10.5.16 (a) Starting with a one-dimensional inhomogeneous differential equation (Eq. (10.89)), assume that $\psi(x)$ and $\rho(x)$ may be represented by eigenfunction expansions. Without any use of the Dirac delta function or its representations, show that

$$\psi(x) = \sum_{n=0}^{\infty} \frac{\int_a^b \rho(t)\varphi_n(t)\,dt}{\lambda_n - \lambda}\varphi_n(x).$$

Note that (1) if $\rho = 0$, no solution exists unless $\lambda = \lambda_n$ and (2) if $\lambda = \lambda_n$, no solution exists unless ρ is orthogonal to φ_n. This same behavior will reappear with integral equations in Section 16.4.

(b) Interchanging summation and integration, show that you have constructed the Green's function corresponding to Eq. (10.90).

10.5.17 The eigenfunctions of the Schrödinger equation are often complex. In this case the orthogonality integral, Eq. (10.40), is replaced by

$$\int_a^b \varphi_i^*(x)\varphi_j(x)w(x)\,dx = \delta_{ij}.$$

Instead of Eq. (1.189), we have

$$\delta(\mathbf{r}_1 - \mathbf{r}_2) = \sum_{n=0}^{\infty} \varphi_n(\mathbf{r}_1)\varphi_n^*(\mathbf{r}_2).$$

Show that the Green's function, Eq. (10.87), becomes

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=0}^{\infty} \frac{\varphi_n(\mathbf{r}_1)\varphi_n^*(\mathbf{r}_2)}{k_n^2 - k^2} = G^*(\mathbf{r}_2, \mathbf{r}_1).$$

Additional Readings

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CHAPTER 11

BESSEL FUNCTIONS

11.1 BESSEL FUNCTIONS OF THE FIRST KIND, $J_\nu(x)$

Bessel functions appear in a wide variety of physical problems. In Section 9.3, separation of the Helmholtz, or wave, equation in circular cylindrical coordinates led to Bessel's equation. In Section 11.7 we will see that the Helmholtz equation in spherical polar coordinates also leads to a form of Bessel's equation. Bessel functions may also appear in integral form — integral representations. This may result from integral transforms (Chapter 15) or from the mathematical elegance of starting the study of Bessel functions with Hankel functions, Section 11.4.

Bessel functions and closely related functions form a rich area of mathematical analysis with many representations, many interesting and useful properties, and many interrelations. Some of the major interrelations are developed in Section 11.1 and in succeeding sections. Note that Bessel functions are not restricted to Chapter 11. The asymptotic forms are developed in Section 7.3 as well as in Section 11.6. The confluent hypergeometric representations appear in Section 13.5.

Generating Function for Integral Order

Although Bessel functions are of interest primarily as solutions of differential equations, it is instructive and convenient to develop them from a completely different approach, that of the generating function.[1] This approach also has the advantage of focusing on the functions themselves rather than on the differential equations they satisfy. Let us introduce a function of two variables,

$$g(x, t) = e^{(x/2)(t-1/t)}. \tag{11.1}$$

[1]Generating functions have already been used in Chapter 5. In Section 5.6 the generating function $(1+x)^n$ was used to derive the binomial coefficients. In Section 5.9 the generating function $x(e^x - 1)^{-1}$ was used to derive the Bernoulli numbers.

Expanding this function in a Laurent series (Section 6.5), we obtain

$$\boxed{e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n.} \tag{11.2}$$

It is instructive to compare Eq. (11.2) with the equivalent Eqs. (11.23) and (11.25).

The coefficient of t^n, $J_n(x)$, is defined to be a Bessel function of the first kind, of integral order n. Expanding the exponentials, we have a product of Maclaurin series in $xt/2$ and $-x/2t$, respectively,

$$e^{xt/2} \cdot e^{-x/2t} = \sum_{r=0}^{\infty} \left(\frac{x}{2}\right)^r \frac{t^r}{r!} \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^s \frac{t^{-s}}{s!}. \tag{11.3}$$

Here, the summation index r is changed to n, with $n = r - s$ and summation limits $n = -s$ to ∞, and the order of the summations is interchanged, which is justified by absolute convergence. The range of the summation over n becomes $-\infty$ to ∞, while the summation over s extends from $\max(-n, 0)$ to ∞. For a given s we get $t^n (n \geq 0)$ from $r = n + s$:

$$\left(\frac{x}{2}\right)^{n+s} \frac{t^{n+s}}{(n+s)!} (-1)^s \left(\frac{x}{2}\right)^s \frac{t^{-s}}{s!}. \tag{11.4}$$

The coefficient of t^n is then[2]

$$\boxed{J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s} = \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}(n+1)!} + \cdots.} \tag{11.5}$$

This series form exhibits is behavior of the Bessel function $J_n(x)$ for small x and permits numerical evaluation of $J_n(x)$. The results for J_0, J_1, and J_2 are shown in Fig. 11.1. From Section 5.3 the error in using only a finite number of terms of this alternating series in numerical evaluation is less than the first term omitted. For instance, if we want $J_n(x)$

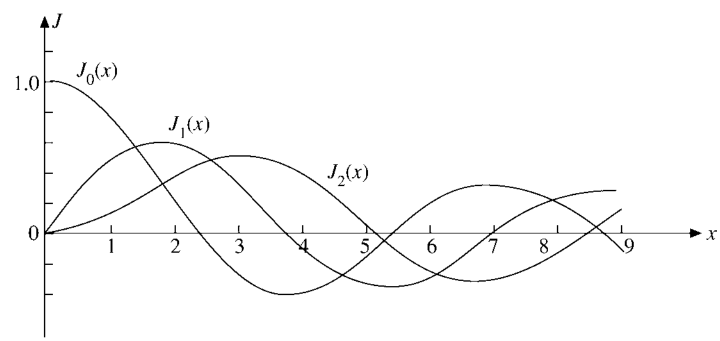

FIGURE 11.1 Bessel functions, $J_0(x)$, $J_1(x)$, and $J_2(x)$.

[2]From the steps leading to this series and from its convergence characteristics it should be clear that this series may be used with x replaced by z and with z any point in the finite complex plane.

to ±1% accuracy, the first term alone of Eq. (11.5) will suffice, provided the ratio of the second term to the first is less than 1% (in magnitude) or $x < 0.2(n+1)^{1/2}$. The Bessel functions oscillate but are **not** periodic — except in the limit as $x \to \infty$ (Section 11.6). The amplitude of $J_n(x)$ is not constant but decreases asymptotically as $x^{-1/2}$. (See Eq.(11.137) for this envelope.)

For $n < 0$, Eq. (11.5) gives

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s-n)!} \left(\frac{x}{2}\right)^{2s-n}. \tag{11.6}$$

Since n is an integer (here), $(s-n)! \to \infty$ for $s = 0, \ldots, (n-1)$. Hence the series may be considered to start with $s = n$. Replacing s by $s+n$, we obtain

$$J_{-n}(x) = \sum_{s=0}^{\infty} \frac{(-1)^{s+n}}{s!(s+n)!} \left(\frac{x}{2}\right)^{n+2s}, \tag{11.7}$$

showing immediately that $J_n(x)$ and $J_{-n}(x)$ are not independent but are related by

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (\text{integral } n). \tag{11.8}$$

These series expressions (Eqs. (11.5) and (11.6)) may be used with n replaced by ν to **define** $J_\nu(x)$ and $J_{-\nu}(x)$ for nonintegral ν (compare Exercise 11.1.7).

Recurrence Relations

The recurrence relations for $J_n(x)$ and its derivatives may all be obtained by operating on the series, Eq. (11.5), although this requires a bit of clairvoyance (or a lot of trial and error). Verification of the known recurrence relations is straightforward, Exercise 11.1.7. Here it is convenient to obtain them from the generating function, $g(x,t)$. Differentiating both sides of Eq. (11.1) with respect to t, we find that

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= \frac{1}{2} x \left(1 + \frac{1}{t^2}\right) e^{(x/2)(t-1/t)} \\ &= \sum_{n=-\infty}^{\infty} n J_n(x) t^{n-1}, \end{aligned} \tag{11.9}$$

and substituting Eq. (11.2) for the exponential and equating the coefficients of like powers of t,[3] we obtain

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x). \tag{11.10}$$

This is a three-term recurrence relation. Given J_0 and J_1, for example, J_2 (and any other integral order J_n) may be computed.

[3]This depends on the fact that the power-series representation is unique (Sections 5.7 and 6.5).

Differentiating Eq. (11.1) with respect to x, we have

$$\frac{\partial}{\partial x} g(x,t) = \frac{1}{2}\left(t - \frac{1}{t}\right) e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n'(x) t^n. \tag{11.11}$$

Again, substituting in Eq. (11.2) and equating the coefficients of like powers of t, we obtain the result

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x). \tag{11.12}$$

As a special case of this general recurrence relation,

$$J_0'(x) = -J_1(x). \tag{11.13}$$

Adding Eqs. (11.10) and (11.12) and dividing by 2, we have

$$J_{n-1}(x) = \frac{n}{x} J_n(x) + J_n'(x). \tag{11.14}$$

Multiplying by x^n and rearranging terms produces

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x). \tag{11.15}$$

Subtracting Eq. (11.12) from Eq. (11.10) and dividing by 2 yields

$$J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x). \tag{11.16}$$

Multiplying by x^{-n} and rearranging terms, we obtain

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x). \tag{11.17}$$

Bessel's Differential Equation

Suppose we consider a set of functions $Z_\nu(x)$ that satisfies the basic recurrence relations (Eqs. (11.10) and (11.12)), but with ν not necessarily an integer and Z_ν not necessarily given by the series (Eq. (11.5)). Equation (11.14) may be rewritten ($n \to \nu$) as

$$xZ_\nu'(x) = xZ_{\nu-1}(x) - \nu Z_\nu(x). \tag{11.18}$$

On differentiating with respect to x, we have

$$xZ_\nu''(x) + (\nu+1)Z_\nu' - xZ_{\nu-1}' - Z_{\nu-1} = 0. \tag{11.19}$$

Multiplying by x and then subtracting Eq. (11.18) multiplied by ν gives us

$$x^2 Z_\nu'' + xZ_\nu' - \nu^2 Z_\nu + (\nu-1)xZ_{\nu-1} - x^2 Z_{\nu-1}' = 0. \tag{11.20}$$

Now we rewrite Eq. (11.16) and replace n by $\nu - 1$:

$$xZ_{\nu-1}' = (\nu-1)Z_{\nu-1} - xZ_\nu. \tag{11.21}$$

Using Eq. (11.21) to eliminate $Z_{\nu-1}$ and $Z_{\nu-1}'$ from Eq. (11.20), we finally get

$$\boxed{x^2 Z_\nu'' + xZ_\nu' + \left(x^2 - \nu^2\right) Z_\nu = 0,} \tag{11.22}$$

which is **Bessel**'s ODE. Hence any functions $Z_\nu(x)$ that satisfy the recurrence relations (Eqs. (11.10) and (11.12), (11.14) and (11.16), or (11.15) and (11.17)) satisfy Bessel's equation; that is, the unknown Z_ν are Bessel functions. In particular, we have shown that the functions $J_n(x)$, defined by our generating function, satisfy Bessel's ODE. If the argument is $k\rho$ rather than x, Eq. (11.22) becomes

$$\boxed{\rho^2 \frac{d^2}{d\rho^2} Z_\nu(k\rho) + \rho \frac{d}{d\rho} Z_\nu(k\rho) + \left(k^2\rho^2 - \nu^2\right) Z_\nu(k\rho) = 0.} \tag{11.22a}$$

Integral Representation

A particularly useful and powerful way of treating Bessel functions employs integral representations. If we return to the generating function (Eq. (11.2)), and substitute $t = e^{i\theta}$, we get

$$\begin{aligned} e^{ix\sin\theta} &= J_0(x) + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right] \\ &\quad + 2i\left[J_1(x)\sin\theta + J_3(x)\sin 3\theta + \cdots\right], \end{aligned} \tag{11.23}$$

in which we have used the relations

$$\begin{aligned} J_1(x)e^{i\theta} + J_{-1}(x)e^{-i\theta} &= J_1(x)\left(e^{i\theta} - e^{-i\theta}\right) \\ &= 2iJ_1(x)\sin\theta, \\ J_2(x)e^{2i\theta} + J_{-2}(x)e^{-2i\theta} &= 2J_2(x)\cos 2\theta, \end{aligned} \tag{11.24}$$

and so on.

In summation notation,

$$\begin{aligned} \cos(x\sin\theta) &= J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta), \\ \sin(x\sin\theta) &= 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin\left[(2n-1)\theta\right], \end{aligned} \tag{11.25}$$

equating real and imaginary parts of Eq. (11.23).

By employing the orthogonality properties of cosine and sine,[4]

$$\int_0^\pi \cos n\theta \cos m\theta \, d\theta = \frac{\pi}{2}\delta_{nm}, \tag{11.26a}$$

$$\int_0^\pi \sin n\theta \sin m\theta \, d\theta = \frac{\pi}{2}\delta_{nm}, \tag{11.26b}$$

[4]They are eigenfunctions of a self-adjoint equation (linear oscillator equation) and satisfy appropriate boundary conditions (compare Sections 10.2 and 14.1).

in which n and m are **positive** integers (zero is excluded),[5] we obtain

$$\frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\cos n\theta\,d\theta = \begin{cases} J_n(x), & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \tag{11.27}$$

$$\frac{1}{\pi}\int_0^{\pi}\sin(x\sin\theta)\sin n\theta\,d\theta = \begin{cases} 0, & n \text{ even}, \\ J_n(x), & n \text{ odd}. \end{cases} \tag{11.28}$$

If these two equations are added together,

$$\begin{aligned} J_n(x) &= \frac{1}{\pi}\int_0^{\pi}\big[\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta\big]\,d\theta \\ &= \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta, \qquad n = 0, 1, 2, 3, \ldots. \end{aligned} \tag{11.29}$$

As a special case (integrate Eq. (11.25) over $(0, \pi)$ to get)

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\,d\theta. \tag{11.30}$$

Noting that $\cos(x\sin\theta)$ repeats itself in all four quadrants, we may write Eq. (11.30) as

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta)\,d\theta. \tag{11.30a}$$

On the other hand, $\sin(x\sin\theta)$ reverses its sign in the third and fourth quadrants, so

$$\frac{1}{2\pi}\int_0^{2\pi}\sin(x\sin\theta)\,d\theta = 0. \tag{11.30b}$$

Adding Eq. (11.30a) and i times Eq. (11.30b), we obtain the complex exponential representation

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta}\,d\theta. \tag{11.30c}$$

This integral representation (Eq. (11.29)) may be obtained somewhat more directly by employing contour integration (compare Exercise 11.1.16).[6] Many other integral representations exist (compare Exercise 11.1.18).

Example 11.1.1 Fraunhofer Diffraction, Circular Aperture

In the theory of diffraction through a circular aperture we encounter the integral

$$\Phi \sim \int_0^a r\,dr \int_0^{2\pi} e^{ibr\cos\theta}\,d\theta \tag{11.31}$$

[5]Equations (11.26a) and (11.26b) hold for either m or $n = 0$. If both m and $n = 0$, the constant in (11.26a) becomes π; the constant in Eq. (11.26b) becomes 0.

[6]For $n = 0$ a simple integration over θ from 0 to 2π will convert Eq. (11.23) into Eq. (11.30c).

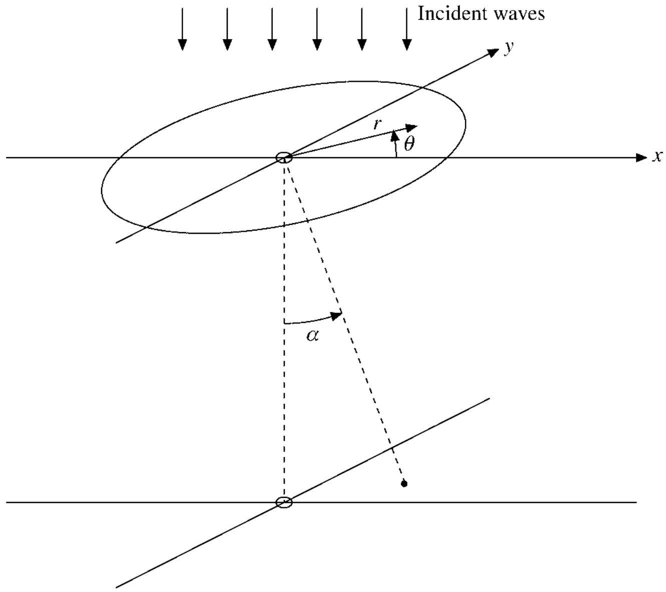

FIGURE 11.2 Fraunhofer diffraction, circular aperture.

for Φ, the amplitude of the diffracted wave.[7] Here θ is an azimuth angle in the plane of the circular aperture of radius a, and α is the angle defined by a point on a screen below the circular aperture relative to the normal through the center point. The parameter b is given by

$$b = \frac{2\pi}{\lambda} \sin\alpha, \tag{11.32}$$

with λ the wavelength of the incident wave. The other symbols are defined by Fig. 11.2. From Eq. (11.30c) we get[8]

$$\Phi \sim 2\pi \int_0^a J_0(br) r\, dr. \tag{11.33}$$

Equation (11.15) enables us to integrate Eq. (11.33) immediately to obtain

$$\Phi \sim \frac{2\pi ab}{b^2} J_1(ab) \sim \frac{\lambda a}{\sin\alpha} J_1\left(\frac{2\pi a}{\lambda}\sin\alpha\right). \tag{11.34}$$

Note here that $J_1(0) = 0$. The intensity of the light in the diffraction pattern is proportional to Φ^2 and

$$\Phi^2 \sim \left\{\frac{J_1[(2\pi a/\lambda)\sin\alpha]}{\sin\alpha}\right\}^2. \tag{11.35}$$

[7] The exponent $ibr\cos\theta$ gives the phase of the wave on the distant screen at angle α relative to the phase of the wave incident on the aperture at the point (r, θ). The imaginary exponential form of this integrand means that the integral is technically a Fourier transform, Chapter 15. In general, the Fraunhofer diffraction pattern is given by the Fourier transform of the aperture.

[8] We could also refer to Exercise 11.1.16(b).

Table 11.1 Zeros of the Bessel Functions and Their First Derivatives

Number of zero	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178
	$J_0'(x)^a$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$		
1	3.8317	1.8412	3.0542	4.2012		
2	7.0156	5.3314	6.7061	8.0152		
3	10.1735	8.5363	9.9695	11.3459		

[a] $J_0'(x) = -J_1(x)$.

From Table 11.1, which lists the zeros of the Bessel functions and their first derivatives,[9] Eq. (11.35) will have a zero at

$$\frac{2\pi a}{\lambda}\sin\alpha = 3.8317\ldots, \tag{11.36}$$

or

$$\sin\alpha = \frac{3.8317\lambda}{2\pi a}. \tag{11.37}$$

For green light, $\lambda = 5.5 \times 10^{-5}$ cm. Hence, if $a = 0.5$ cm,

$$\alpha \approx \sin\alpha = 6.7 \times 10^{-5} \text{ (radian)} \approx 14 \text{ seconds of arc}, \tag{11.38}$$

which shows that the bending or spreading of the light ray is extremely small. If this analysis had been known in the seventeenth century, the arguments against the wave theory of light would have collapsed. In mid-twentieth century this same diffraction pattern appears in the scattering of nuclear particles by atomic nuclei — a striking demonstration of the wave properties of the nuclear particles. ■

A further example of the use of Bessel functions and their roots is provided by the electromagnetic resonant cavity (Example 11.1.2) and the example and exercises of Section 11.2.

Example 11.1.2 Cylindrical Resonant Cavity

The propagation of electromagnetic waves in hollow metallic cylinders is important in many practical devices. If the cylinder has end surfaces, it is called a **cavity**. Resonant cavities play a crucial role in many particle accelerators.

[9] Additional roots of the Bessel functions and their first derivatives may be found in C. L. Beattie, Table of first 700 zeros of Bessel functions. *Bell Syst. Tech. J.* **37**: 689 (1958), and Bell Monogr. **3055**. Roots may be accessed in Mathematica and other symbolic software and are on the Web.

