2\pi - \frac{1}{2}$.

**7.64.** Prove that (a) $\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$, (b) $\sum_{n=1}^\infty \frac{1}{n^6} = \frac{\pi^6}{945}$.

**7.65.** Prove that $\sum_{n=1}^\infty \frac{(-1)^{n-1} n \sin n\theta}{n^2 + \alpha^2} = \frac{\pi}{2}\frac{\sinh \alpha\theta}{\sinh \alpha\pi}$, $-\pi < \theta < \pi$.

**7.66.** Prove that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

**7.67.** Prove that $\sum_{n=-\infty}^{\infty} \frac{1}{n^4+4a^4} = \frac{\pi}{4a^3}\left\{\frac{\sinh 2\pi a + \sin 2\pi a}{\cosh 2\pi a - \cos 2\pi a}\right\}$.

**7.68.** Prove that $\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{(m^2+a^2)(n^2+b^2)} = \frac{\pi^2}{ab}\coth \pi a \coth \pi b$.

### Mittag–Leffler's Expansion Theorem

**7.69.** Prove that $\csc z = \frac{1}{z} - 2z\left(\frac{1}{z^2-\pi^2} - \frac{1}{z^2-4\pi^2} + \frac{1}{z^2-9\pi^2} - \cdots\right)$.

**7.70.** Prove that $\operatorname{sech} z = \pi\left(\frac{1}{(\pi/2)^2+z^2} - \frac{3}{(3\pi/2)^2+z^2} + \frac{5}{(5\pi/2)^2+z^2} - \cdots\right)$.

**7.71.** (a) Prove that $\tan z = 2z\left(\frac{1}{(\pi/2)^2-z^2} + \frac{1}{(3\pi/2)^2-z^2} + \frac{1}{(5\pi/2)^2-z^2} + \cdots\right)$.

(b) Use the result in (a) to show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

**7.72.** Prove the expansions (a) 2, (b) 4, (c) 5, (d) 7, (e) 8 on page 209.

**7.73.** Prove that $\sum_{k=1}^{\infty} \frac{1}{z^2+4k^2\pi^2} = \frac{1}{2z}\left\{\frac{1}{2} - \frac{1}{z} + \frac{1}{e^z-1}\right\}$.

**7.74.** Prove that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$.

### Miscellaneous Problems

**7.75.** Prove that Cauchy's theorem and integral formulas can be obtained as special cases of the residue theorem.

**7.76.** Prove that the sum of the residues of the function $\frac{2z^5-4z^2+5}{3z^6-8z+10}$ at all the poles is 2/3.

**7.77.** Let $n$ be a positive integer. Prove that $\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\,d\theta = 2\pi/n!$.

**7.78.** Evaluate $\oint_C z^3 e^{1/z}\,dz$ around the circle $C$ with equation $|z-1| = 4$.

**7.79.** Prove that under suitably stated conditions on the function:

(a) $\int_0^{2\pi} f(e^{i\theta})\,d\theta = 2\pi f(0)$, (b) $\int_0^{2\pi} f(e^{i\theta})\cos\theta\,d\theta = -\pi f'(0)$.

**7.80.** Show that: (a) $\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)\,d\theta = 2\pi$ (b) $\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\cos\theta\,d\theta = \pi$.

**7.81.** Prove that $\int_0^{\infty} \frac{\sin ax}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\frac{a}{2} - \frac{1}{2a}$.

[*Hint.* Integrate $e^{aiz}/(e^{2\pi z}-1)$ around a rectangle with vertices at $0, R, R+i, i$ and let $R \to \infty$.]

**7.82.** Prove that $\int_0^{\infty} \frac{\sin ax}{e^x+1}\,dx = \frac{1}{2a} - \frac{\pi}{2\sinh \pi a}$.

**7.83.** Given $a$, $p$, and $t$ are positive constants. Prove that $\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{z^2+p^2}\,dz = \frac{\sin pt}{p}$.

**7.84.** Prove that $\int_0^{\infty} \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi \ln a}{2a}$.

**7.85.** Suppose $-\pi < a < \pi$. Prove that $\displaystyle\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\sinh ax}{\sinh \pi x}\,dx = \frac{\sin a}{\cos a + \cosh \lambda}$.

**7.86.** Prove that $\displaystyle\int_0^{\infty} \frac{dx}{(4x^2+\pi^2)\cosh x} = \frac{\ln 2}{2\pi}$.

**7.87.** Prove that (a) $\displaystyle\int_0^{\infty} \frac{\ln x}{x^4+1}\,dx = \frac{-\pi^2\sqrt{2}}{16}$, (b) $\displaystyle\int_0^{\infty} \frac{(\ln x)^2}{x^4+1}\,dx = \frac{3\pi^3\sqrt{2}}{64}$.

[*Hint.* Consider $\displaystyle\oint_C \frac{(\ln z)^2}{z^4+1}\,dz$ around a semicircle properly indented at $z = 0$.]

**7.88.** Evaluate $\displaystyle\int_0^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx$.

**7.89.** Prove that if $|a| < 1$ and $b > 0$, $\displaystyle\int_0^{\infty} \frac{\sinh ax}{\sinh x}\cos bx\,dx = \frac{\pi}{2}\left(\frac{\sin a\pi}{\cos a\pi + \cosh b\pi}\right)$.

**7.90.** Prove that if $-1 < p < 1$, $\displaystyle\int_0^{\infty} \frac{\cos px}{\cosh x}\,dx = \frac{\pi}{2\cosh(p\pi/2)}$.

**7.91.** Prove that $\displaystyle\int_0^{\infty} \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi \ln 2}{2}$.

**7.92.** Suppose $\alpha > 0$ and $-\pi/2 < \beta < \pi/2$. Prove that

(a) $\displaystyle\int_0^{\infty} e^{-\alpha x^2\cos\beta}\cos(\alpha x^2\sin\beta)\,dx = \tfrac{1}{2}\sqrt{\pi/\alpha}\cos(\beta/2)$.