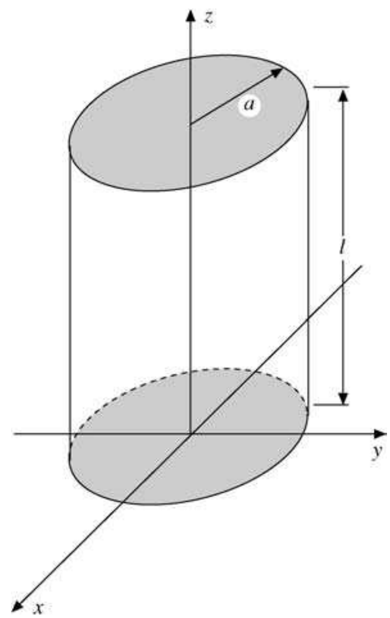

FIGURE 11.3 Cylindrical resonant cavity.

We take the z-axis along the center of the cavity with end surfaces at $z = 0$ and $z = l$ and use cylindrical coordinates suggested by the geometry. Its walls are perfect conductors, so the tangential electric field vanishes on them (as in Fig. 11.3):

$$E_z = 0 = E_\varphi \qquad \text{for } \rho = a, \qquad E_\rho = 0 = E_\varphi \qquad \text{for } z = 0, l.$$

Inside the cavity we have a vacuum, so $\varepsilon_0\mu_0 = 1/c^2$. In the interior of a resonant cavity, electromagnetic waves oscillate with harmonic time dependence $e^{-i\omega t}$, which follows from separating the time from the spatial variables in Maxwell's equations (Section 1.9), so

$$\nabla \times \nabla \times \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \alpha^2 \mathbf{E}, \qquad \alpha = \frac{\omega}{c}.$$

With $\nabla \cdot \mathbf{E} = 0$ (vacuum, no charges) and Eq. (1.85), we obtain for the space part of the electric field

$$\nabla^2 \mathbf{E} + \alpha^2 \mathbf{E} = 0,$$

which is called the **vector Helmholtz PDE**. The z-component (E_z, space part only) satisfies the scalar Helmholtz equation,

$$\nabla^2 E_z + \alpha^2 E_z = 0. \tag{11.39}$$

The transverse electric field components $\mathbf{E}_\perp = (E_\rho, E_\varphi)$ obey the same PDE but different boundary conditions, given earlier. Once E_z is known, Maxwell's equations determine E_φ fully. See Jackson, *Electrodynamics* in Additional Readings for details.

We separate the z variable from ρ and φ, because there are no mixed derivatives $\frac{\partial^2 E_z}{\partial z\,\partial\rho}$, etc. The product solution, $E_z = v(\rho,\varphi)w(z)$, is substituted into the Helmholtz PDE for E_z using Eq. (2.35) for ∇^2 in cylindrical coordinates, and then we divide by vw, yielding

$$\frac{1}{w(z)}\frac{d^2w}{dz^2} + \frac{1}{v}\left(\frac{\partial^2 v}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial\varphi^2} + \alpha^2\right)v(\rho,\varphi) = 0.$$

This implies

$$-\frac{1}{w(z)}\frac{d^2w}{dz^2} = \frac{1}{v(\rho,\varphi)}\left(\frac{\partial^2 v}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial\varphi^2} + \alpha^2 v\right) = k^2.$$

Here, k^2 is a separation constant, because the left- and right-hand sides depend on different variables. For $w(z)$ we find the harmonic oscillator ODE with standing wave solution (not transients) that we seek,

$$w(z) = A\sin kz + B\cos kz,$$

with A, B constants. For $v(\rho,\varphi)$ we obtain

$$\frac{\partial^2 v}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial\varphi^2} + \gamma^2 v = 0, \qquad \gamma^2 = \alpha^2 - k^2.$$

In this PDE we can separate the ρ and φ variables, because there is no mixed term $\frac{\partial^2 v}{\partial\rho\,\partial\varphi}$. The product form $v = u(\rho)\Phi(\varphi)$ yields

$$\frac{\rho^2}{u(\rho)}\left(\frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} + \gamma^2\right) = -\frac{1}{\Phi(\varphi)}\frac{d^2\Phi}{d\varphi^2} = m^2,$$

where the **separation constant m^2 must be an integer**, because the angular solution $\Phi = e^{im\varphi}$ of the ODE

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0$$

must be periodic in the azimuthal angle.

This leaves us with the radial ODE

$$\frac{d^2u}{d\rho^2} + \frac{1}{\rho}\frac{du}{d\rho} + \left(\gamma^2 - \frac{m^2}{\rho^2}\right)u = 0.$$

Dimensional arguments suggest rescaling $\rho \to r = \gamma\rho$ and dividing by γ^2, which yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(1 - \frac{m^2}{r^2}\right)u = 0.$$

This is Bessel's ODE for $\nu = m$. We use the regular solution $J_m(\gamma\rho)$ because the (irregular) second independent solution is singular at the origin, which is unacceptable here. The complete solution is

$$E_z = J_m(\gamma\rho)e^{im\varphi}(A\sin kz + B\cos kz), \tag{11.40a}$$

where the constant γ is determined from the **boundary condition** $E_z = 0$ on the cavity surface $\rho = a$, that is, that γa be a root of the Bessel function J_m (see Table 11.1). This gives a discrete set of values $\gamma = \gamma_{mn}$, where n designates the nth root of J_m (see Table 11.1).

For the transverse magnetic or TM mode of oscillation with $H_z = 0$ Maxwell's equations imply. (See again *Resonant Cavities* in J. D. Jackson's *Electrodynamics* in Additional Readings.)

$$\mathbf{E}_\perp \sim \nabla_\perp \frac{\partial E_z}{\partial z}, \qquad \nabla_\perp = \left(\frac{\partial}{\partial \rho}, \frac{1}{\rho}\frac{\partial}{\partial \varphi}\right).$$

The form of this result suggests $E_z \sim \cos kz$, that is, setting $A = 0$ so that $\mathbf{E}_\perp \sim \sin kz = 0$ at $z = 0, l$ can be satisfied by

$$k = \frac{p\pi}{l}, \qquad p = 0, 1, 2, \ldots. \tag{11.41}$$

Thus, the **tangential** electric fields E_ρ and E_φ vanish at $z = 0$ and l. In other words, $A = 0$ corresponds to $dE_z/dz = 0$ at $z = 0$ and $z = l$ for the TM mode. Altogether then, we have

$$\gamma^2 = \frac{\omega^2}{c^2} - k^2 = \frac{\omega^2}{c^2} - \frac{p^2\pi^2}{l^2}, \tag{11.42}$$

with

$$\gamma = \gamma_{mn} = \frac{\alpha_{mn}}{a}, \tag{11.43}$$

where α_{mn} is the nth zero of J_m. The general solution

$$E_z = \sum_{m,n,p} J_m(\gamma_{mn}\rho) e^{\pm im\varphi} B_{mnp} \cos\frac{p\pi z}{l}, \tag{11.40b}$$

with constants B_{mnp}, now follows from the superposition principle.

The result of the two boundary conditions and the separation constant m^2 is that the angular frequency of our oscillation depends on three discrete parameters:

$$\omega_{mnp} = c\sqrt{\frac{\alpha_{mn}^2}{a^2} + \frac{p^2\pi^2}{l^2}}, \qquad \begin{cases} m = 0, 1, 2, \ldots, \\ n = 1, 2, 3, \ldots, \\ p = 0, 1, 2 \ldots. \end{cases} \tag{11.44}$$

These are the allowable resonant frequencies for our TM mode. The TE mode of oscillation is the topic of Exercise 11.1.26. ■

Alternate Approaches

Bessel functions are introduced here by means of a generating function, Eq. (11.2). Other approaches are possible. Listing the various possibilities, we have:

1. Generating function (magic), Eq. (11.2).
2. Series solution of Bessel's differential equation, Section 9.5.
3. Contour integrals: Some writers prefer to start with contour integral definitions of the Hankel functions, Section 7.3 and 11.4, and develop the Bessel function $J_\nu(x)$ from the Hankel functions.

4. Direct solution of physical problems: Example 11.1.1. Fraunhofer diffraction with a circular aperture, illustrates this. Incidentally, Eq. (11.31) can be treated by series expansion, if desired. Feynman[10] develops Bessel functions from a consideration of cavity resonators.

In case the generating function seems too arbitrary, it can be derived from a contour integral, Exercise 11.1.16, or from the Bessel function recurrence relations, Exercise 11.1.6. Note that the contour integral is not limited to integer ν, thus providing a starting point for developing Bessel functions.

Bessel Functions of Nonintegral Order

These different approaches are not exactly equivalent. The generating function approach is very convenient for deriving two recurrence relations, Bessel's differential equation, integral representations, addition theorems (Exercise 11.1.2), and upper and lower bounds (Exercise 11.1.1). However, you will probably have noticed that the generating function defined only Bessel functions of integral order, J_0, J_1, J_2, and so on. This is a limitation of the generating function approach that can be avoided by using the contour integral in Exercise 11.1.16 instead, thus leading to foregoing approach (3). But the Bessel function of the first kind, $J_\nu(x)$, may easily be defined for nonintegral ν by using the series (Eq. (11.5)) as a new definition.

The recurrence relations may be verified by substituting in the series form of $J_\nu(x)$ (Exercise 11.1.7). From these relations Bessel's equation follows. In fact, if ν is not an integer, there is actually an important simplification. It is found that J_ν and $J_{-\nu}$ are independent, for no relation of the form of Eq. (11.8) exists. On the other hand, for $\nu = n$, an integer, we need another solution. The development of this second solution and an investigation of its properties form the subject of Section 11.3.

Exercises

11.1.1 From the product of the generating functions $g(x,t) \cdot g(x,-t)$ show that

$$1 = \left[J_0(x)\right]^2 + 2\left[J_1(x)\right]^2 + 2\left[J_2(x)\right]^2 + \cdots$$

and therefore that $|J_0(x)| \le 1$ and $|J_n(x)| \le 1/\sqrt{2}$, $n = 1, 2, 3, \ldots$.
Hint. Use uniqueness of power series, Section 5.7.

11.1.2 Using a generating function $g(x,t) = g(u+v,t) = g(u,t) \cdot g(v,t)$, show that

(a) $$J_n(u+v) = \sum_{s=-\infty}^{\infty} J_s(u) \cdot J_{n-s}(v),$$

(b) $$J_0(u+v) = J_0(u)J_0(v) + 2\sum_{s=1}^{\infty} J_s(u)J_{-s}(v).$$

[10] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. II. Reading, MA: Addison-Wesley (1964), Chapter 23.

These are addition theorems for the Bessel functions.

11.1.3 Using only the generating function

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$$

and not the explicit series form of $J_n(x)$, show that $J_n(x)$ has odd or even parity according to whether n is odd or even, that is,[11]

$$J_n(x) = (-1)^n J_n(-x).$$

11.1.4 Derive the Jacobi–Anger expansion

$$e^{iz\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(z)e^{im\theta}.$$

This is an expansion of a plane wave in a series of cylindrical waves.

11.1.5 Show that

(a) $\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$,

(b) $\sin x = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$.

11.1.6 To help remove the generating function from the realm of magic, show that it can be **derived** from the recurrence relation, Eq. (11.10).
Hint.

(a) Assume a generating function of the form

$$g(x,t) = \sum_{m=-\infty}^{\infty} J_m(x)t^m.$$

(b) Multiply Eq. (11.10) by t^n and sum over n.
(c) Rewrite the preceding result as

$$\left(t + \frac{1}{t}\right)g(x,t) = \frac{2t}{x}\frac{\partial g(x,t)}{\partial t}.$$

(d) Integrate and adjust the "constant" of integration (a function of x) so that the coefficient of the zeroth power, t^0, is $J_0(x)$, as given by Eq. (11.5).

11.1.7 Show, by direct differentiation, that

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+\nu)!}\left(\frac{x}{2}\right)^{\nu+2s}$$

[11] This is easily seen from the series form (Eq. (11.5)).

satisfies the two recurrence relations

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x),$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x),$$

and Bessel's differential equation

$$x^2 J''_\nu(x) + x J'_\nu(x) + \left(x^2 - \nu^2\right) J_\nu(x) = 0.$$

11.1.8 Prove that

$$\frac{\sin x}{x} = \int_0^{\pi/2} J_0(x\cos\theta)\cos\theta\, d\theta, \qquad \frac{1-\cos x}{x} = \int_0^{\pi/2} J_1(x\cos\theta)\, d\theta.$$

Hint. The definite integral

$$\int_0^{\pi/2} \cos^{2s+1}\theta\, d\theta = \frac{2\cdot 4\cdot 6\cdots(2s)}{1\cdot 3\cdot 5\cdots(2s+1)}$$

may be useful.

11.1.9 Show that

$$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}} dt.$$

This integral is a Fourier cosine transform (compare Section 15.3). The corresponding Fourier sine transform,

$$J_0(x) = \frac{2}{\pi}\int_1^\infty \frac{\sin xt}{\sqrt{t^2-1}} dt,$$

is established in Section 11.4 (Exercise 11.4.6) using a Hankel function integral representation.

11.1.10 Derive

$$J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x).$$

Hint. Try mathematical induction.

11.1.11 Show that between any two consecutive zeros of $J_n(x)$ there is one and only one zero of $J_{n+1}(x)$.
Hint. Equations (11.15) and (11.17) may be useful.

11.1.12 An analysis of antenna radiation patterns for a system with a circular aperture involves the equation

$$g(u) = \int_0^1 f(r) J_0(ur) r\, dr.$$

If $f(r) = 1 - r^2$, show that

$$g(u) = \frac{2}{u^2} J_2(u).$$

11.1.13 The differential cross section in a nuclear scattering experiment is given by $d\sigma/d\Omega = |f(\theta)|^2$. An approximate treatment leads to

$$f(\theta) = \frac{-ik}{2\pi}\int_0^{2\pi}\int_0^R \exp[ik\rho\sin\theta\sin\varphi]\rho\,d\rho\,d\varphi.$$

Here θ is an angle through which the scattered particle is scattered. R is the nuclear radius. Show that

$$\frac{d\sigma}{d\Omega} = \left(\pi R^2\right)\frac{1}{\pi}\left[\frac{J_1(kR\sin\theta)}{\sin\theta}\right]^2.$$

11.1.14 A set of functions $C_n(x)$ satisfies the recurrence relations

$$C_{n-1}(x) - C_{n+1}(x) = \frac{2n}{x}C_n(x),$$

$$C_{n-1}(x) + C_{n+1}(x) = 2C_n'(x).$$

(a) What linear second-order ODE does the $C_n(x)$ satisfy?
(b) By a change of variable transform your ODE into Bessel's equation. This suggests that $C_n(x)$ may be expressed in terms of Bessel functions of transformed argument.

11.1.15 A particle (mass m) is contained in a right circular cylinder (pillbox) of radius R and height H. The particle is described by a wave function satisfying the Schrödinger wave equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\rho,\varphi,z) = E\psi(\rho,\varphi,z)$$

and the condition that the wave function go to zero over the surface of the pillbox. Find the lowest (zero point) permitted energy.

$$\text{ANS. } E = \frac{\hbar^2}{2m}\left[\left(\frac{z_{pq}}{R}\right)^2 + \left(\frac{n\pi}{H}\right)^2\right],$$

$$E_{\min} = \frac{\hbar^2}{2m}\left[\left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2\right],$$

where z_{pq} is the qth zero of J_p and the index p is fixed by the azimuthal dependence.

11.1.16 (a) Show by direct differentiation and substitution that

$$J_\nu(x) = \frac{1}{2\pi i}\int_C e^{(x/2)(t-1/t)}t^{-\nu-1}\,dt$$

or that the equivalent equation,

$$J_\nu(x) = \frac{1}{2\pi i}\left(\frac{x}{2}\right)^\nu\int e^{s-x^2/4s}s^{-\nu-1}\,ds,$$

satisfies Bessel's equation. C is the contour shown in Fig. 11.4. The negative real axis is the cut line.

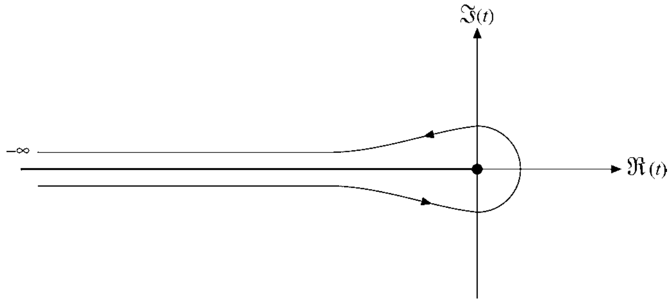

FIGURE 11.4 Bessel function contour.

Hint. Show that the total integrand (after substituting in Bessel's differential equation) may be written as a total derivative:

$$\frac{d}{dt}\left\{\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]t^{-\nu}\left[\nu+\frac{x}{2}\left(t+\frac{1}{t}\right)\right]\right\}.$$

(b) Show that the first integral (with n an integer) may be transformed into

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\theta - n\theta)}\,d\theta = \frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{i(x\cos\theta + n\theta)}\,d\theta.$$

11.1.17 The contour C in Exercise 11.1.16 is deformed to the path $-\infty$ to -1, unit circle $e^{-i\pi}$ to $e^{i\pi}$, and finally -1 to $-\infty$. Show that

$$J_\nu(x) = \frac{1}{\pi}\int_0^{\pi}\cos(\nu\theta - x\sin\theta)\,d\theta - \frac{\sin\nu\pi}{\pi}\int_0^{\infty} e^{-\nu\theta - x\sinh\theta}\,d\theta.$$

This is Bessel's integral.
Hint. The negative values of the variable of integration u may be handled by using

$$u = te^{\pm ix}.$$

11.1.18 (a) Show that

$$J_\nu(x) = \frac{2}{\pi^{1/2}(\nu-\frac{1}{2})!}\left(\frac{x}{2}\right)^{\nu}\int_0^{\pi/2}\cos(x\sin\theta)\cos^{2\nu}\theta\,d\theta,$$

where $\nu > -\frac{1}{2}$.
Hint. Here is a chance to use series expansion and term-by-term integration. The formulas of Section 8.4 will prove useful.

(b) Transform the integral in part (a) into

$$\begin{aligned} J_\nu(x) &= \frac{1}{\pi^{1/2}(\nu-\frac{1}{2})!}\left(\frac{x}{2}\right)^\nu \int_0^\pi \cos(x\cos\theta)\sin^{2\nu}\theta\, d\theta \\ &= \frac{1}{\pi^{1/2}(\nu-\frac{1}{2})!}\left(\frac{x}{2}\right)^\nu \int_0^\pi e^{\pm ix\cos\theta}\sin^{2\nu}\theta\, d\theta \\ &= \frac{1}{\pi^{1/2}(\nu-\frac{1}{2})!}\left(\frac{x}{2}\right)^\nu \int_{-1}^1 e^{\pm ipx}(1-p^2)^{\nu-1/2}\, dp. \end{aligned}$$

These are alternate integral representations of $J_\nu(x)$.

11.1.19 (a) From

$$J_\nu(x) = \frac{1}{2\pi i}\left(\frac{x}{2}\right)^\nu \int t^{-\nu-1}e^{t-x^2/4t}\, dt$$

derive the recurrence relation

$$J'_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x).$$

(b) From

$$J_\nu(x) = \frac{1}{2\pi i}\int t^{-\nu-1}e^{(x/2)(t-1/t)}\, dt$$

derive the recurrence relation

$$J'_\nu(x) = \tfrac{1}{2}\bigl[J_{\nu-1}(x) - J_{\nu+1}(x)\bigr].$$

11.1.20 Show that the recurrence relation

$$J'_n(x) = \tfrac{1}{2}\bigl[J_{n-1}(x) - J_{n+1}(x)\bigr]$$

follows directly from differentiation of

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta.$$

11.1.21 Evaluate

$$\int_0^\infty e^{-ax}J_0(bx)\, dx, \qquad a, b > 0.$$

Actually the results hold for $a \geq 0$, $-\infty < b < \infty$. This is a Laplace transform of J_0. *Hint*. Either an integral representation of J_0 or a series expansion will be helpful.

11.1.22 Using trigonometric forms, verify that

$$J_0(br) = \frac{1}{2\pi}\int_0^{2\pi} e^{ibr\sin\theta}\, d\theta.$$

11.1.23 (a) Plot the intensity (Φ^2 of Eq. (11.35)) as a function of $(\sin\alpha/\lambda)$ along a diameter of the circular diffraction pattern. Locate the first two minima.

(b) What fraction of the total light intensity falls within the central maximum?
Hint. $[J_1(x)]^2/x$ may be written as a derivative and the area integral of the intensity integrated by inspection.

11.1.24 The fraction of light incident on a circular aperture (normal incidence) that is transmitted is given by

$$T = 2\int_0^{2ka} J_2(x)\frac{dx}{x} - \frac{1}{2ka}\int_0^{2ka} J_2(x)\,dx.$$

Here a is the radius of the aperture and k is the wave number, $2\pi/\lambda$. Show that

$$\text{(a)}\quad T = 1 - \frac{1}{ka}\sum_{n=0}^{\infty} J_{2n+1}(2ka), \qquad \text{(b)}\quad T = 1 - \frac{1}{2ka}\int_0^{2ka} J_0(x)\,dx.$$

11.1.25 The amplitude $U(\rho, \varphi, t)$ of a vibrating circular membrane of radius a satisfies the wave equation

$$\nabla^2 U - \frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} = 0.$$

Here v is the phase velocity of the wave fixed by the elastic constants and whatever damping is imposed.

(a) Show that a solution is

$$U(\rho, \varphi, t) = J_m(k\rho)\big(a_1 e^{im\varphi} + a_2 e^{-im\varphi}\big)\big(b_1 e^{i\omega t} + b_2 e^{-i\omega t}\big).$$

(b) From the Dirichlet boundary condition, $J_m(ka) = 0$, find the allowable values of the wavelength $\lambda (k = 2\pi/\lambda)$.

Note. There are other Bessel functions besides J_m, but they all diverge at $\rho = 0$. This is shown explicitly in Section 11.3. The divergent behavior is actually implicit in Eq. (11.6).

11.1.26 Example 11.1.2 describes the TM modes of electromagnetic cavity oscillation. The transverse electric (TE) modes differ, in that we work from the z component of the magnetic induction $\mathbf{B}$:

$$\nabla^2 B_z + \alpha^2 B_z = 0$$

with boundary conditions

$$B_z(0) = B_z(l) = 0 \qquad \text{and} \qquad \left.\frac{\partial B_z}{\partial \rho}\right|_{\rho=0} = 0.$$

Show that the TE resonant frequencies are given by

$$\omega_{mnp} = c\sqrt{\frac{\beta_{mn}^2}{a^2} + \frac{p^2\pi^2}{l^2}}, \qquad p = 1, 2, 3, \ldots.$$

11.1.27 Plot the three lowest TM and the three lowest TE angular resonant frequencies, ω_{mnp}, as a function of the radius/length (a/l) ratio for $0 \le a/l \le 1.5$.
Hint. Try plotting ω^2 (in units of c^2/a^2) versus $(a/l)^2$. Why this choice?

11.1.28 A thin conducting disk of radius a carries a charge q. Show that the potential is described by

$$\varphi(r,z) = \frac{q}{4\pi\varepsilon_0 a}\int_0^\infty e^{-k|z|} J_0(kr)\frac{\sin ka}{k}\,dk,$$

where J_0 is the usual Bessel function and r and z are the familiar cylindrical coordinates.
Note. This is a difficult problem. One approach is through Fourier transforms such as Exercise 15.3.11. For a discussion of the physical problem see Jackson (*Classical Electrodynamics* in Additional Readings).

11.1.29 Show that

$$\int_0^a x^m J_n(x)\,dx, \qquad m \ge n \ge 0,$$

(a) is integrable in terms of Bessel functions and powers of x (such as $a^p J_q(a)$) for $m+n$ odd;
(b) may be reduced to integrated terms plus $\int_0^a J_0(x)dx$ for $m+n$ even.

11.1.30 Show that

$$\int_0^{\alpha_{0n}}\left(1-\frac{y}{\alpha_{0n}}\right)J_0(y)y\,dy = \frac{1}{\alpha_{0n}}\int_0^{\alpha_{0n}} J_0(y)\,dy.$$

Here α_{0n} is the nth root of $J_0(y)$. This relation is useful (see Exercise 11.2.11): The expression on the right is easier and quicker to evaluate — and much more accurate. Taking the difference of two terms in the expression on the left leads to a large relative error.

11.1.31 The circular aperature diffraction amplitude Φ of Eq. (17.35) is proportional to $f(z) = J_1(z)/z$. The corresponding single slit diffraction amplitude is proportional to $g(z) = \sin z/z$.

(a) Calculate and plot $f(z)$ and $g(z)$ for $z = 0.0(0.2)12.0$.
(b) Locate the two lowest values of $z (z > 0)$ for which $f(z)$ takes on an extreme value. Calculate the corresponding values of $f(z)$.
(c) Locate the two lowest values of $z (z > 0)$ for which $g(z)$ takes on an extreme value. Calculate the corresponding values of $g(z)$.

11.1.32 Calculate the electrostatic potential of a charged disk $\varphi(r,z)$ from the integral form of Exercise 11.1.28. Calculate the potential for $r/a = 0.0(0.5)2.0$ and $z/a = 0.25(0.25)1.25$. Why is $z/a = 0$ omitted? Exercise 12.3.17 is a spherical harmonic version of this same problem.

11.2 ORTHOGONALITY

If Bessel's equation, Eq. (11.22a), is divided by ρ, we see that it becomes self-adjoint, and therefore, by the Sturm–Liouville theory, Section 10.2, the solutions are expected to be orthogonal — if we can arrange to have appropriate boundary conditions satisfied. To take care of the boundary conditions for a finite interval $[0, a]$, we introduce parameters a and $\alpha_{\nu m}$ into the argument of J_ν to get $J_\nu(\alpha_{\nu m}\rho/a)$. Here a is the upper limit of the cylindrical radial coordinate ρ. From Eq. (11.22a),

$$\rho\frac{d^2}{d\rho^2}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)+\frac{d}{d\rho}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)+\left(\frac{\alpha_{\nu m}^2\rho}{a^2}-\frac{\nu^2}{\rho}\right)J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)=0. \tag{11.45}$$

Changing the parameter $\alpha_{\nu m}$ to $\alpha_{\nu n}$, we find that $J_\nu(\alpha_{\nu n}\rho/a)$ satisfies

$$\rho\frac{d^2}{d\rho^2}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)+\frac{d}{d\rho}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)+\left(\frac{\alpha_{\nu n}^2\rho}{a^2}-\frac{\nu^2}{\rho}\right)J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)=0. \tag{11.45a}$$

Proceeding as in Section 10.2, we multiply Eq. (11.45) by $J_\nu(\alpha_{\nu n}\rho/a)$ and Eq. (11.45a) by $J_\nu(\alpha_{\nu m}\rho/a)$ and subtract, obtaining

$$\begin{aligned}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]-J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\right]\\ =\frac{\alpha_{\nu n}^2-\alpha_{\nu m}^2}{a^2}\rho J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right).\end{aligned} \tag{11.46}$$

Integrating from $\rho = 0$ to $\rho = a$, we obtain

$$\begin{aligned}\int_0^a J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]d\rho-\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\right]d\rho\\ =\frac{\alpha_{\nu n}^2-\alpha_{\nu m}^2}{a^2}\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\rho\,d\rho.\end{aligned} \tag{11.47}$$

Upon integrating by parts, we see that the left-hand side of Eq. (11.47) becomes

$$\left|\rho J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\frac{d}{d\rho}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right|_0^a-\left|\rho J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\frac{d}{d\rho}J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\right|_0^a. \tag{11.48}$$

For $\nu \ge 0$ the factor ρ guarantees a zero at the lower limit, $\rho = 0$. Actually the lower limit on the index ν may be extended down to $\nu > -1$, Exercise 11.2.4.[12] At $\rho = a$, each expression vanishes if we choose the parameters $\alpha_{\nu n}$ and $\alpha_{\nu m}$ to be zeros, or roots of J_ν; that is, $J_\nu(\alpha_{\nu m}) = 0$. The subscripts now become meaningful: $\alpha_{\nu m}$ is the mth zero of J_ν.

With this choice of parameters, the left-hand side vanishes (the Sturm–Liouville boundary conditions are satisfied) and for $m \ne n$,

$$\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\rho\,d\rho = 0. \tag{11.49}$$

This gives us orthogonality over the interval $[0, a]$.

[12] The case $\nu = -1$ reverts to $\nu = +1$, Eq. (11.8).

Normalization

The normalization integral may be developed by returning to Eq. (11.48), setting $\alpha_{\nu n} = \alpha_{\nu m} + \varepsilon$, and taking the limit $\varepsilon \to 0$ (compare Exercise 11.2.2). With the aid of the recurrence relation, Eq. (11.16), the result may be written as

$$\int_0^a \left[J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]^2 \rho\, d\rho = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu m})\right]^2. \tag{11.50}$$

Bessel Series

If we assume that the set of Bessel functions $J_\nu(\alpha_{\nu m}\rho/a))(\nu$ fixed, $m = 1, 2, 3, \ldots)$ is complete, then any well-behaved but otherwise arbitrary function $f(\rho)$ may be expanded in a Bessel series (Bessel–Fourier or Fourier–Bessel)

$$f(\rho) = \sum_{m=1}^{\infty} c_{\nu m} J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right), \qquad 0 \le \rho \le a, \qquad \nu > -1. \tag{11.51}$$

The coefficients $c_{\nu m}$ are determined by using Eq. (11.50),

$$c_{\nu m} = \frac{2}{a^2[J_{\nu+1}(\alpha_{\nu m})]^2}\int_0^a f(\rho) J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\rho\, d\rho. \tag{11.52}$$

A similar series expansion involving $J_\nu(\beta_{\nu m}\rho/a)$ with $(d/d\rho)J_\nu(\beta_{\nu m}\rho/a)|_{\rho=a} = 0$ is included in Exercises 11.2.3 and 11.2.6(b).

Example 11.2.1 ELECTROSTATIC POTENTIAL IN A HOLLOW CYLINDER

From Table 9.3 of Section 9.3 (with α replaced by k), our solution of Laplace's equation in circular cylindrical coordinates is a linear combination of

$$\psi_{km}(\rho, \varphi, z) = J_m(k\rho)[a_m \sin m\varphi + b_m \cos m\varphi]\left[c_1 e^{kz} + c_2 e^{-kz}\right]. \tag{11.53}$$

The particular linear combination is determined by the boundary conditions to be satisfied. Our cylinder here has a radius a and a height l. The top end section has a potential distribution $\psi(\rho, \varphi)$. Elsewhere on the surface the potential is zero.[13] The problem is to find the electrostatic potential

$$\psi(\rho, \varphi, z) = \sum_{k,m} \psi_{km}(\rho, \varphi, z) \tag{11.54}$$

everywhere in the interior.

For convenience, the circular cylindrical coordinates are placed as shown in Fig. 11.3. Since $\psi(\rho, \varphi, 0) = 0$, we take $c_1 = -c_2 = \frac{1}{2}$. The z dependence becomes $\sinh kz$, vanishing at $z = 0$. The requirement that $\psi = 0$ on the cylindrical sides is met by requiring the separation constant k to be

$$k = k_{mn} = \frac{\alpha_{mn}}{a}, \tag{11.55}$$

[13] If $\psi = 0$ at $z = 0, l$, but $\psi \neq 0$ for $\rho = a$, the modified Bessel functions, Section 11.5, are involved.

where the first subscript, m, gives the index of the Bessel function, whereas the second subscript identifies the particular zero of J_m.

The electrostatic potential becomes

$$\psi(\rho, \varphi, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\alpha_{mn}\frac{\rho}{a}\right) \cdot [a_{mn}\sin m\varphi + b_{mn}\cos m\varphi] \cdot \sinh\left(\alpha_{mn}\frac{z}{a}\right). \tag{11.56}$$

Equation (11.56) is a double series: a Bessel series in ρ and a Fourier series in φ.

At $z = l$, $\psi = \psi(\rho, \varphi)$, a known function of ρ and φ. Therefore

$$\psi(\rho, \varphi) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\alpha_{mn}\frac{\rho}{a}\right) \cdot [a_{mn}\sin m\varphi + b_{mn}\cos m\varphi] \cdot \sinh\left(\alpha_{mn}\frac{l}{a}\right). \tag{11.57}$$

The constants a_{mn} and b_{mn} are evaluated by using Eqs. (11.49) and (11.50) and the corresponding equations for $\sin\varphi$ and $\cos\varphi$ (Example 10.2.1 and Eqs. (14.2), (14.3), (14.15) to (14.17)). We find[14]

$$\left.\begin{matrix} a_{mn} \\ b_{mn} \end{matrix}\right\} = 2\left[\pi a^2 \sinh\left(\alpha_{mn}\frac{l}{a}\right) J_{m+1}^2(\alpha_{mn})\right]^{-1} \cdot \int_0^{2\pi}\int_0^a \psi(\rho, \varphi) J_m\left(\alpha_{mn}\frac{\rho}{a}\right)\left\{\begin{matrix} \sin m\varphi \\ \cos m\varphi \end{matrix}\right\}\rho\, d\rho\, d\varphi. \tag{11.58}$$

These are definite integrals, that is, numbers. Substituting back into Eq. (11.56), the series is specified and the potential $\psi(\rho, \varphi, z)$ is determined. ■

Continuum Form

The Bessel series, Eq. (11.51), and Exercise 11.2.6 apply to expansions over the finite interval $[0, a]$. If $a \to \infty$, then the series forms may be expected to go over into integrals. The discrete roots $\alpha_{\nu m}$ become a continuous variable α. A similar situation is encountered in the Fourier series, Section 15.2. The development of the Bessel integral from the Bessel series is left as Exercise 11.2.8.

For operations with a continuum of Bessel functions, $J_\nu(\alpha\rho)$, a key relation is the Bessel function **closure** equation,

$$\int_0^{\infty} J_\nu(\alpha\rho) J_\nu(\alpha'\rho)\rho\, d\rho = \frac{1}{\alpha}\delta(\alpha - \alpha'), \qquad \nu > -\frac{1}{2}. \tag{11.59}$$

This may be proved by the use of Hankel transforms, Section 15.1. An alternate approach, starting from a relation similar to Eq. (10.82), is given by Morse and Feshbach, Section 6.3. A second kind of orthogonality (varying the index) is developed for spherical Bessel functions in Section 11.7.

[14] If $m = 0$, the factor 2 is omitted (compare Eq. (14.16)).

Exercises

11.2.1 Show that

$$\left(a^2-b^2\right)\int_0^P J_\nu(ax)J_\nu(bx)x\,dx = P\left[bJ_\nu(aP)J'_\nu(bP)-aJ'_\nu(aP)J_\nu(bP)\right],$$

with

$$J'_\nu(aP)=\frac{d}{d(ax)}J_\nu(ax)\Big|_{x=P},$$

$$\int_0^P \left[J_\nu(ax)\right]^2 x\,dx = \frac{P^2}{2}\left\{\left[J'_\nu(aP)\right]^2+\left(1-\frac{\nu^2}{a^2P^2}\right)\left[J_\nu(aP)\right]^2\right\}, \qquad \nu>-1.$$

These two integrals are usually called the **first and second Lommel integrals**.
Hint. We have the development of the orthogonality of the Bessel functions as an analogy.

11.2.2 Show that

$$\int_0^a\left[J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]^2\rho\,d\rho = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu m})\right]^2, \qquad \nu>-1.$$

Here $\alpha_{\nu m}$ is the mth zero of J_ν.
Hint. With $\alpha_{\nu n}=\alpha_{\nu m}+\varepsilon$, expand $J_\nu[(\alpha_{\nu m}+\varepsilon)\rho/a]$ about $\alpha_{\nu m}\rho/a$ by a Taylor expansion.

11.2.3 (a) If $\beta_{\nu m}$ is the mth zero of $(d/d\rho)J_\nu(\beta_{\nu m}\rho/a)$, show that the Bessel functions are orthogonal over the interval $[0,a]$ with an orthogonality integral

$$\int_0^a J_\nu\left(\beta_{\nu m}\frac{\rho}{a}\right)J_\nu\left(\beta_{\nu n}\frac{\rho}{a}\right)\rho\,d\rho=0, \qquad m\neq n, \qquad \nu>-1.$$

(b) Derive the corresponding normalization integral $(m=n)$.

$$\text{ANS. } \frac{a^2}{2}\left(1-\frac{\nu^2}{\beta_{\nu m}^2}\right)\left[J_\nu(\beta_{\nu m})\right]^2, \qquad \nu>-1.$$

11.2.4 Verify that the orthogonality equation, Eq. (11.49), and the normalization equation, Eq. (11.50), hold for $\nu>-1$.
Hint. Using power-series expansions, examine the behavior of Eq. (11.48) as $\rho\to 0$.

11.2.5 From Eq. (11.49) develop a proof that $J_\nu(z), \nu>-1$, has no complex roots (with nonzero imaginary part).
Hint.

(a) Use the series form of $J_\nu(z)$ to exclude pure imaginary roots.
(b) Assume $\alpha_{\nu m}$ to be complex and take $\alpha_{\nu n}$ to be $\alpha^*_{\nu m}$.

11.2.6 (a) In the series expansion

$$f(\rho)=\sum_{m=1}^{\infty}c_{\nu m}J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right), \qquad 0\le\rho\le a, \qquad \nu>-1,$$

with $J_\nu(\alpha_{\nu m}) = 0$, show that the coefficients are given by

$$c_{\nu m} = \frac{2}{a^2[J_{\nu+1}(\alpha_{\nu m})]^2} \int_0^a f(\rho) J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\rho\, d\rho.$$

(b) In the series expansion

$$f(\rho) = \sum_{m=1}^{\infty} d_{\nu m} J_\nu\left(\beta_{\nu m}\frac{\rho}{a}\right), \qquad 0 \le \rho \le a, \qquad \nu > -1,$$

with $(d/d\rho)J_\nu(\beta_{\nu m}\rho/a)\,|_{\rho=a} = 0$, show that the coefficients are given by

$$d_{\nu m} = \frac{2}{a^2(1-\nu^2/\beta_{\nu m}^2)[J_\nu(\beta_{\nu m})]^2} \int_0^a f(\rho) J_\nu\left(\beta_{\nu m}\frac{\rho}{a}\right)\rho\, d\rho.$$

11.2.7 A right circular cylinder has an electrostatic potential of $\psi(\rho, \varphi)$ on both ends. The potential on the curved cylindrical surface is zero. Find the potential at all interior points. *Hint*. Choose your coordinate system and adjust your z dependence to exploit the symmetry of your potential.

11.2.8 For the continuum case, show that Eqs. (11.51) and (11.52) are replaced by

$$f(\rho) = \int_0^\infty a(\alpha) J_\nu(\alpha\rho)\, d\alpha,$$

$$a(\alpha) = \alpha \int_0^\infty f(\rho) J_\nu(\alpha\rho)\rho\, d\rho.$$

Hint. The corresponding case for sines and cosines is worked out in Section 15.2. These are Hankel transforms. A derivation for the special case $\nu = 0$ is the topic of Exercise 15.1.1.

11.2.9 A function $f(x)$ is expressed as a Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(\alpha_{mn} x),$$

with α_{mn} the nth root of J_m. Prove the Parseval relation,

$$\int_0^1 \left[f(x)\right]^2 x\, dx = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \left[J_{m+1}(\alpha_{mn})\right]^2.$$

11.2.10 Prove that

$$\sum_{n=1}^{\infty} (\alpha_{mn})^{-2} = \frac{1}{4(m+1)}.$$

Hint. Expand x^m in a Bessel series and apply the Parseval relation.

11.2.11 A right circular cylinder of length l has a potential

$$\psi\left(z = \pm\frac{l}{2}\right) = 100\left(1 - \frac{\rho}{a}\right),$$

where a is the radius. The potential over the curved surface (side) is zero. Using the Bessel series from Exercise 11.2.7, calculate the electrostatic potential for $\rho/a = 0.0(0.2)1.0$ and $z/l = 0.0(0.1)0.5$. Take $a/l = 0.5$.

Hint. From Exercise 11.1.30 you have

$$\int_0^{\alpha_{0n}} \left(1 - \frac{y}{\alpha_{0n}}\right) J_0(y)\, y\, dy.$$

Show that this equals

$$\frac{1}{\alpha_{0n}} \int_0^{\alpha_{0n}} J_0(y)\, dy.$$

Numerical evaluation of this latter form rather than the former is both faster and more accurate.

Note. For $\rho/a = 0.0$ and $z/l = 0.5$ the convergence is slow, 20 terms giving only 98.4 rather than 100.

Check value. For $\rho/a = 0.4$ and $z/l = 0.3$, $\psi = 24.558$.

11.3 Neumann Functions, Bessel Functions of the Second Kind

From the theory of ODEs it is known that Bessel's equation has two independent solutions. Indeed, for nonintegral order ν we have already found two solutions and labeled them $J_\nu(x)$ and $J_{-\nu}(x)$, using the infinite series (Eq. (11.5)). The trouble is that when ν is integral, Eq. (11.8) holds and we have but one independent solution. A second solution may be developed by the methods of Section 9.6. This yields a perfectly good second solution of Bessel's equation but is not the standard form.