(b) $\displaystyle\int_0^{\infty} e^{-\alpha x^2\cos\beta}\sin(\alpha x^2\sin\beta)\,dx = \tfrac{1}{2}\sqrt{\pi/\alpha}\sin(\beta/2)$.

**7.93.** Prove that $\displaystyle\csc^2 z = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n\pi)^2}$.

**7.94.** Suppose $a$ and $p$ are real and such that $0 < |p| < 1$ and $0 < |\alpha| < z$. Prove that

$$\int_0^{\infty} \frac{x^{-p}\,dx}{x^2 + 2x\cos\alpha + 1} = \left(\frac{\pi}{\sin p\pi}\right)\left(\frac{\sin p\alpha}{\sin\alpha}\right)$$

**7.95.** Prove $\displaystyle\int_0^{1} \frac{dx}{\sqrt[3]{x^2-x^3}} = \frac{2\pi}{\sqrt{3}}$. [Consider contour of Fig. 7-18.]

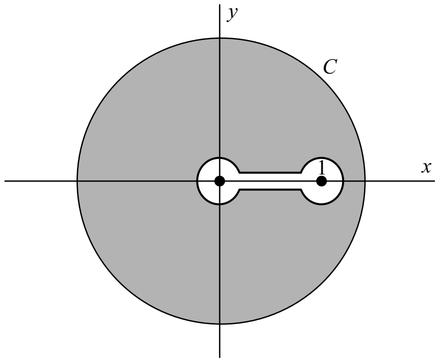

Fig. 7-18

**7.96.** Prove the residue theorem for multiply-connected regions.

**7.97.** Find sufficient conditions under which the residue theorem (Problem 7.2) is valid if $C$ encloses infinitely many isolated singularities.

**7.98.** Let $C$ be a circle with equation $|z| = 4$. Determine the value of the integral

$$\oint_C z^2 \csc\frac{1}{z}\,dz$$

if it exists.

**7.99.** Give an analytical proof that $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \pi/2$.

[*Hint.* Consider the derivative of $(\sin\theta)/\theta$, showing that it is a decreasing function.]

**7.100.** Prove that $\displaystyle\int_0^\infty \frac{x}{\sinh \pi x}\,dx = \frac{1}{4}$.

**7.101.** Verify that the integral around $\Gamma$ in equation (2) of Problem 7.22 goes to zero as $R \to \infty$.

**7.102.** (a) Suppose $r$ is real. Prove that $\displaystyle\int_0^\pi \ln(1 - 2r\cos\theta + r^2)\,d\theta = \begin{cases} 0 & \text{if } |r| \leq 1 \\ \pi \ln r^2 & \text{if } |r| \geq 1 \end{cases}$.

(b) Use the result in (a) to evaluate $\displaystyle\int_0^{\pi/2} \ln\sin\theta\,d\theta$ (see Problem 7.23).

**7.103.** Complete the proof of Case 2 in Problem 7.25.

**7.104.** Let $0 < p < 1$. Prove that $\displaystyle\int_0^\infty \frac{x^{-p}}{x-1}\,dx = \pi\cot p\pi$ in the Cauchy principal value sense.

**7.105.** Show that $\sum_{n=-\infty}^{\infty} \dfrac{1}{n^4 + n^2 + 1} = \dfrac{\pi\sqrt{3}}{3}\tanh\left(\dfrac{\pi\sqrt{3}}{2}\right)$.

**7.106.** Verify that as $N \to \infty$, the integral on the left of (1) in Problem 7.29 goes to zero.

**7.107.** Prove that $\dfrac{1}{1^5} - \dfrac{1}{3^5} + \dfrac{1}{5^5} - \dfrac{1}{7^5} + \cdots = \dfrac{5\pi^5}{1536}$.

**7.108.** Prove the results given on page 209 for (a) $\displaystyle\sum_{-\infty}^{\infty} f\left(\frac{2n+1}{2}\right)$ and (b) $\displaystyle\sum_{-\infty}^{\infty} (-1)^n f\left(\frac{2n+1}{2}\right)$.

**7.109.** Given $-\pi \leq \theta \leq \pi$. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sin n\theta}{n^3} = \frac{\theta(\pi-\theta)(\pi+\theta)}{12}$.

**7.110.** Prove that the function $\cot z - 1/z$ of Problem 7.34 is bounded on the circles $C_N$.

**7.111.** Show that the second, fourth, and sixth integrals in equation (3) of Problem 7.36 approach zero as $\epsilon \to 0$ and $R \to \infty$.

**7.112.** Prove that $\dfrac{1}{\cosh(\pi/2)} - \dfrac{1}{3\cosh(3\pi/2)} + \dfrac{1}{5\cosh(5\pi/2)} - \cdots = \dfrac{\pi}{8}$.

**7.113.** Prove that $\displaystyle\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{\sqrt{z}}\,dz = \frac{1}{\sqrt{\pi t}}$ where $a$ and $t$ are any positive constants.

**7.114.** Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^7} = \frac{19\pi^7}{56{,}700}$.

**7.115.** Prove that $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)\cosh \pi x} = \frac{4-\pi}{2}$.

**7.116.** Prove that $\dfrac{1}{1^3\sinh\pi} - \dfrac{1}{2^3\sinh 2\pi} + \dfrac{1}{3^3\sinh 3\pi} - \cdots = \dfrac{\pi^3}{360}$.

**7.117.** Prove that if $a$ and $t$ are any positive constants,

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{zt}\cot^{-1} z\,dz = \frac{\sin t}{t}$$