Definition and Series Form

As an alternate approach, we take the particular linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$

$$\boxed{N_\nu(x) = \frac{\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}.} \tag{11.60}$$

This is the Neumann function (Fig. 11.5).[15] For nonintegral ν, $N_\nu(x)$ clearly satisfies Bessel's equation, for it is a linear combination of known solutions $J_\nu(x)$ and $J_{-\nu}(x)$.

[15] In AMS-55 (see footnote 4 in Chapter 5 or Additional Readings of Chapter 8 p. for this ref.) and in most mathematics tables, this is labeled $Y_\nu(x)$.

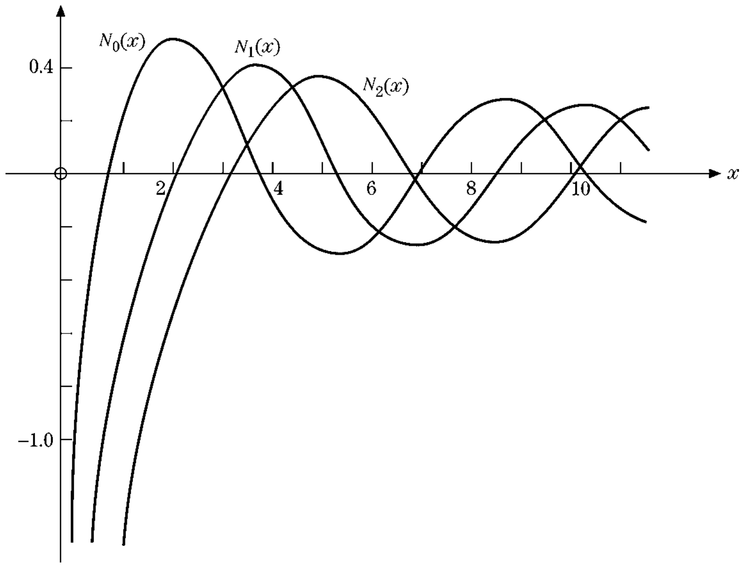

FIGURE 11.5 Neumann functions $N_0(x)$, $N_1(x)$, and $N_2(x)$.

Substituting the power-series Eq. (11.6) for $n \to \nu$ (given in Exercise 11.1.7) yields

$$N_\nu(x) = -\frac{(\nu-1)!}{\pi}\left(\frac{2}{x}\right)^\nu + \cdots,^{16} \tag{11.61}$$

for $\nu > 0$. However, for integral ν, $\nu = n$, Eq. (11.8) applies and Eq. (11.60)[16] becomes indeterminate. The definition of $N_\nu(x)$ was chosen deliberately for this indeterminate property. Again substituting the power series and evaluating $N_\nu(x)$ for $\nu \to 0$ by l'Hôpital's rule for indeterminate forms, we obtain the limiting value

$$N_0(x) = \frac{2}{\pi}(\ln x + \gamma - \ln 2) + O\left(x^2\right) \tag{11.62}$$

for $n = 0$ and $x \to 0$, using

$$\nu!(-\nu)! = \frac{\pi\nu}{\sin\pi\nu} \tag{11.63}$$

from Eq. (8.32). The first and third terms in Eq. (11.62) come from using $(d/d\nu)(x/2)^\nu = (x/2)^\nu \ln(x/2)$, while γ comes from $(d/d\nu)\nu!$ for $\nu \to 0$ using Eqs. (8.38) and (8.40). For $n > 0$ we obtain similarly

$$N_n(x) = -\frac{1}{\pi}(n-1)!\left(\frac{2}{x}\right)^n + \cdots + \frac{2}{\pi}\left(\frac{x}{2}\right)^n \frac{1}{n!}\ln\left(\frac{x}{2}\right) + \cdots. \tag{11.64}$$

Equations (11.62) and (11.64) exhibit the logarithmic dependence that was to be expected. This, of course, verifies the independence of J_n and N_n.

[16] Note that this limiting form applies to both integral and nonintegral values of the index ν.

Other Forms

As with all the other Bessel functions, $N_\nu(x)$ has integral representations. For $N_0(x)$ we have

$$N_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\,dt = -\frac{2}{\pi}\int_1^\infty \frac{\cos(xt)}{(t^2-1)^{1/2}}dt, \qquad x > 0.$$

These forms can be derived as the imaginary part of the Hankel representations of Exercise 11.4.7. The latter form is a Fourier cosine transform.

To verify that $N_\nu(x)$, our Neumann function (Fig. 11.5) or Bessel function of the second kind, actually does satisfy Bessel's equation for integral n, we may proceed as follows. L'Hôpital's rule applied to Eq. (11.60) yields

$$\begin{aligned} N_n(x) &= \left.\frac{(d/d\nu)[\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)]}{(d/d\nu)\sin\nu\pi}\right|_{\nu=n} \\ &= \frac{-\pi\sin n\pi\, J_n(x) + [\cos n\pi\,\partial J_\nu/\partial\nu - \partial J_{-\nu}/\partial\nu]|_{\nu=n}}{\pi\cos n\pi} \\ &= \frac{1}{\pi}\left[\frac{\partial J_\nu(x)}{\partial\nu} - (-1)^n\frac{\partial J_{-\nu}(x)}{\partial\nu}\right]\Bigg|_{\nu=n}. \end{aligned} \tag{11.65}$$

Differentiating Bessel's equation for $J_{\pm\nu}(x)$ with respect to ν, we have

$$x^2\frac{d^2}{dx^2}\left(\frac{\partial J_{\pm\nu}}{\partial\nu}\right) + x\frac{d}{dx}\left(\frac{\partial J_{\pm\nu}}{\partial\nu}\right) + \left(x^2-\nu^2\right)\frac{\partial J_{\pm\nu}}{\partial\nu} = 2\nu J_{\pm\nu}. \tag{11.66}$$

Multiplying the equation for $J_{-\nu}$ by $(-1)^\nu$, subtracting from the equation for J_ν (as suggested by Eq. (11.65)), and taking the limit $\nu \to n$, we obtain

$$x^2\frac{d^2}{dx^2}N_n + x\frac{d}{dx}N_n + \left(x^2-n^2\right)N_n = \frac{2n}{\pi}\left[J_n - (-1)^n J_{-n}\right]. \tag{11.67}$$

For $\nu = n$, an integer, the right-hand side vanishes by Eq. (11.8) and $N_n(x)$ is seen to be a solution of Bessel's equation. The most general solution for any ν can therefore be written as

$$y(x) = AJ_\nu(x) + BN_\nu(x). \tag{11.68}$$

It is seen from Eqs. (11.62) and (11.64) that N_n diverges, at least logarithmically. Any boundary condition that requires the solution to be finite at the origin (as in our vibrating circular membrane (Section 11.1)) automatically excludes $N_n(x)$. Conversely, in the absence of such a requirement, $N_n(x)$ must be considered.

To a certain extent the definition of the Neumann function $N_n(x)$ is arbitrary. Equations (11.62) and (11.64) contain terms of the form $a_nJ_n(x)$. Clearly, any finite value of the constant a_n would still give us a second solution of Bessel's equation. Why should a_n have the particular value implicit in Eqs. (11.62) and (11.64)? The answer involves the asymptotic dependence developed in Section 11.6. If J_n corresponds to a cosine wave, then N_n corresponds to a sine wave. This simple and convenient asymptotic phase relationship is a consequence of the particular admixture of J_n in N_n.

Recurrence Relations

Substituting Eq. (11.60) for $N_\nu(x)$ (nonintegral ν) into the recurrence relations (Eqs. (11.10) and (11.12) for $J_n(x)$, we see immediately that $N_\nu(x)$ satisfies these same recurrence relations. This actually constitutes another proof that N_ν is a solution. Note that the converse is not necessarily true. All solutions need not satisfy the same recurrence relations. An example of this sort of trouble appears in Section 11.5.

Wronskian Formulas

From Section 9.6 and Exercise 10.1.4 we have the Wronskian formula[17] for solutions of the Bessel equation,

$$u_\nu(x)v'_\nu(x) - u'_\nu(x)v_\nu(x) = \frac{A_\nu}{x}, \tag{11.69}$$

in which A_ν is a parameter that depends on the particular Bessel functions $u_\nu(x)$ and $v_\nu(x)$ being considered. A_ν is a constant in the sense that it is independent of x. Consider the special case

$$u_\nu(x) = J_\nu(x), \qquad v_\nu(x) = J_{-\nu}(x), \tag{11.70}$$

$$J_\nu J'_{-\nu} - J'_\nu J_{-\nu} = \frac{A_\nu}{x}. \tag{11.71}$$

Since A_ν is a constant, it may be identified at any convenient point, such as $x = 0$. Using the first terms in the series expansions (Eqs. (11.5) and (11.6)), we obtain

$$\begin{aligned} J_\nu &\to \frac{x^\nu}{2^\nu \nu!}, \qquad & J_{-\nu} &\to \frac{2^\nu x^{-\nu}}{(-\nu)!} \\ J'_\nu &\to \frac{\nu x^{\nu-1}}{2^\nu \nu!}, \qquad & J'_{-\nu} &\to -\frac{\nu 2^\nu x^{-\nu-1}}{(-\nu)!}. \end{aligned} \tag{11.72}$$

Substitution into Eq. (11.69) yields

$$J_\nu(x)J'_{-\nu}(x) - J'_\nu(x)J_{-\nu}(x) = \frac{-2\nu}{x\nu!(-\nu)!} = -\frac{2\sin\nu\pi}{\pi x}, \tag{11.73}$$

using Eq. (8.32). Note that A_ν vanishes for integral ν, as it must, since the nonvanishing of the Wronskian is a test of the independence of the two solutions. By Eq. (11.73), J_n and J_{-n} are clearly linearly dependent.

Using our recurrence relations, we may readily develop a large number of alternate forms, among which are

$$J_\nu J_{-\nu+1} + J_{-\nu}J_{\nu-1} = \frac{2\sin\nu\pi}{\pi x}, \tag{11.74}$$

[17]This result depends on $P(x)$ of Section 9.5 being equal to $p'(x)/p(x)$, the corresponding coefficient of the self-adjoint form of Section 10.1.

$$J_\nu J_{-\nu-1} + J_{-\nu} J_{\nu+1} = -\frac{2\sin\nu\pi}{\pi x}, \tag{11.75}$$

$$J_\nu N'_\nu - J'_\nu N_\nu = \frac{2}{\pi x}, \tag{11.76}$$

$$J_\nu N_{\nu+1} - J_{\nu+1} N_\nu = -\frac{2}{\pi x}. \tag{11.77}$$

Many more will be found in the references given at chapter's end.

You will recall that in Chapter 9 Wronskians were of great value in two respects: (1) in establishing the linear independence or linear dependence of solutions of differential equations and (2) in developing an integral form of a second solution. Here the specific forms of the Wronskians and Wronskian-derived combinations of Bessel functions are useful primarily to illustrate the general behavior of the various Bessel functions. Wronskians are of great use in checking tables of Bessel functions. In Section 10.5 Wronskians appeared in connection with Green's functions.

Example 11.3.1 Coaxial Wave Guides

We are interested in an electromagnetic wave confined between the concentric, conducting cylindrical surfaces $\rho = a$ and $\rho = b$. Most of the mathematics is worked out in Section 9.3 and Example 11.1.2. To go from the standing wave of these examples to the traveling wave here, we let $A = iB$, $A = a_{mn}$, $B = b_{mn}$ in Eq. (11.40a) and obtain

$$E_z = \sum_{m,n} b_{mn} J_m(\gamma\rho) e^{\pm im\varphi} e^{i(kz-\omega t)}. \tag{11.78}$$

Additional properties of the components of the electromagnetic wave in the simple cylindrical wave guide are explored in Exercises 11.3.8 and 11.3.9. For the coaxial wave guide one generalization is needed. The origin, $\rho = 0$, is now excluded ($0 < a \le \rho \le b$). Hence the Neumann function $N_m(\gamma\rho)$ may not be excluded. $E_z(\rho, \varphi, z, t)$ becomes

$$E_z = \sum_{m,n} \big[b_{mn} J_m(\gamma\rho) + c_{mn} N_m(\gamma\rho)\big] e^{\pm im\varphi} e^{i(kz-\omega t)}. \tag{11.79}$$

With the condition

$$H_z = 0, \tag{11.80}$$

we have the basic equations for a TM (transverse magnetic) wave.

The (tangential) electric field must vanish at the conducting surfaces (Dirichlet boundary condition), or

$$b_{mn} J_m(\gamma a) + c_{mn} N_m(\gamma a) = 0, \tag{11.81}$$

$$b_{mn} J_m(\gamma b) + c_{mn} N_m(\gamma b) = 0. \tag{11.82}$$

These transcendental equations may be solved for $\gamma(\gamma_{mn})$ and the ratio c_{mn}/b_{mn}. From Example 11.1.2,

$$k^2 = \omega^2 \mu_0 \varepsilon_0 - \gamma^2 = \frac{\omega^2}{c^2} - \gamma^2. \tag{11.83}$$

Since k^2 must be positive for a real wave, the minimum frequency that will be propagated (in this TM mode) is

$$\omega = \gamma c, \tag{11.84}$$

with γ fixed by the boundary conditions, Eqs. (11.81) and (11.82). This is the cutoff frequency of the wave guide.

There is also a TE (transverse electric) mode, with $E_z = 0$ and H_z given by Eq. (11.79). Then we have Neumann boundary conditions in place of Eqs. (11.81) and (11.82). Finally, for the coaxial guide (not for the plain cylindrical guide, $a = 0$), a TEM (transverse electromagnetic) mode, $E_z = H_z = 0$, is possible. This corresponds to a plane wave, as in free space.

The simpler cases (no Neumann functions, simpler boundary conditions) of a circular wave guide are included as Exercises 11.3.8 and 11.3.9.

To conclude this discussion of Neumann functions, we introduce the Neumann function $N_\nu(x)$ for the following reasons:

1. It is a second, independent solution of Bessel's equation, which completes the general solution.
2. It is required for specific physical problems such as electromagnetic waves in coaxial cables and quantum mechanical scattering theory.
3. It leads to a Green's function for the Bessel equation (Sections 9.7 and 10.5).
4. It leads directly to the two Hankel functions (Section 11.4).

■

Exercises

11.3.1 Prove that the Neumann functions N_n (with n an integer) satisfy the recurrence relations

$$N_{n-1}(x) + N_{n+1}(x) = \frac{2n}{x} N_n(x),$$
$$N_{n-1}(x) - N_{n+1}(x) = 2N_n'(x).$$

Hint. These relations may be proved by differentiating the recurrence relations for J_ν or by using the limit form of N_ν but **not** dividing everything by zero.

11.3.2 Show that

$$N_{-n}(x) = (-1)^n N_n(x).$$

11.3.3 Show that

$$N_0'(x) = -N_1(x).$$

11.3.4 If Y and Z are any two solutions of Bessel's equation, show that

$$Y_\nu(x)Z'_\nu(x) - Y'_\nu(x)Z_\nu(x) = \frac{A_\nu}{x},$$

in which A_ν may depend on ν but is independent of x. This is a special case of Exercise 10.1.4.

11.3.5 Verify the Wronskian formulas

$$J_\nu(x)J_{-\nu+1}(x) + J_{-\nu}(x)J_{\nu-1}(x) = \frac{2\sin\nu\pi}{\pi x},$$

$$J_\nu(x)N'_\nu(x) - J'_\nu(x)N_\nu(x) = \frac{2}{\pi x}.$$

11.3.6 As an alternative to letting x approach zero in the evaluation of the Wronskian constant, we may invoke uniqueness of power series (Section 5.7). The coefficient of x^{-1} in the series expansion of $u_\nu(x)v'_\nu(x) - u'_\nu(x)v_\nu(x)$ is then A_ν. Show by series expansion that the coefficients of x^0 and x^1 of $J_\nu(x)J'_{-\nu}(x) - J'_\nu(x)J_{-\nu}(x)$ are each zero.

11.3.7 (a) By differentiating and substituting into Bessel's ODE, show that

$$\int_0^\infty \cos(x\cosh t)\,dt$$

is a solution.
Hint. You can rearrange the final integral as

$$\int_0^\infty \frac{d}{dt}\left\{x\sin(x\cosh t)\sinh t\right\}dt.$$

(b) Show that

$$N_0(x) = -\frac{2}{\pi}\int_0^\infty \cos(x\cosh t)\,dt$$

is linearly independent of $J_0(x)$.

11.3.8 A cylindrical wave guide has radius r_0. Find the nonvanishing components of the electric and magnetic fields for

(a) TM_{01}, transverse magnetic wave ($H_z = H_\rho = E_\varphi = 0$),
(b) TE_{01}, transverse electric wave ($E_z = E_\rho = H_\varphi = 0$).

The subscripts 01 indicate that the longitudinal component (E_z or H_z) involves J_0 and the boundary condition is satisfied by the **first** zero of J_0 or J'_0.
Hint. All components of the wave have the same factor: $\exp i(kz - \omega t)$.

11.3.9 For a given mode of oscillation the **minimum** frequency that will be passed by a circular cylindrical wave guide (radius r_0) is

$$\nu_{\min} = \frac{c}{\lambda_c},$$

in which λ_c is fixed by the boundary condition

$$J_n\left(\frac{2\pi r_0}{\lambda_c}\right) = 0 \qquad \text{for TM}_{nm} \text{ mode,}$$

$$J_n'\left(\frac{2\pi r_0}{\lambda_c}\right) = 0 \qquad \text{for TE}_{nm} \text{ mode.}$$

The subscript n denotes the order of the Bessel function and m indicates the zero used. Find this cutoff wavelength λ_c for the three TM and three TE modes with the longest cutoff wavelengths. Explain your results in terms of the graph of J_0, J_1, and J_2 (Fig. 11.1).

11.3.10 Write a program that will compute successive roots of the Neumann function $N_n(x)$, that is α_{ns}, where $N_n(\alpha_{ns}) = 0$. Tabulate the first five roots of N_0, N_1, and N_2. Check your values for the roots against those listed in AMS-55 (see Additional Readings of Chapter 8 for the full ref.).

Check value. $\alpha_{12} = 5.42968$.

11.3.11 For the case $m = 0$, $a = 1$, and $b = 2$, the coaxial wave guide boundary conditions lead to

$$f(x) = \frac{J_0(2x)}{N_0(2x)} - \frac{J_0(x)}{N_0(x)}$$

(Fig. 11.6).

(a) Calculate $f(x)$ for $x = 0.0(0.1)10.0$ and plot $f(x)$ versus x to find the approximate location of the roots.

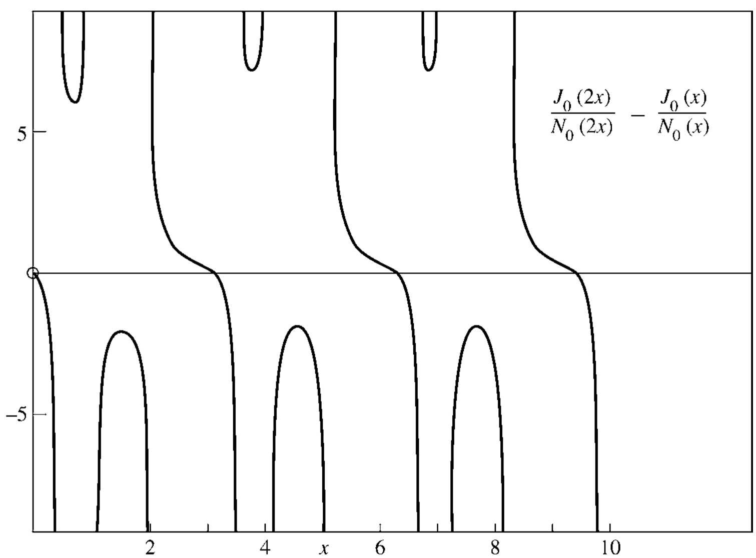

FIGURE 11.6 $f(x)$ of Exercise 11.3.11.

(b) Call a root-finding subroutine to determine the first three roots to higher precision.

ANS. 3.1230, 6.2734, 9.4182.

Note. The higher roots can be expected to appear at intervals whose length approaches n. Why? AMS-55 (see Additional Readings of Chapter 8 for the reference), gives an approximate formula for the roots. The function $g(x) = J_0(x)N_0(2x) - J_0(2x)N_0(x)$ is much better behaved than $f(x)$ previously discussed.

11.4 HANKEL FUNCTIONS

Many authors prefer to introduce the Hankel functions by means of integral representations and then to use them to define the Neumann function $N_\nu(z)$. An outline of this approach is given at the end of this section.

Definitions

Because we have already obtained the Neumann function by more elementary (and less powerful) techniques, we may use it to define the Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$:

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) \tag{11.85}$$

and

$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x). \tag{11.86}$$

This is exactly analogous to taking

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta. \tag{11.87}$$

For real arguments, $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are complex conjugates. The extent of the analogy will be seen even better when the asymptotic forms are considered (Section 11.6). Indeed, it is their asymptotic behavior that makes the Hankel functions useful.

Series expansion of $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ may be obtained by combining Eqs. (11.5) and (11.63). Often only the first term is of interest; it is given by

$$H_0^{(1)}(x) \approx i\frac{2}{\pi}\ln x + 1 + i\frac{2}{\pi}(\gamma - \ln 2) + \cdots, \tag{11.88}$$

$$H_\nu^{(1)}(x) \approx -i\frac{(\nu-1)!}{\pi}\left(\frac{2}{x}\right)^\nu + \cdots, \qquad \nu > 0, \tag{11.89}$$

$$H_0^{(2)}(x) \approx -i\frac{2}{\pi}\ln x + 1 - i\frac{2}{\pi}(\gamma - \ln 2) + \cdots, \tag{11.90}$$

$$H_\nu^{(2)}(x) \approx i\frac{(\nu-1)!}{\pi}\left(\frac{2}{x}\right)^\nu + \cdots, \qquad \nu > 0. \tag{11.91}$$

Since the Hankel functions are linear combinations (with constant coefficients) of J_ν and N_ν, they satisfy the same recurrence relations (Eqs. (11.10) and (11.12))

$$H_{\nu-1}(x) + H_{\nu+1}(x) = \frac{2\nu}{x} H_\nu(x), \tag{11.92}$$

$$H_{\nu-1}(x) - H_{\nu+1}(x) = 2H'_\nu(x), \tag{11.93}$$

for both $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$.

A variety of Wronskian formulas can be developed:

$$H_\nu^{(2)} H_{\nu+1}^{(1)} - H_\nu^{(1)} H_{\nu+1}^{(2)} = \frac{4}{i\pi x}, \tag{11.94}$$

$$J_{\nu-1} H_\nu^{(1)} - J_\nu H_{\nu-1}^{(1)} = \frac{2}{i\pi x}, \tag{11.95}$$

$$J_\nu H_{\nu-1}^{(2)} - J_{\nu-1} H_\nu^{(2)} = \frac{2}{i\pi x}. \tag{11.96}$$

Example 11.4.1 CYLINDRICAL TRAVELING WAVES

As an illustration of the use of Hankel functions, consider a two-dimensional wave problem similar to the vibrating circular membrane of Exercise 11.1.25. Now imagine that the waves are generated at $r = 0$ and move outward to infinity. We replace our standing waves by traveling ones. The differential equation remains the same, but the boundary conditions change. We now demand that for large r the wave behave like

$$U \sim e^{i(kr-\omega t)} \tag{11.97}$$

to describe an outgoing wave. As before, k is the wave number. This assumes, for simplicity, that there is no azimuthal dependence, that is, no angular momentum, or $m = 0$. In Sections 7.3 and 11.6, $H_0^{(1)}(kr)$ is shown to have the asymptotic behavior (for $r \to \infty$)

$$H_0^{(1)}(kr) \sim e^{ikr}. \tag{11.98}$$

This boundary condition at infinity then determines our wave solution as

$$U(r,t) = H_0^{(1)}(kr) e^{-i\omega t}. \tag{11.99}$$

This solution diverges as $r \to 0$, which is the behavior to be expected with a source at the origin.

The choice of a two-dimensional wave problem to illustrate the Hankel function $H_0^{(1)}(z)$ is not accidental. Bessel functions may appear in a variety of ways, such as in the separation of conical coordinates. However, they enter most commonly in the radial equations from the separation of variables in the Helmholtz equation in cylindrical and in spherical polar coordinates. We have taken a degenerate form of cylindrical coordinates for this illustration. Had we used spherical polar coordinates (spherical waves), we should have encountered index $\nu = n + \frac{1}{2}$, n an integer. These special values yield the spherical Bessel functions to be discussed in Section 11.7. ■

Contour Integral Representation of the Hankel Functions

The integral representation (Schlaefli integral)

$$J_\nu(x) = \frac{1}{2\pi i} \oint_C e^{(x/2)(t-1/t)} \frac{dt}{t^{\nu+1}} \tag{11.100}$$

may easily be established as a Cauchy integral for $\nu = n$, an integer (by recognizing that the numerator is the generating function (Eq. (11.1)) and integrating around the origin). If ν is not an integer, the integrand is not single-valued and a cut line is needed in our complex plane. Choosing the negative real axis as the cut line and using the contour shown in Fig. 11.7, we can extend Eq. (11.100) to nonintegral ν. Substituting Eq. (11.100) into Bessel's ODE, we can represent the combined integrand by an exact differential that vanishes as $t \to \infty e^{\pm i\pi}$ (compare Exercise 11.1.16).

We now deform the contour so that it approaches the origin along the positive real axis, as shown in Fig. 11.8. For $x > 0$, this particular approach guarantees that the exact differential mentioned will vanish as $t \to 0$ because of the $e^{-x/2t} \to 0$ factor. Hence each of the separate portions ($\infty\, e^{-i\pi}$ to 0) and (0 to $\infty\, e^{i\pi}$) is a solution of Bessel's equation. We define

$$H_\nu^{(1)}(x) = \frac{1}{\pi i} \int_0^{\infty e^{i\pi}} e^{(x/2)(t-1/t)} \frac{dt}{t^{\nu+1}}, \tag{11.101}$$

$$H_\nu^{(2)}(x) = \frac{1}{\pi i} \int_{\infty e^{-i\pi}}^{0} e^{(x/2)(t-1/t)} \frac{dt}{t^{\nu+1}}. \tag{11.102}$$

These expressions are particularly convenient because they may be handled by the method of steepest descents (Section 7.3). $H_\nu^{(1)}(x)$ has a saddle point at $t = +i$, whereas $H_\nu^{(2)}(x)$ has a saddle point at $t = -i$.

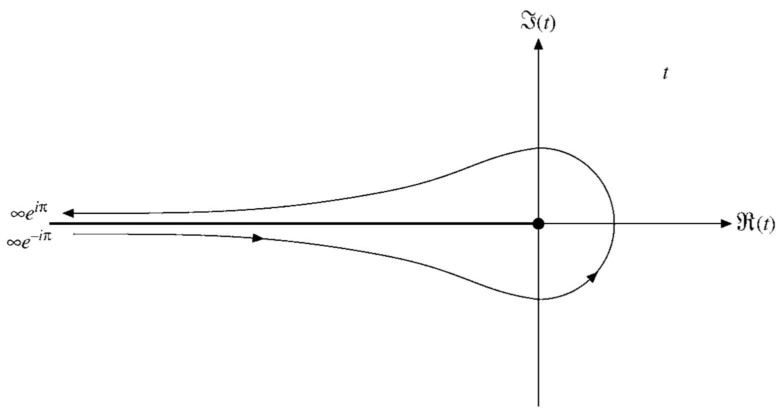

FIGURE 11.7 Bessel function contour.

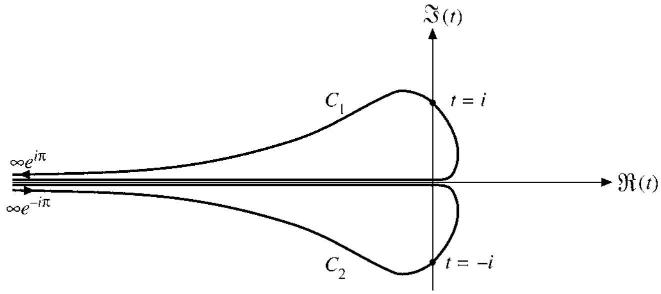

FIGURE 11.8 Hankel function contours.

The problem of relating Eqs. (11.101) and (11.102) to our earlier definition of the Hankel function (Eqs. (11.85) and (11.86)) remains. Since Eqs. (11.100) to (11.102) combined yield

$$J_\nu(x) = \frac{1}{2}\left[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)\right] \tag{11.103}$$

by inspection, we need only show that

$$N_\nu(x) = \frac{1}{2i}\left[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)\right]. \tag{11.104}$$

This may be accomplished by the following steps:

1. With the substitutions $t = e^{i\pi}/s$ for $H_\nu^{(1)}$ and $t = e^{-i\pi}/s$ for $H_\nu^{(2)}$, we obtain

$$H_\nu^{(1)}(x) = e^{-i\nu\pi} H_{-\nu}^{(1)}(x), \tag{11.105}$$

$$H_\nu^{(2)}(x) = e^{i\nu\pi} H_{-\nu}^{(2)}(x). \tag{11.106}$$

2. From Eqs. (11.103) ($\nu \to -\nu$), (11.105), and (11.106),

$$J_{-\nu}(x) = \frac{1}{2}\left[e^{i\nu\pi} H_\nu^{(1)}(x) + e^{-i\nu\pi} H_\nu^{(2)}(x)\right]. \tag{11.107}$$

3. Finally substitute J_ν (Eq. (11.103)) and $J_{-\nu}$ (Eq. (11.107)) into the defining equation for N_ν, Eq. (11.60). This leads to Eq. (11.104) and establishes the contour integrals Eqs. (11.101) and (11.102) as the Hankel functions.

Integral representations have appeared before: Eq. (8.35) for $\Gamma(z)$ and various representations of $J_\nu(z)$ in Section 11.1. With these integral representations of the Hankel functions, it is perhaps appropriate to ask why we are interested in integral representations. There are at least four reasons. The first is simply aesthetic appeal. Second, the integral representations help to distinguish between two linearly independent solutions. In Fig. 11.6, the contours C_1 and C_2 cross **different** saddle points (Section 7.3). For the Legendre functions the contour for $P_n(z)$ (Fig. 12.11) and that for $Q_n(z)$ encircle **different** singular points.

Third, the integral representations facilitate manipulations, analysis, and the development of relations among the various special functions. Fourth, and probably most important of all, the integral representations are extremely useful in developing asymptotic expansions. One approach, the method of steepest descents, appears in Section 7.3. A second approach, the direct expansion of an integral representation is given in Section 11.6 for the modified Bessel function $K_\nu(z)$. This same technique may be used to obtain asymptotic expansions of the confluent hypergeometric functions M and U — Exercise 13.5.13.

In conclusion, the Hankel functions are introduced here for the following reasons:

- As analogs of $e^{\pm ix}$ they are useful for describing traveling waves.
- They offer an alternate (contour integral) and a rather elegant definition of Bessel functions.
- $H_\nu^{(1)}$ is used to define the modified Bessel function K_ν of Section 11.5.

Exercises

11.4.1 Verify the Wronskian formulas

(a) $J_\nu(x)H_\nu^{(1)\prime}(x) - J_\nu'(x)H_\nu^{(1)}(x) = \frac{2i}{\pi x}$,

(b) $J_\nu(x)H_\nu^{(2)\prime}(x) - J_\nu'(x)H_\nu^{(2)}(x) = \frac{-2i}{\pi x}$,

(c) $N_\nu(x)H_\nu^{(1)\prime}(x) - N_\nu'(x)H_\nu^{(1)}(x) = \frac{-2}{\pi x}$,

(d) $N_\nu(x)H_\nu^{(2)\prime}(x) - N_\nu'(x)H_\nu^{(2)}(x) = \frac{-2}{\pi x}$,

(e) $H_\nu^{(1)}(x)H_\nu^{(2)\prime}(x) - H_\nu^{(1)\prime}(x)H_\nu^{(2)}(x) = \frac{-4i}{\pi x}$,

(f) $H_\nu^{(2)}(x)H_{\nu+1}^{(1)}(x) - H_\nu^{(1)}(x)H_{\nu+1}^{(2)}(x) = \frac{4}{i\pi x}$,

(g) $J_{\nu-1}(x)H_\nu^{(1)}(x) - J_\nu(x)H_{\nu-1}^{(1)}(x) = \frac{2}{i\pi x}$.

11.4.2 Show that the integral forms

(a) $$\frac{1}{i\pi}\int_{0C_1}^{\infty e^{i\pi}} e^{(x/2)(t-1/t)}\frac{dt}{t^{\nu+1}} = H_\nu^{(1)}(x),$$

(b) $$\frac{1}{i\pi}\int_{\infty e^{-i\pi}C_2}^{0} e^{(x/2)(t-1/t)}\frac{dt}{t^{\nu+1}} = H_\nu^{(2)}(x)$$

satisfy Bessel's ODE. The contours C_1 and C_2 are shown in Fig. 11.8.

11.4.3 Using the integrals and contours given in problem 11.4.2, show that

$$\frac{1}{2i}\left[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)\right] = N_\nu(x).$$

11.4.4 Show that the integrals in Exercise 11.4.2 may be transformed to yield

(a) $$H_\nu^{(1)}(x) = \frac{1}{\pi i}\int_{C_3} e^{x\sinh\gamma-\nu\gamma}\,d\gamma,$$ (b) $$H_\nu^{(2)}(x) = \frac{1}{\pi i}\int_{C_4} e^{x\sinh\gamma-\nu\gamma}\,d\gamma$$

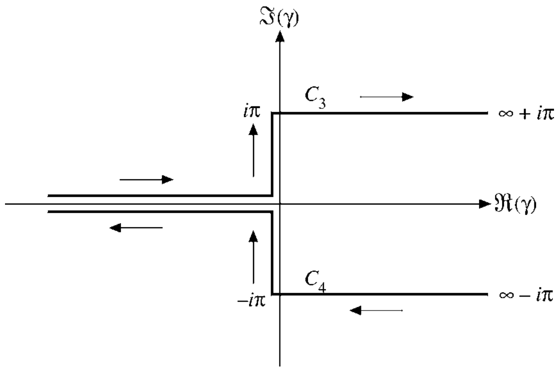

FIGURE 11.9 Hankel function contours.

(see Fig. 11.9).

11.4.5 (a) Transform $H_0^{(1)}(x)$, Eq. (11.101), into

$$H_0^{(1)}(x) = \frac{1}{i\pi}\int_C e^{ix\cosh s}\,ds,$$

where the contour C runs from $-\infty - i\pi/2$ through the origin of the s-plane to $\infty + i\pi/2$.

(b) Justify rewriting $H_0^{(1)}(x)$ as

$$H_0^{(1)}(x) = \frac{2}{i\pi}\int_0^{\infty + i\pi/2} e^{ix\cosh s}\,ds.$$

(c) Verify that this integral representation actually satisfies Bessel's differential equation. (The $i\pi/2$ in the upper limit is not essential. It serves as a convergence factor. We can replace it by $ia\pi/2$ and take the limit.)

11.4.6 From

$$H_0^{(1)}(x) = \frac{2}{i\pi}\int_0^{\infty} e^{ix\cosh s}\,ds$$

show that

(a) $J_0(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\infty} \sin(x\cosh s)\,ds,$ (b) $J_0(x) = \dfrac{2}{\pi}\displaystyle\int_1^{\infty} \frac{\sin(xt)}{\sqrt{t^2-1}}dt.$

This last result is a Fourier sine transform.

11.4.7 From (see Exercises 11.4.4 and 11.4.5)

$$H_0^{(1)}(x) = \frac{2}{i\pi}\int_0^{\infty} e^{ix\cosh s} ds$$

show that

(a) $N_0(x) = -\dfrac{2}{\pi}\displaystyle\int_0^{\infty} \cos(x\cosh s)\,ds.$

(b) $N_0(x) = -\frac{2}{\pi}\int_1^\infty \frac{\cos(xt)}{\sqrt{t^2-1)}}dt.$

These are the integral representations in Section 11.3 (Other Forms).
This last result is a Fourier cosine transform.

11.5 Modified Bessel Functions, $I_\nu(x)$ and $K_\nu(x)$

The Helmholtz equation,

$$\nabla^2\psi + k^2\psi = 0,$$

separated in circular cylindrical coordinates, leads to Eq. (11.22a), the Bessel equation. Equation (11.22a) is satisfied by the Bessel and Neumann functions $J_\nu(k\rho)$ and $N_\nu(k\rho)$ and any linear combination, such as the Hankel functions $H_\nu^{(1)}(k\rho)$ and $H_\nu^{(2)}(k\rho)$. Now, the Helmholtz equation describes the space part of wave phenomena. If instead we have a diffusion problem, then the Helmholtz equation is replaced by

$$\boxed{\nabla^2\psi - k^2\psi = 0.} \tag{11.108}$$

The analog to Eq. (11.22a) is

$$\rho^2\frac{d^2}{d\rho^2}Y_\nu(k\rho) + \rho\frac{d}{d\rho}Y_\nu(k\rho) - \left(k^2\rho^2 + \nu^2\right)Y_\nu(k\rho) = 0. \tag{11.109}$$

The Helmholtz equation may be transformed into the diffusion equation by the transformation $k \to ik$. Similarly, $k \to ik$ changes Eq. (11.22a) into Eq. (11.109) and shows that

$$Y_\nu(k\rho) = Z_\nu(ik\rho).$$

The solutions of Eq. (11.109) are Bessel functions of imaginary argument. To obtain a solution that is regular at the origin, we take Z_ν as the regular Bessel function J_ν. It is customary (and convenient) to choose the normalization so that

$$Y_\nu(x) = I_\nu(x) \equiv i^{-\nu}J_\nu(ix). \tag{11.110}$$

(Here the variable $k\rho$ is being replaced by x for simplicity.) The extra $i^{-\nu}$ normalization cancels the i^ν from each term and leaves $I_\nu(x)$ real. Often this is written as

$$\boxed{I_\nu(x) = e^{-\nu\pi i/2}J_\nu\left(xe^{i\pi/2}\right).} \tag{11.111}$$

I_0 and I_1 are shown in Fig. 11.10.

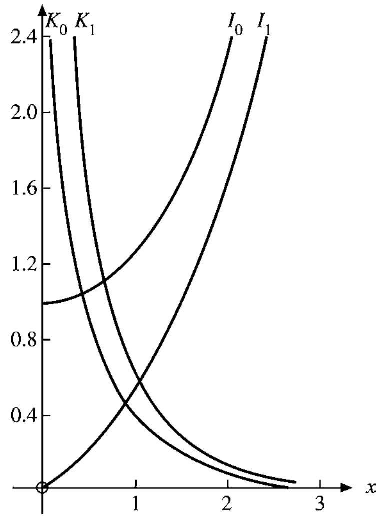

FIGURE 11.10 Modified Bessel functions.

Series Form

In terms of infinite series this is equivalent to removing the $(-1)^s$ sign in Eq. (11.5) and writing

$$I_\nu(x) = \sum_{s=0}^{\infty} \frac{1}{s!(s+\nu)!}\left(\frac{x}{2}\right)^{2s+\nu}, \qquad I_{-\nu}(x) = \sum_{s=0}^{\infty} \frac{1}{s!(s-\nu)!}\left(\frac{x}{2}\right)^{2s-\nu}. \tag{11.112}$$

For integral ν this yields

$$I_n(x) = I_{-n}(x). \tag{11.113}$$

Recurrence Relations

The recurrence relations satisfied by $I_\nu(x)$ may be developed from the series expansions, but it is perhaps easier to work from the existing recurrence relations for $J_\nu(x)$. Let us replace x by $-ix$ and rewrite Eq. (11.110) as

$$J_\nu(x) = i^\nu I_\nu(-ix). \tag{11.114}$$

Then Eq. (11.10) becomes

$$i^{\nu-1} I_{\nu-1}(-ix) + i^{\nu+1} I_{\nu+1}(-ix) = \frac{2\nu}{x} i^\nu I_\nu(-ix).$$

Replacing x by ix, we have a recurrence relation for $I_\nu(x)$,

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x). \tag{11.115}$$

Equation (11.12) transforms to

$$I_{\nu-1}(x) + I_{\nu+1}(x) = 2I'_\nu(x). \tag{11.116}$$

These are the recurrence relations used in Exercise 11.1.14. It is worth emphasizing that although two recurrence relations, Eqs. (11.115) and (11.116) or Exercise 11.5.7, specify the second-order ODE, the converse is not true. The ODE does not uniquely fix the recurrence relations. Equations (11.115) and (11.116) and Exercise 11.5.7 provide an example.

From Eq. (11.113) it is seen that we have but one independent solution when ν is an integer, exactly as in the Bessel functions J_ν. The choice of a second, independent solution of Eq. (11.108) is essentially a matter of convenience. The second solution given here is selected on the basis of its asymptotic behavior — as shown in the next section. The confusion of choice and notation for this solution is perhaps greater than anywhere else in this field.[18] Many authors[19] choose to define a second solution in terms of the Hankel function $H_\nu^{(1)}(x)$ by

$$K_\nu(x) \equiv \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) = \frac{\pi}{2} i^{\nu+1}\big[J_\nu(ix) + iN_\nu(ix)\big]. \tag{11.117}$$

The factor $i^{\nu+1}$ makes $K_\nu(x)$ real when x is real. Using Eqs. (11.60) and (11.110), we may transform Eq. (11.117) to[20]

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}, \tag{11.118}$$

analogous to Eq. (11.60) for $N_\nu(x)$. The choice of Eq. (11.117) as a definition is somewhat unfortunate in that the function $K_\nu(x)$ does not satisfy the same recurrence relations as $I_\nu(x)$ (compare Exercises 11.5.7 and 11.5.8). To avoid this annoyance, other authors[21] have included an additional factor of $\cos\nu\pi$. This permits K_ν to satisfy the same recurrence relations as I_ν, but it has the disadvantage of making $K_\nu = 0$ for $\nu = \frac{1}{2}, \frac{3}{5}, \frac{5}{2}, \ldots$.

The series expansion of $K_\nu(x)$ follows directly from the series form of $H_\nu^{(1)}(ix)$. The lowest-order terms are (cf. Eqs. (11.61) and (11.62))

$$\begin{aligned} K_0(x) &= -\ln x - \gamma + \ln 2 + \cdots, \\ K_\nu(x) &= 2^{\nu-1}(\nu-1)!x^{-\nu} + \cdots. \end{aligned} \tag{11.119}$$

Because the modified Bessel function I_ν is related to the Bessel function J_ν, much as sinh is related to sine, I_ν and the second solution K_ν are sometimes referred to as hyperbolic Bessel functions. K_0 and K_1 are shown in Fig. 11.10.

$I_0(x)$ and $K_0(x)$ have the integral representations

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \cosh(x\cos\theta)\,d\theta, \tag{11.120}$$

$$K_0(x) = \int_0^\infty \cos(x\sinh t)\,dt = \int_0^\infty \frac{\cos(xt)\,dt}{(t^2+1)^{1/2}}, \qquad x > 0. \tag{11.121}$$

[18] A discussion and comparison of notations will be found in *Math. Tables Aids Comput.* **1**: 207–308 (1944).

[19] Watson, Morse and Feshbach, Jeffreys and Jeffreys (without the $\pi/2$).

[20] For integral index n we take the limit as $\nu \to n$.

[21] Whittaker and Watson, see Additional Readings of Chapter 13.

Equation (11.120) may be derived from Eq. (11.30) for $J_0(x)$ or may be taken as a special case of Exercise 11.5.4, $\nu = 0$. The integral representation of K_0, Eq. (11.121), is a Fourier transform and may best be derived with Fourier transforms, Chapter 15, or with Green's functions Section 9.7. A variety of other forms of integral representations (including $\nu \neq 0$) appear in the exercises. These integral representations are useful in developing asymptotic forms (Section 11.6) and in connection with Fourier transforms, Chapter 15.

To put the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ in proper perspective, we introduce them here because:

- These functions are solutions of the frequently encountered modified Bessel equation.
- They are needed for specific physical problems, such as diffusion problems.
- $K_\nu(x)$ provides a Green's function, Section 9.7.
- $K_\nu(x)$ leads to a convenient determination of asymptotic behavior (Section 11.6).

Exercises

11.5.1 Show that

$$e^{(x/2)(t+1/t)} = \sum_{n=-\infty}^{\infty} I_n(x)t^n,$$

thus generating modified Bessel functions, $I_n(x)$.

11.5.2 Verify the following identities

(a) $1 = I_0(x) + 2\sum_{n=1}^{\infty}(-1)^n I_{2n}(x),$

(b) $e^x = I_0(x) + 2\sum_{n=1}^{\infty} I_n(x),$

(c) $e^{-x} = I_0(x) + 2\sum_{n=1}^{\infty}(-1)^n I_n(x),$

(d) $\cosh x = I_0(x) + 2\sum_{n=1}^{\infty} I_{2n}(x),$

(e) $\sinh x = 2\sum_{n=1}^{\infty} I_{2n-1}(x).$

11.5.3 (a) From the generating function of Exercise 11.5.1 show that

$$I_n(x) = \frac{1}{2\pi i}\oint \exp\bigl[(x/2)(t+1/t)\bigr]\frac{dt}{t^{n+1}}.$$

(b) For $n = \nu$, not an integer, show that the preceding integral representation may be generalized to

$$I_\nu(x) = \frac{1}{2\pi i}\int_C \exp\big[(x/2)(t + 1/t)\big]\frac{dt}{t^{\nu+1}}.$$

The contour C is the same as that for $J_\nu(x)$, Fig. 11.7.

11.5.4 For $\nu > -\frac{1}{2}$ show that $I_\nu(z)$ may be represented by

$$\begin{aligned} I_\nu(z) &= \frac{1}{\pi^{1/2}(\nu-\frac{1}{2})!}\left(\frac{z}{2}\right)^\nu \int_0^\pi e^{\pm z\cos\theta}\sin^{2\nu}\theta\, d\theta \\ &= \frac{1}{\pi^{1/2}(\nu-\frac{1}{2})!}\left(\frac{z}{2}\right)^\nu \int_{-1}^1 e^{\pm zp}\big(1-p^2\big)^{\nu-1/2}\,dp \\ &= \frac{2}{\pi^{1/2}(\nu-\frac{1}{2})!}\left(\frac{z}{2}\right)^\nu \int_0^{\pi/2} \cosh(z\cos\theta)\sin^{2\nu}\theta\, d\theta. \end{aligned}$$

11.5.5 A cylindrical cavity has a radius a and height l, Fig. 11.3. The ends, $z = 0$ and l, are at zero potential. The cylindrical walls, $\rho = a$, have a potential $V = V(\varphi, z)$.

(a) Show that the electrostatic potential $\Phi(\rho, \varphi, z)$ has the functional form

$$\Phi(\rho,\varphi,z) = \sum_{m=0}^\infty \sum_{n=1}^\infty I_m(k_n\rho)\sin k_n z\cdot(a_{mn}\sin m\varphi + b_{mn}\cos m\varphi),$$

where $k_n = n\pi/l$.

(b) Show that the coefficients a_{mn} and b_{mn} are given by[22]

$$\left.\begin{matrix} a_{mn} \\ b_{mn}\end{matrix}\right\} = \frac{2}{\pi l I_m(k_n a)}\int_0^{2\pi}\int_0^l V(\varphi,z)\sin k_n z\cdot\left\{\begin{matrix}\sin m\varphi \\ \cos m\varphi\end{matrix}\right\} dz\,d\varphi.$$

Hint. Expand $V(\varphi, z)$ as a double series and use the orthogonality of the trigonometric functions.

11.5.6 Verify that $K_\nu(x)$ is given by

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$$

and from this show that

$$K_\nu(x) = K_{-\nu}(x).$$

11.5.7 Show that $K_\nu(x)$ satisfies the recurrence relations

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x}K_\nu(x),$$

$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2K'_\nu(x).$$

[22] When $m = 0$, the 2 in the coefficient is replaced by 1.

11.5.8 If $\mathcal{K}_\nu = e^{\nu\pi i} K_\nu$, show that $\mathcal{K}_\nu$ satisfies the same recurrence relations as I_ν.

11.5.9 For $\nu > -\frac{1}{2}$ show that $K_\nu(z)$ may be represented by

$$K_\nu(z) = \frac{\pi^{1/2}}{(\nu - \frac{1}{2})!}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-z\cosh t} \sinh^{2\nu} t\, dt, \qquad -\frac{\pi}{2} < \arg z < \frac{\pi}{2}$$
$$= \frac{\pi^{1/2}}{(\nu - \frac{1}{2})!}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zp} (p^2 - 1)^{\nu - 1/2}\, dp.$$

11.5.10 Show that $I_\nu(x)$ and $K_\nu(x)$ satisfy the Wronskian relation

$$I_\nu(x)K_\nu'(x) - I_\nu'(x)K_\nu(x) = -\frac{1}{x}.$$

This result is quoted in Section 9.7 in the development of a Green's function.

11.5.11 If $r = (x^2 + y^2)^{1/2}$, prove that

$$\frac{1}{r} = \frac{2}{\pi}\int_0^\infty \cos(xt) K_0(yt)\, dt.$$

This is a Fourier cosine transform of K_0.

11.5.12 (a) Verify that

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \cosh(x\cos\theta)\, d\theta$$

satisfies the modified Bessel equation, $\nu = 0$.

(b) Show that this integral contains no admixture of $K_0(x)$, the irregular second solution.

(c) Verify the normalization factor $1/\pi$.

11.5.13 Verify that the integral representations

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos t} \cos(nt)\, dt,$$
$$K_\nu(z) = \int_0^\infty e^{-z\cosh t} \cosh(\nu t)\, dt, \qquad \Re(z) > 0,$$

satisfy the modified Bessel equation by direct substitution into that equation. How can you show that the first form does not contain an admixture of K_n and that the second form does not contain an admixture of I_ν? How can you check the normalization?

11.5.14 Derive the integral representation

$$I_n(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta} \cos(n\theta)\, d\theta.$$

Hint. Start with the corresponding integral representation of $J_n(x)$. Equation (11.120) is a special case of this representation.

11.5.15 Show that

$$K_0(z) = \int_0^\infty e^{-z\cosh t}\, dt$$

satisfies the modified Bessel equation. How can you establish that this form is linearly independent of $I_0(z)$?

11.5.16 Show that

$$e^{ax} = I_0(a)T_0(x) + 2\sum_{n=1}^{\infty} I_n(a)T_n(x), \qquad -1 \le x \le 1.$$

$T_n(x)$ is the nth-order Chebyshev polynomial, Section 13.3.
Hint. Assume a Chebyshev series expansion. Using the orthogonality and normalization of the $T_n(x)$, solve for the coefficients of the Chebyshev series.

11.5.17 (a) Write a double precision subroutine to calculate $I_n(x)$ to 12-decimal-place accuracy for $n = 0, 1, 2, 3, \ldots$ and $0 \le x \le 1$. Check your results against the 10-place values given in AMS-55, Table 9.11, see Additional Readings of Chapter 8 for the reference.
(b) Referring to Exercise 11.5.16, calculate the coefficients in the Chebyshev expansions of $\cosh x$ and of $\sinh x$.

11.5.18 The cylindrical cavity of Exercise 11.5.5 has a potential along the cylinder walls:

$$V(z) = \begin{cases} 100\frac{z}{l}, & 0 \le \frac{z}{l} \le \frac{1}{2}, \\ 100\left(1 - \frac{z}{l}\right), & \frac{1}{2} \le \frac{z}{l} \le 1. \end{cases}$$

With the radius–height ratio $a/l = 0.5$, calculate the potential for $z/l = 0.1(0.1)0.5$ and $\rho/a = 0.0(0.2)1.0$.

Check value. For $z/l = 0.3$ and $\rho/a = 0.8$, $V = 26.396$.

11.6 Asymptotic Expansions

Frequently in physical problems there is a need to know how a given Bessel or modified Bessel function behaves for large values of the argument, that is, the asymptotic behavior. This is one occasion when computers are not very helpful. One possible approach is to develop a power-series solution of the differential equation, as in Section 9.5, but now using negative powers. This is Stokes' method, Exercise 11.6.5. The limitation is that starting from some positive value of the argument (for convergence of the series), we do not know what mixture of solutions or multiple of a given solution we have. The problem is to relate the asymptotic series (useful for large values of the variable) to the power-series or related definition (useful for small values of the variable). This relationship can be established by introducing a suitable **integral representation** and then using either the method of steepest descent, Section 7.3, or the direct expansion as developed in this section.

Expansion of an Integral Representation

As a direct approach, consider the integral representation (Exercise 11.5.9)

$$K_\nu(z) = \frac{\pi^{1/2}}{(\nu - \frac{1}{2})!}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zx}\left(x^2-1\right)^{\nu-1/2}\,dx, \qquad \nu > -\frac{1}{2}. \tag{11.122}$$

For the present let us take z to be real, although Eq. (11.122) may be established for $-\pi/2 < \arg z < \pi/2$ ($\Re(z) > 0$). We have three tasks:

1. To show that K_ν as given in Eq. (11.122) actually satisfies the modified Bessel equation (11.109).
2. To show that the regular solution I_ν is absent.
3. To show that Eq. (11.122) has the proper normalization.

1. The fact that Eq. (11.122) is a solution of the modified Bessel equation may be verified by direct substitution. We obtain

$$z^{\nu+1}\int_1^\infty \frac{d}{dx}\left[e^{-zx}\left(x^2-1\right)^{\nu+1/2}\right]dx = 0,$$

which transforms the combined integrand into the derivative of a function that vanishes at both endpoints. Hence the integral is some linear combination of I_ν and K_ν.

2. The rejection of the possibility that this solution contains I_ν constitutes Exercise 11.6.1.

3. The normalization may be verified by showing that, in the limit $z \to 0$, $K_\nu(z)$ is in agreement with Eq. (11.119). By substituting $x = 1 + t/z$,

$$\begin{aligned}
&\frac{\pi^{1/2}}{(\nu - \frac{1}{2})!}\left(\frac{z}{2}\right)^\nu \int_1^\infty e^{-zx}\left(x^2-1\right)^{\nu-1/2}\,dx \\
&\quad = \frac{\pi^{1/2}}{(\nu - \frac{1}{2})!}\left(\frac{z}{2}\right)^\nu e^{-z}\int_0^\infty e^{-t}\left(\frac{t^2}{z^2}+\frac{2t}{z}\right)^{\nu-1/2}\frac{dt}{z} \qquad (11.123\text{a})\\
&\quad = \frac{\pi^{1/2}}{(\nu - \frac{1}{2})!}\frac{e^{-z}}{2^\nu z^\nu}\int_0^\infty e^{-t}t^{2\nu-1}\left(1+\frac{2z}{t}\right)^{\nu-1/2}dt, \qquad (11.123\text{b})
\end{aligned}$$

taking out t^2/z^2 as a factor. This substitution has changed the limits of integration to a more convenient range and has isolated the negative exponential dependence e^{-z}. The integral in Eq. (11.123b) may be evaluated for $z = 0$ to yield $(2\nu - 1)!$. Then, using the duplication formula (Section 8.4), we have

$$\lim_{z\to 0} K_\nu(z) = \frac{(\nu-1)!2^{\nu-1}}{z^\nu}, \qquad \nu > 0, \tag{11.124}$$

in agreement with Eq. (11.119), which thus checks the normalization.[23]

[23] For $\nu \to 0$ the integral diverges logarithmically, in agreement with the logarithmic divergence of $K_0(z)$ for $z \to 0$ (Section 11.5).

Now, to develop an asymptotic series for $K_\nu(z)$, we may rewrite Eq. (11.123a) as

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} \frac{e^{-z}}{(\nu - \frac{1}{2})!} \int_0^\infty e^{-t} t^{\nu-1/2} \left(1 + \frac{t}{2z}\right)^{\nu-1/2} dt \tag{11.125}$$

(taking out $2t/z$ as a factor).

We expand $(1 + t/2z)^{\nu-1/2}$ by the binomial theorem to obtain

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} \frac{e^{-z}}{(\nu - \frac{1}{2})!} \sum_{r=0}^{\infty} \frac{(\nu - \frac{1}{2})!}{r!(\nu - r - \frac{1}{2})!} (2z)^{-r} \int_0^\infty e^{-t} t^{\nu+r-1/2}\, dt. \tag{11.126}$$

Term-by-term integration (valid for asymptotic series) yields the desired asymptotic expansion of $K_\nu(z)$:

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{(4\nu^2 - 1^2)}{1!8z} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2!(8z)^2} + \cdots\right]. \tag{11.127}$$

Although the integral of Eq. (11.122), integrating along the real axis, was convergent only for $-\pi/2 < \arg z < \pi/2$, Eq. (11.127) may be extended to $-3\pi/2 < \arg z < 3\pi/2$. Considered as an infinite series, Eq. (11.127) is actually divergent.[24] However, this series is asymptotic, in the sense that for large enough z, $K_\nu(z)$ may be approximated to any fixed degree of accuracy with a small number of terms. (Compare Section 5.10 for a definition and discussion of asymptotic series.)

It is convenient to rewrite Eq. (11.127) as

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[P_\nu(iz) + i Q_\nu(iz)\right], \tag{11.128}$$

where

$$P_\nu(z) \sim 1 - \frac{(\mu - 1)(\mu - 9)}{2!(8z)^2} + \frac{(\mu - 1)(\mu - 9)(\mu - 25)(\mu - 49)}{4!(8z)^4} - \cdots, \tag{11.129a}$$

$$Q_\nu(z) \sim \frac{\mu - 1}{1!(8z)} - \frac{(\mu - 1)(\mu - 9)(\mu - 25)}{3!(8z)^3} + \cdots, \tag{11.129b}$$

and

$$\mu = 4\nu^2.$$

It should be noted that although $P_\nu(z)$ of Eq. (11.129a) and $Q_\nu(z)$ of Eq. (11.129b) have alternating signs, the series for $P_\nu(iz)$ and $Q_\nu(iz)$ of Eq. (11.128) have all signs positive. Finally, for z large, P_ν dominates.

Then with the asymptotic form of $K_\nu(z)$, Eq. (11.128), we can obtain expansions for all other Bessel and hyperbolic Bessel functions by defining relations:

[24]Our binomial expansion is valid only for $t < 2z$ and we have integrated t out to infinity. The exponential decrease of the integrand prevents a disaster, but the resultant series is still only asymptotic, not convergent. By Table 9.3, $z = \infty$ is an essential singularity of the Bessel (and modified Bessel) equations. Fuchs' theorem does not guarantee a convergent series and we do not get a convergent series.

1. From

$$\frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(iz) = K_\nu(z) \tag{11.130}$$

we have

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left\{i\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right\} \cdot \left[P_\nu(z) + iQ_\nu(z)\right], \qquad -\pi < \arg z < 2\pi. \tag{11.131}$$

2. The second Hankel function is just the complex conjugate of the first (for real argument),

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left\{-i\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right\} \cdot \left[P_\nu(z) - iQ_\nu(z)\right], \qquad -2\pi < \arg z < \pi. \tag{11.132}$$

An alternate derivation of the asymptotic behavior of the Hankel functions appears in Section 7.3 as an application of the method of steepest descents.

3. Since $J_\nu(z)$ is the real part of $H_\nu^{(1)}(z)$ for real z,

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left\{P_\nu(z) \cos\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] - Q_\nu(z) \sin\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right\}, \qquad -\pi < \arg z < \pi, \tag{11.133}$$

holds for real z, that is, $\arg z = 0, \pi$. Once Eq. (11.133) is established for real z, the relation is valid for complex z in the given range of argument.

4. The Neumann function is the imaginary part of $H_\nu^{(1)}(z)$ for real z, or

$$N_\nu(z) = \sqrt{\frac{2}{\pi z}} \left\{P_\nu(z) \sin\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] + Q_\nu(z) \cos\left[z - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right]\right\}, \qquad -\pi < \arg z < \pi. \tag{11.134}$$

Initially, this relation is established for real z, but it may be extended to the complex domain as shown.

5. Finally, the regular hyperbolic or modified Bessel function $I_\nu(z)$ is given by

$$I_\nu(z) = i^{-\nu} J_\nu(iz) \tag{11.135}$$

or

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left[P_\nu(iz) - iQ_\nu(iz)\right], \qquad -\frac{\pi}{2} < \arg z < \frac{\pi}{2}. \tag{11.136}$$

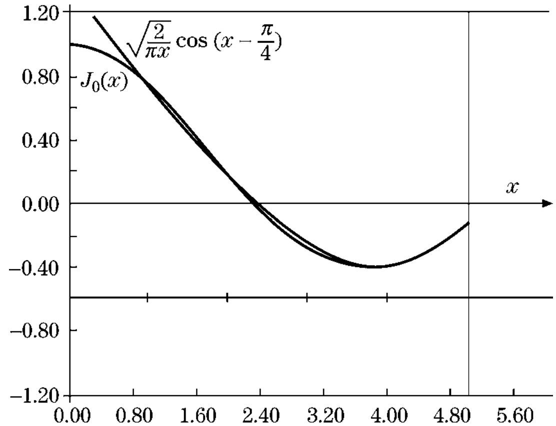

FIGURE 11.11 Asymptotic approximation of $J_0(x)$.

This completes our determination of the asymptotic expansions. However, it is perhaps worth noting the primary characteristics. Apart from the ubiquitous $z^{-1/2}$, J_ν and N_ν behave as cosine and sine, respectively. The zeros are **almost** evenly spaced at intervals of π; the spacing becomes exactly π in the limit as $z \to \infty$. The Hankel functions have been defined to behave like the imaginary exponentials, and the modified Bessel functions I_ν and K_ν go into the positive and negative exponentials. This asymptotic behavior may be sufficient to eliminate immediately one of these functions as a solution for a physical problem. We should also note that the asymptotic series $P_\nu(z)$ and $Q_\nu(z)$, Eqs. (11.129a) and (11.129b), terminate for $\nu = \pm 1/2, \pm 3/2, \ldots$ and become polynomials (in negative powers of z). For these special values of ν the asymptotic approximations become exact solutions.

It is of some interest to consider the accuracy of the asymptotic forms, taking just the first term, for example (Fig. 11.11),

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left[x - \left(n + \frac{1}{2}\right)\left(\frac{\pi}{2}\right)\right]. \tag{11.137}$$

Clearly, the condition for the validity of Eq. (11.137) is that the sine term be negligible; that is,

$$8x \gg 4n^2 - 1. \tag{11.138}$$

For n or $\nu > 1$ the asymptotic region may be far out.

As pointed out in Section 11.3, the asymptotic forms may be used to evaluate the various Wronskian formulas (compare Exercise 11.6.3).

Exercises

11.6.1 In checking the normalization of the integral representation of $K_\nu(z)$ (Eq. (11.122)), we assumed that $I_\nu(z)$ was not present. How do we know that the integral representation (Eq. (11.122)) does not yield $K_\nu(z) + \varepsilon I_\nu(z)$ with $\varepsilon \neq 0$?

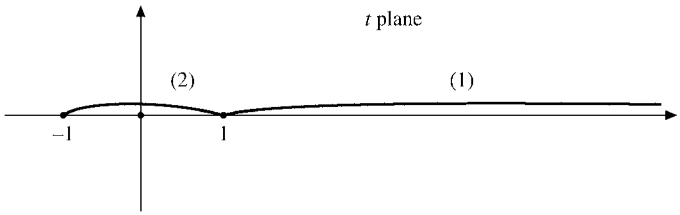

FIGURE 11.12 Modified Bessel function contours.

11.6.2 (a) Show that

$$y(z) = z^{\nu} \int e^{-zt} \left(t^2 - 1\right)^{\nu - 1/2} dt$$

satisfies the modified Bessel equation, provided the contour is chosen so that

$$e^{-zt} \left(t^2 - 1\right)^{\nu + 1/2}$$

has the same value at the initial and final points of the contour.

(b) Verify that the contours shown in Fig. 11.12 are suitable for this problem.

11.6.3 Use the asymptotic expansions to verify the following Wronskian formulas:

(a) $J_\nu(x)J_{-\nu-1}(x) + J_{-\nu}(x)J_{\nu+1}(x) = -2\sin\nu\pi/\pi x$,
(b) $J_\nu(x)N_{\nu+1}(x) - J_{\nu+1}(x)N_\nu(x) = -2/\pi x$,
(c) $J_\nu(x)H^{(2)}_{\nu-1}(x) - J_{\nu-1}(x)H^{(2)}_\nu(x) = 2/i\pi x$,
(d) $I_\nu(x)K'_\nu(x) - I'_\nu(x)K_\nu(x) = -1/x$,
(e) $I_\nu(x)K_{\nu+1}(x) + I_{\nu+1}(x)K_\nu(x) = 1/x$.

11.6.4 From the asymptotic form of $K_\nu(z)$, Eq. (11.127), derive the asymptotic form of $H^{(1)}_\nu(z)$, Eq. (11.131). Note particularly the phase, $(\nu + \frac{1}{2})\pi/2$.

11.6.5 Stokes' method.

(a) Replace the Bessel function in Bessel's equation by $x^{-1/2}y(x)$ and show that $y(x)$ satisfies

$$y''(x) + \left(1 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right) y(x) = 0.$$

(b) Develop a power-series solution with negative powers of x starting with the assumed form

$$y(x) = e^{ix} \sum_{n=0}^{\infty} a_n x^{-n}.$$

Determine the recurrence relation giving a_{n+1} in terms of a_n. Check your result against the asymptotic series, Eq. (11.131).

(c) From the results of Section 7.4 determine the initial coefficient, a_0.

11.6.6 Calculate the first 15 partial sums of $P_0(x)$ and $Q_0(x)$, Eqs. (11.129a) and (11.129b). Let x vary from 4 to 10 in unit steps. Determine the number of terms to be retained for maximum accuracy and the accuracy achieved as a function of x. Specifically, how small may x be without raising the error above 3×10^{-6}?

ANS. $x_{\min} = 6$.

11.6.7 (a) Using the asymptotic series (partial sums) $P_0(x)$ and $Q_0(x)$ determined in Exercise 11.6.6, write a function subprogram FCT(X) that will calculate $J_0(x)$, x real, for $x \geq x_{\min}$.

(b) Test your function by comparing it with the $J_0(x)$ (tables or computer library subroutine) for $x = x_{\min}(10)x_{\min} + 10$.

Note. A more accurate and perhaps simpler asymptotic form for $J_0(x)$ is given in AMS-55, Eq. (9.4.3), see Additional Readings of Chapter 8 for the reference.

11.7 SPHERICAL BESSEL FUNCTIONS

When the Helmholtz equation is separated in spherical coordinates, the radial equation has the form

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + \left[k^2 r^2 - n(n+1)\right] R = 0. \tag{11.139}$$

This is Eq. (9.65) of Section 9.3. The parameter k enters from the original Helmholtz equation, while $n(n+1)$ is a separation constant. From the behavior of the polar angle function (Legendre's equation, Sections 9.5 and 12.5), the separation constant must have this form, with n a nonnegative integer. Equation (11.139) has the virtue of being self-adjoint, but clearly it is not Bessel's equation. However, if we substitute

$$R(kr) = \frac{Z(kr)}{(kr)^{1/2}},$$

Equation (11.139) becomes

$$r^2 \frac{d^2 Z}{dr^2} + r \frac{dZ}{dr} + \left[k^2 r^2 - \left(n + \frac{1}{2}\right)^2\right] Z = 0, \tag{11.140}$$

which **is** Bessel's equation. Z is a Bessel function of order $n + \frac{1}{2}$ (n an integer). Because of the importance of spherical coordinates, this combination, that is,

$$\frac{Z_{n+1/2}(kr)}{(kr)^{1/2}},$$

occurs quite often.

Definitions

It is convenient to label these functions spherical Bessel functions with the following defining equations:

$$\boxed{\begin{aligned} j_n(x) &= \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \\ n_n(x) &= \sqrt{\frac{\pi}{2x}} N_{n+1/2}(x) = (-1)^{n+1}\sqrt{\frac{\pi}{2x}} J_{-n-1/2}(x),^{25} \\ h_n^{(1)}(x) &= \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(1)}(x) = j_n(x) + i n_n(x), \\ h_n^{(2)}(x) &= \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(2)}(x) = j_n(x) - i n_n(x). \end{aligned}} \tag{11.141}$$

These spherical Bessel functions (Figs. 11.13 and 11.14) can be expressed in series form by using the series (Eq. (11.5)) for J_n, replacing n with $n + \frac{1}{2}$:

$$J_{n+1/2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+n+\frac{1}{2})!}\left(\frac{x}{2}\right)^{2s+n+1/2}. \tag{11.142}$$

Using the Legendre duplication formula,

$$z!(z+\tfrac{1}{2})! = 2^{-2z-1}\pi^{1/2}(2z+1)!, \tag{11.143}$$

we have

$$\begin{aligned} j_n(x) &= \sqrt{\frac{\pi}{2x}} \sum_{s=0}^{\infty} \frac{(-1)^s 2^{2s+2n+1}(s+n)!}{\pi^{1/2}(2s+2n+1)!s!}\left(\frac{x}{2}\right)^{2s+n+1/2} \\ &= 2^n x^n \sum_{s=0}^{\infty} \frac{(-1)^s (s+n)!}{s!(2s+2n+1)!} x^{2s}. \end{aligned} \tag{11.144}$$

Now, $N_{n+1/2}(x) = (-1)^{n+1} J_{-n-1/2}(x)$ and from Eq. (11.5) we find that

$$J_{-n-1/2}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s-n-\frac{1}{2})!}\left(\frac{x}{2}\right)^{2s-n-1/2}. \tag{11.145}$$

This yields

$$n_n(x) = (-1)^{n+1}\frac{2^n \pi^{1/2}}{x^{n+1}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s-n-\frac{1}{2})!}\left(\frac{x}{2}\right)^{2s}. \tag{11.146}$$

[25] This is possible because $\cos(n+\frac{1}{2})\pi = 0$, see Eq. (11.60).

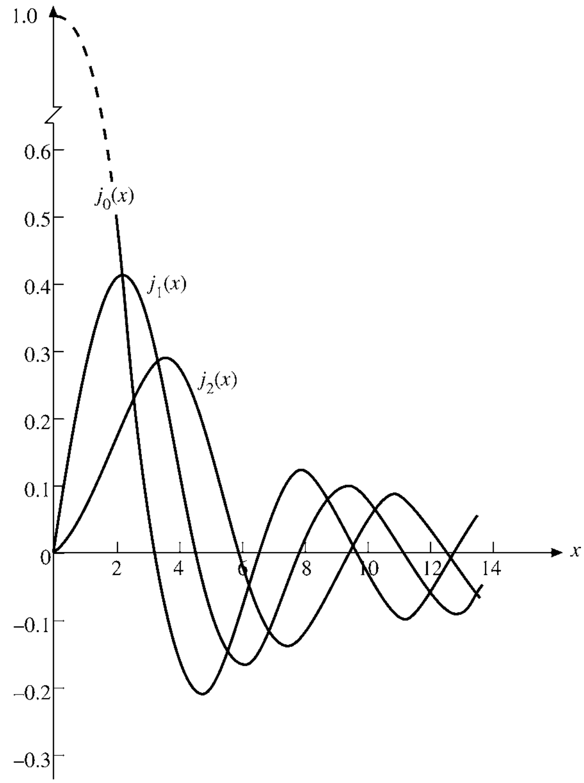

FIGURE 11.13 Spherical Bessel functions.

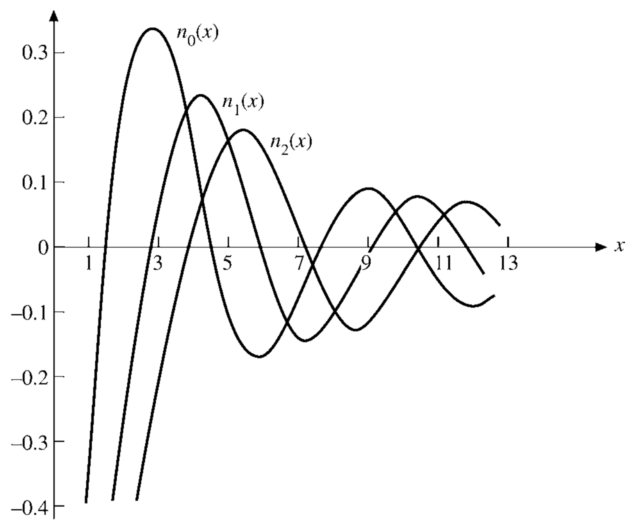

FIGURE 11.14 Spherical Neumann functions.

The Legendre duplication formula can be used again to give

$$n_n(x) = \frac{(-1)^{n+1}}{2^n x^{n+1}} \sum_{s=0}^{\infty} \frac{(-1)^s (s-n)!}{s!(2s-2n)!} x^{2s}. \tag{11.147}$$

These series forms, Eqs. (11.144) and (11.147), are useful in three ways: (1) limiting values as $x \to 0$, (2) closed-form representations for $n = 0$, and, as an extension of this, (3) an indication that the spherical Bessel functions are closely related to sine and cosine.

For the special case $n = 0$ we find from Eq. (11.144) that

$$j_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} x^{2s} = \frac{\sin x}{x}, \tag{11.148}$$

whereas for n_0, Eq. (11.147) yields

$$n_0(x) = -\frac{\cos x}{x}. \tag{11.149}$$

From the definition of the spherical Hankel functions (Eq. (11.141)),

$$\begin{aligned} h_0^{(1)}(x) &= \frac{1}{x}(\sin x - i\cos x) = -\frac{i}{x} e^{ix}, \\ h_0^{(2)}(x) &= \frac{1}{x}(\sin x + i\cos x) = \frac{i}{x} e^{-ix}. \end{aligned} \tag{11.150}$$

Equations (11.148) and (11.149) suggest expressing all spherical Bessel functions as combinations of sine and cosine. The appropriate combinations can be developed from the power-series solutions, Eqs. (11.144) and (11.147), but this approach is awkward. Actually the trigonometric forms are already available as the asymptotic expansion of Section 11.6. From Eqs. (11.131) and (11.129a),

$$\begin{aligned} h_n^{(1)}(x) &= \sqrt{\frac{\pi}{2z}} H_{n+1/2}^{(1)}(z) \\ &= (-i)^{n+1} \frac{e^{iz}}{z} \left\{ P_{n+1/2}(z) + i Q_{n+1/2}(z) \right\}. \end{aligned} \tag{11.151}$$

Now, $P_{n+1/2}$ and $Q_{n+1/2}$ are **polynomials**. This means that Eq. (11.151) is mathematically exact, not simply an asymptotic approximation. We obtain

$$\begin{aligned} h_n^{(1)}(z) &= (-i)^{n+1} \frac{e^{iz}}{z} \sum_{s=0}^{n} \frac{i^s}{s!(8z)^s} \frac{(2n+2s)!!}{(2n-2s)!!} \\ &= (-i)^{n+1} \frac{e^{iz}}{z} \sum_{s=0}^{n} \frac{i^s}{s!(2z)^s} \frac{(n+s)!}{(n-s)!}. \end{aligned} \tag{11.152}$$

Often a factor $(-i)^n = (e^{-i\pi/2})^n$ will be combined with the e^{iz} to give $e^{i(z-n\pi/2)}$. For z real, $j_n(z)$ is the real part of this, $n_n(z)$ the imaginary part, and $h_n^{(2)}(z)$ the complex conjugate. Specifically,

$$h_1^{(1)}(x) = e^{ix}\left(-\frac{1}{x} - \frac{i}{x^2}\right), \tag{11.153a}$$

$$h_2^{(1)}(x) = e^{ix}\left(\frac{i}{x} - \frac{3}{x^2} - \frac{3i}{x^3}\right), \tag{11.153b}$$

$$\begin{aligned} j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}, \\ j_2(x) &= \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x, \end{aligned} \tag{11.154}$$

$$\begin{aligned} n_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \\ n_2(x) &= -\left(\frac{3}{x^3} - \frac{1}{x}\right)\cos x - \frac{3}{x^2}\sin x, \end{aligned} \tag{11.155}$$

and so on.

Limiting Values

For $x \ll 1$,[26] Eqs. (11.144) and (11.147) yield

$$j_n(x) \approx \frac{2^n n!}{(2n+1)!}x^n = \frac{x^n}{(2n+1)!!}, \tag{11.156}$$

$$\begin{aligned} n_n(x) &\approx \frac{(-1)^{n+1}}{2^n}\cdot\frac{(-n)!}{(-2n)!}x^{-n-1} \\ &= -\frac{(2n)!}{2^n n!}x^{-n-1} = -(2n-1)!!x^{-n-1}. \end{aligned} \tag{11.157}$$

The transformation of factorials in the expressions for $n_n(x)$ employs Exercise 8.1.3. The limiting values of the spherical Hankel functions go as $\pm i n_n(x)$.

The asymptotic values of $j_n, n_n, h_n^{(2)}$, and $h_n^{(1)}$ may be obtained from the Bessel asymptotic forms, Section 11.6. We find

$$j_n(x) \sim \frac{1}{x}\sin\left(x - \frac{n\pi}{2}\right), \tag{11.158}$$

$$n_n(x) \sim -\frac{1}{x}\cos\left(x - \frac{n\pi}{2}\right), \tag{11.159}$$

$$h_n^{(1)}(x) \sim (-i)^{n+1}\frac{e^{ix}}{x} = -i\frac{e^{i(x-n\pi/2)}}{x}, \tag{11.160a}$$

$$h_n^{(2)}(x) \sim i^{n+1}\frac{e^{-ix}}{x} = i\frac{e^{-i(x-n\pi/2)}}{x}. \tag{11.160b}$$

[26]The condition that the second term in the series be negligible compared to the first is actually $x \ll 2[(2n+2)(2n+3)/(n+1)]^{1/2}$ for $j_n(x)$.

The condition for these spherical Bessel forms is that $x \gg n(n+1)/2$. From these asymptotic values we see that $j_n(x)$ and $n_n(x)$ are appropriate for a description of **standing spherical waves**; $h_n^{(1)}(x)$ and $h_n^{(2)}(x)$ correspond to **traveling spherical waves**. If the time dependence for the traveling waves is taken to be $e^{-i\omega t}$, then $h_n^{(1)}(x)$ yields an outgoing traveling spherical wave, $h_n^{(2)}(x)$ an incoming wave. Radiation theory in electromagnetism and scattering theory in quantum mechanics provide many applications.

Recurrence Relations

The recurrence relations to which we now turn provide a convenient way of developing the higher-order spherical Bessel functions. These recurrence relations may be derived from the series, but, as with the modified Bessel functions, it is easier to substitute into the known recurrence relations (Eqs. (11.10) and (11.12)). This gives

$$f_{n-1}(x) + f_{n+1}(x) = \frac{2n+1}{x} f_n(x), \tag{11.161}$$

$$n f_{n-1}(x) - (n+1) f_{n+1}(x) = (2n+1) f_n'(x). \tag{11.162}$$

Rearranging these relations (or substituting into Eqs. (11.15) and (11.17)), we obtain

$$\frac{d}{dx}\left[x^{n+1} f_n(x)\right] = x^{n+1} f_{n-1}(x), \tag{11.163}$$

$$\frac{d}{dx}\left[x^{-n} f_n(x)\right] = -x^{-n} f_{n+1}(x). \tag{11.164}$$

Here f_n may represent $j_n, n_n, h_n^{(1)}$, or $h_n^{(2)}$.

The specific forms, Eqs. (11.154) and (11.155), may also be readily obtained from Eq. (11.164).

By mathematical induction we may establish the Rayleigh formulas

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right), \tag{11.165}$$

$$n_n(x) = -(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\cos x}{x}\right), \tag{11.166}$$

$$\begin{aligned} h_n^{(1)}(x) &= -i(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{ix}}{x}\right), \\ h_n^{(2)}(x) &= i(-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{e^{-ix}}{x}\right). \end{aligned} \tag{11.167}$$

Orthogonality

We may take the orthogonality integral for the ordinary Bessel functions (Eqs. (11.49) and (11.50)),

$$\int_0^a J_\nu\left(\alpha_{\nu p}\frac{\rho}{a}\right) J_\nu\left(\alpha_{\nu q}\frac{\rho}{a}\right)\rho\, d\rho = \frac{a^2}{2}\left[J_{\nu+1}(\alpha_{\nu p})\right]^2 \delta_{pq}, \tag{11.168}$$

and substitute in the expression for j_n to obtain

$$\int_0^a j_n\left(\alpha_{np}\frac{\rho}{a}\right) j_n\left(\alpha_{nq}\frac{\rho}{a}\right)\rho^2\, d\rho = \frac{a^3}{2}\left[j_{n+1}(\alpha_{np})\right]^2 \delta_{pq}. \tag{11.169}$$

Here α_{np} and α_{nq} are roots of j_n.

This represents orthogonality with respect to the roots of the Bessel functions. An illustration of this sort of orthogonality is provided in Example 11.7.1, the problem of a particle in a sphere. Equation (11.169) guarantees orthogonality of the wave functions $j_n(r)$ for fixed n. (If n varies, the accompanying spherical harmonic will provide orthogonality.)

Example 11.7.1 PARTICLE IN A SPHERE

An illustration of the use of the spherical Bessel functions is provided by the problem of a quantum mechanical particle in a sphere of radius a. Quantum theory requires that the wave function ψ, describing our particle, satisfy

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi, \tag{11.170}$$

and the boundary conditions (1) $\psi(r \le a)$ remains finite, (2) $\psi(a) = 0$. This corresponds to a square-well potential $V = 0$, $r \le a$, and $V = \infty$, $r > a$. Here $\hbar$ is Planck's constant divided by $2\pi, m$ is the mass of our particle, and E is, its energy. Let us determine the **minimum** value of the energy for which our wave equation has an acceptable solution. Equation (11.170) is Helmholtz's equation with a radial part (compare Section 9.3 for separation of variables):

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[k^2 - \frac{n(n+1)}{r^2}\right]R = 0, \tag{11.171}$$

with $k^2 = 2mE/\hbar^2$. Hence by Eq. (11.139), with $n = 0$,

$$R = Aj_0(kr) + Bn_0(kr).$$

We choose the orbital angular momentum index $n = 0$, for any angular dependence would raise the energy. The spherical Neumann function is rejected because of its divergent behavior at the origin. To satisfy the second boundary condition (for all angles), we require

$$ka = \frac{\sqrt{2mE}}{\hbar}a = \alpha, \tag{11.172}$$

where α is a root of j_0, that is, $j_0(\alpha) = 0$. This has the effect of limiting the allowable energies to a certain discrete set, or, in other words, application of boundary condition (2) quantizes the energy E. The smallest α is the first zero of j_0,

$$\alpha = \pi,$$

and

$$E_{\min} = \frac{\pi^2 \hbar^2}{2ma^2} = \frac{h^2}{8ma^2}, \tag{11.173}$$

which means that for any finite sphere the particle energy will have a positive minimum or zero-point energy. This is an illustration of the Heisenberg uncertainty principle for Δp with $\Delta r \le a$.

In solid-state physics, astrophysics, and other areas of physics, we may wish to know how many different solutions (energy states) correspond to energies less than or equal to some fixed energy E_0. For a cubic volume (Exercise 9.3.5) the problem is fairly simple. The considerably more difficult spherical case is worked out by R. H. Lambert, *Am. J. Phys.* **36**: 417, 1169 (1968).

The relevant orthogonality relation for the $j_n(kr)$ can be derived from the integral given in Exercise 11.7.23. ■

Another form, orthogonality with respect to the indices, may be written as

$$\int_{-\infty}^{\infty} j_m(x) j_n(x)\, dx = 0, \qquad m \ne n,\ m, n \ge 0. \tag{11.174}$$

The proof is left as Exercise 11.7.10. If $m = n$ (compare Exercise 11.7.11), we have

$$\int_{-\infty}^{\infty} \left[j_n(x) \right]^2 dx = \frac{\pi}{2n+1}. \tag{11.175}$$

Most physical applications of orthogonal Bessel and spherical Bessel functions involve orthogonality with varying roots and an interval $[0, a]$ and Eqs. (11.168) and (11.169) and Exercise 11.7.23 for continuous-energy eigenvalues.

The spherical Bessel functions will enter again in connection with spherical waves, but further consideration is postponed until the corresponding angular functions, the Legendre functions, have been introduced.

Exercises

11.7.1 Show that if

$$n_n(x) = \sqrt{\frac{\pi}{2x}} N_{n+1/2}(x),$$

it automatically equals

$$(-1)^{n+1} \sqrt{\frac{\pi}{2x}} J_{-n-1/2}(x).$$

11.7.2 Derive the trigonometric-polynomial forms of $j_n(z)$ and $n_n(z)$.[27]

$$j_n(z) = \frac{1}{z}\sin\left(z - \frac{n\pi}{2}\right)\sum_{s=0}^{[n/2]}\frac{(-1)^s(n+2s)!}{(2s)!(2z)^{2s}(n-2s)!}$$

$$+ \frac{1}{z}\cos\left(z - \frac{n\pi}{2}\right)\sum_{s=0}^{[(n-1)/2]}\frac{(-1)^s(n+2s+1)!}{(2s+1)!(2z)^{2s}(n-2s-1)!},$$

$$n_n(z) = \frac{(-1)^{n+1}}{z}\cos\left(z + \frac{n\pi}{2}\right)\sum_{s=0}^{[n/2]}\frac{(-1)^s(n+2s)!}{(2s)!(2z)^{2s}(n-2s)!}$$

$$+ \frac{(-1)^{n+1}}{z}\sin\left(z + \frac{n\pi}{2}\right)\sum_{s=0}^{[(n-1)/2]}\frac{(-1)^s(n+2s+1)!}{(2s+1)!(2z)^{2s+1}(n-2s-1)!}.$$

11.7.3 Use the integral representation of $J_\nu(x)$,

$$J_\nu(x) = \frac{1}{\pi^{1/2}(\nu-\frac{1}{2})!}\left(\frac{x}{2}\right)^\nu\int_{-1}^{1}e^{\pm ixp}\left(1-p^2\right)^{\nu-1/2}dp,$$

to show that the spherical Bessel functions $j_n(x)$ are expressible in terms of trigonometric functions; that is, for example,

$$j_0(x) = \frac{\sin x}{x}, \qquad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}.$$

11.7.4 (a) Derive the recurrence relations

$$f_{n-1}(x) + f_{n+1}(x) = \frac{2n+1}{x}f_n(x),$$

$$nf_{n-1}(x) - (n+1)f_{n+1}(x) = (2n+1)f_n'(x)$$

satisfied by the spherical Bessel functions $j_n(x), n_n(x), h_n^{(1)}(x)$, and $h_n^{(2)}(x)$.

(b) Show, from these two recurrence relations, that the spherical Bessel function $f_n(x)$ satisfies the differential equation

$$x^2 f_n''(x) + 2xf_n'(x) + \left[x^2 - n(n+1)\right]f_n(x) = 0.$$

11.7.5 Prove by mathematical induction that

$$j_n(x) = (-1)^n x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$$

for n an arbitrary nonnegative integer.

11.7.6 From the discussion of orthogonality of the spherical Bessel functions, show that a Wronskian relation for $j_n(x)$ and $n_n(x)$ is

$$j_n(x)n_n'(x) - j_n'(x)n_n(x) = \frac{1}{x^2}.$$

[27] The upper limit on the summation $[n/2]$ means the largest **integer** that does not exceed $n/2$.

11.7.7 Verify

$$h_n^{(1)}(x)h_n^{(2)\prime}(x) - h_n^{(1)\prime}(x)h_n^{(2)}(x) = -\frac{2i}{x^2}.$$

11.7.8 Verify Poisson's integral representation of the spherical Bessel function,

$$j_n(z) = \frac{z^n}{2^{n+1}n!}\int_0^\pi \cos(z\cos\theta)\sin^{2n+1}\theta\,d\theta.$$

11.7.9 Show that

$$\int_0^\infty J_\mu(x)J_\nu(x)\frac{dx}{x} = \frac{2}{\pi}\frac{\sin[(\mu-\nu)\pi/2]}{\mu^2-\nu^2}, \qquad \mu+\nu > -1.$$

11.7.10 Derive Eq. (11.174):

$$\int_{-\infty}^\infty j_m(x)j_n(x)\,dx = 0, \quad \begin{array}{l} m \neq n \\ m,n \geq 0. \end{array}$$

11.7.11 Derive Eq. (11.175):

$$\int_{-\infty}^\infty \big[j_n(x)\big]^2\,dx = \frac{\pi}{2n+1}.$$

11.7.12 Set up the orthogonality integral for $j_L(kr)$ in a sphere of radius R with the boundary condition

$$j_L(kR) = 0.$$

The result is used in classifying electromagnetic radiation according to its angular momentum.

11.7.13 The Fresnel integrals (Fig. 11.15 and Exercise 5.10.2) occurring in diffraction theory are given by

$$x(t) = \sqrt{\frac{\pi}{2}}\,C\left(\sqrt{\frac{\pi}{2}}\,t\right) = \int_0^t \cos\big(v^2\big)\,dv, \qquad y(t) = \sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{\pi}{2}}\,t\right) = \int_0^t \sin\big(v^2\big)\,dv.$$

Show that these integrals may be expanded in series of spherical Bessel functions

$$x(s) = \frac{1}{2}\int_0^s j_{-1}(u)u^{1/2}\,du = s^{1/2}\sum_{n=0}^\infty j_{2n}(s),$$

$$y(s) = \frac{1}{2}\int_0^s j_0(u)u^{1/2}\,du = s^{1/2}\sum_{n=0}^\infty j_{2n+1}(s).$$

Hint. To establish the equality of the integral and the sum, you may wish to work with their derivatives. The spherical Bessel analogs of Eqs. (11.12) and (11.14) are helpful.

11.7.14 A hollow sphere of radius a (Helmholtz resonator) contains standing sound waves. Find the minimum frequency of oscillation in terms of the radius a and the velocity of sound v. The sound waves satisfy the wave equation

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

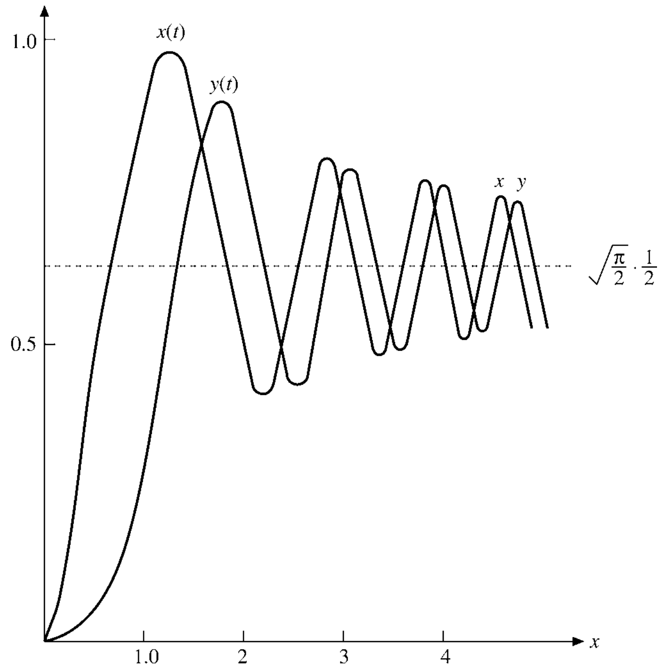

FIGURE 11.15 Fresnel integrals.

and the boundary condition

$$\frac{\partial\psi}{\partial r}=0, \qquad r=a.$$

This is a Neumann boundary condition. Example 11.7.1 has the same PDE but with a Dirichlet boundary condition.

ANS. $\nu_{\min}=0.3313v/a, \qquad \lambda_{\max}=3.018a.$

11.7.15 Defining the spherical modified Bessel functions (Fig. 11.16) by

$$i_n(x)=\sqrt{\frac{\pi}{2x}}I_{n+1/2}(x), \qquad k_n(x)=\sqrt{\frac{2}{\pi x}}K_{n+1/2}(x),$$

show that

$$i_0(x)=\frac{\sinh x}{x}, \qquad k_0(x)=\frac{e^{-x}}{x}.$$

Note that the numerical factors in the definitions of i_n and k_n are not identical.

11.7.16 (a) Show that the parity of $i_n(x)$ is $(-1)^n$.
(b) Show that $k_n(x)$ has no definite parity.

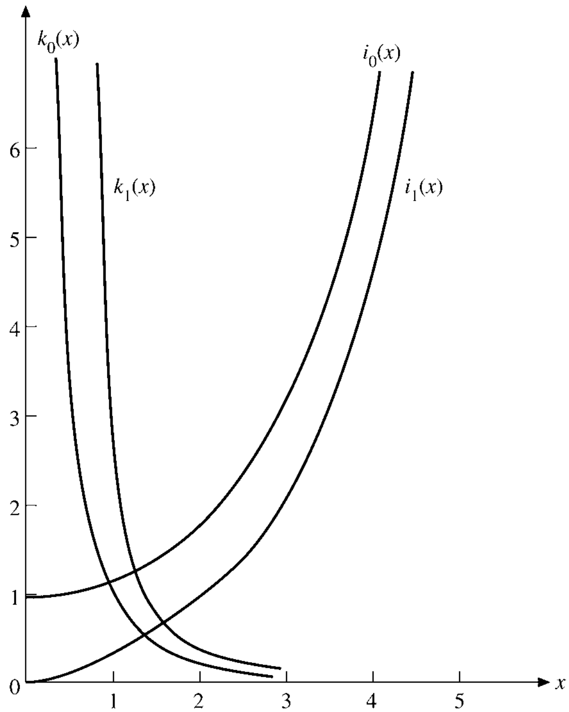

FIGURE 11.16 Spherical modified Bessel functions.

11.7.17 Show that the spherical modified Bessel functions satisfy the following relations:

(a) $i_n(x) = i^{-n} j_n(ix)$,
$k_n(x) = -i^n h_n^{(1)}(ix)$,

(b) $i_{n+1}(x) = x^n \dfrac{d}{dx}\left(x^{-n} i_n\right)$,

$k_{n+1}(x) = -x^n \dfrac{d}{dx}\left(x^{-n} k_n\right)$,

(c) $i_n(x) = x^n \left(\dfrac{1}{x}\dfrac{d}{dx}\right)^n \dfrac{\sinh x}{x}$,

$k_n(x) = (-1)^n x^n \left(\dfrac{1}{x}\dfrac{d}{dx}\right)^n \dfrac{e^{-x}}{x}$.

11.7.18 Show that the recurrence relations for $i_n(x)$ and $k_n(x)$ are

(a) $i_{n-1}(x) - i_{n+1}(x) = \dfrac{2n+1}{x} i_n(x)$,

$n i_{n-1}(x) + (n+1) i_{n+1}(x) = (2n+1) i_n'(x)$,

(b) $k_{n-1}(x) - k_{n+1}(x) = -\frac{2n+1}{x} k_n(x),$

$nk_{n-1}(x) + (n+1)k_{n+1}(x) = -(2n+1)k_n'(x).$

11.7.19 Derive the limiting values for the spherical modified Bessel functions

(a) $$i_n(x) \approx \frac{x^n}{(2n+1)!!}, \qquad k_n(x) \approx \frac{(2n-1)!!}{x^{n+1}}, \qquad x \ll 1.$$

(b) $$i_n(x) \sim \frac{e^x}{2x}, \qquad k_n(x) \sim \frac{e^{-x}}{x}, \qquad x \gg \frac{1}{2}n(n+1).$$

11.7.20 Show that the Wronskian of the spherical modified Bessel functions is given by

$$i_n(x)k_n'(x) - i_n'(x)k_n(x) = -\frac{1}{x^2}.$$

11.7.21 A quantum particle of mass M is trapped in a "square" well of radius a. The Schrödinger equation potential is

$$V(r) = \begin{cases} -V_0, & 0 \le r < a \\ 0, & r > a. \end{cases}$$

The particle's energy E is negative (an eigenvalue).

(a) Show that the radial part of the wave function is given by $j_l(k_1 r)$ for $0 \le r < a$ and $k_l(k_2 r)$ for $r > a$. (We require that $\psi(0)$ be finite and $\psi(\infty) \to 0$.) Here $k_1^2 = 2M(E + V_0)/\hbar^2, k_2^2 = -2ME/\hbar^2$, and l is the angular momentum (n in Eq. (11.139)).

(b) The boundary condition at $r = a$ is that the wave function $\psi(r)$ and its first derivative be continuous. Show that this means

$$\left.\frac{(d/dr)j_l(k_1 r)}{j_l(k_1 r)}\right|_{r=a} = \left.\frac{(d/dr)k_l(k_2 r)}{k_l(k_2 r)}\right|_{r=a}.$$

This equation determines the energy eigenvalues.

Note. This is a generalization of Example 10.1.2.

11.7.22 The quantum mechanical radial wave function for a scattered wave is given by

$$\psi_k = \frac{\sin(kr + \delta_0)}{kr},$$

where k is the wave number, $k = \sqrt{2mE/\hbar}$, and δ_0 is the scattering phase shift. Show that the normalization integral is

$$\int_0^\infty \psi_k(r)\psi_{k'}(r)r^2\,dr = \frac{\pi}{2k}\delta(k - k').$$

Hint. You can use a sine representation of the Dirac delta function. See Exercise 15.3.8.

11.7.23 Derive the spherical Bessel function closure relation

$$\frac{2a^2}{\pi}\int_0^\infty j_n(ar)j_n(br)r^2\,dr = \delta(a-b).$$

Note. An interesting derivation involving Fourier transforms, the Rayleigh plane-wave expansion, and spherical harmonics has been given by P. Ugincius, *Am. J. Phys*. **40**: 1690 (1972).

11.7.24 (a) Write a subroutine that will generate the spherical Bessel functions, $j_n(x)$, that is, will generate the numerical value of $j_n(x)$ given x and n.
Note. One possibility is to use the explicit known forms of j_0 and j_1 and to develop the higher index j_n, by repeated application of the recurrence relation.

(b) Check your subroutine by an independent calculation, such as Eq. (11.154). If possible, compare the machine time needed for this check with the time required for your subroutine.

11.7.25 The wave function of a particle in a sphere (Example 11.7.1) with angular momentum l is $\psi(r,\theta,\varphi) = Aj_l((\sqrt{2ME})r/\hbar)Y_l^m(\theta,\varphi)$. The $Y_l^m(\theta,\varphi)$ is a spherical harmonic, described in Section 12.6. From the boundary condition $\psi(a,\theta,\varphi)=0$ or $j_l((\sqrt{2ME})a/\hbar)=0$ calculate the 10 lowest-energy states. Disregard the m degeneracy ($2l+1$ values of m for each choice of l). Check your results against AMS-55, Table 10.6, see Additional Readings for Chapter 8 for the reference.
Hint. You can use your spherical Bessel subroutine and a root-finding subroutine.

Check values. $j_l(\alpha_{ls})=0,$
$\alpha_{01}=3.1416$
$\alpha_{11}=4.4934$
$\alpha_{21}=5.7635$
$\alpha_{02}=6.2832.$

11.7.26 Let Example 11.7.1 be modified so that the potential is a finite V_0 outside $(r>a)$.

(a) For $E<V_0$ show that

$$\psi_{\text{out}}(r,\theta,\varphi)\sim k_l\left(\frac{r}{\hbar}\sqrt{2M(V_0-E)}\right).$$

(b) The new boundary conditions to be satisfied at $r=a$ are

$$\psi_{\text{in}}(a,\theta,\varphi)=\psi_{\text{out}}(a,\theta,\varphi),$$
$$\frac{\partial}{\partial r}\psi_{\text{in}}(a,\theta,\varphi)=\frac{\partial}{\partial r}\psi_{\text{out}}(a,\theta,\varphi)$$

or

$$\frac{1}{\psi_{\text{in}}}\frac{\partial\psi_{\text{in}}}{\partial r}\bigg|_{r=a}=\frac{1}{\psi_{\text{out}}}\frac{\partial\psi_{\text{out}}}{\partial r}\bigg|_{r=a}.$$

For $l=0$ show that the boundary condition at $r=a$ leads to

$$f(E)=k\left\{\cot ka-\frac{1}{ka}\right\}+k'\left\{1+\frac{1}{k'a}\right\}=0,$$

where $k=\sqrt{2ME}/\hbar$ and $k'=\sqrt{2M(V_0-E)}/\hbar$.

(c) With $a = 4\pi\varepsilon_0\hbar^2/Me^2$ (Bohr radius) and $V_0 = 4Me^4/2\hbar^2$, compute the possible bound states $(0 < E < V_0)$.
Hint. Call a root-finding subroutine after you know the approximate location of the roots of

$$f(E) = 0 \qquad (0 \le E \le V_0).$$

(d) Show that when $a = 4\pi\varepsilon_0\hbar^2/Me^2$ the minimum value of V_0 for which a bound state exists is $V_0 = 2.4674Me^4/2\hbar^2$.

11.7.27 In some nuclear stripping reactions the differential cross section is proportional to $j_l(x)^2$, where l is the angular momentum. The location of the maximum on the curve of experimental data permits a determination of l, if the location of the (first) maximum of $j_l(x)$ is known. Compute the location of the first maximum of $j_1(x)$, $j_2(x)$, and $j_3(x)$.
Note. For better accuracy look for the first zero of $j_l'(x)$. Why is this more accurate than direct location of the maximum?

Additional Readings

Jackson, J. D., *Classical Electrodynamics*, 3rd ed., New York: J. Wiley (1999).

McBride, E. B., *Obtaining Generating Functions*. New York: Springer-Verlag (1971). An introduction to methods of obtaining generating functions.

Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge, UK: Cambridge University Press (1952). This is the definitive text on Bessel functions and their properties. Although difficult reading, it is invaluable as the ultimate reference.

Watson, G. N., *A Treatise on the Theory of Bessel Functions*, 1st ed. Cambridge, UK: Cambridge University Press (1922). See also the references listed at the end of Chapter 13.