7
Green's Functions and Nonhomogeneous Problems

"The young theoretical physicists of a generation or two earlier subscribed to the belief that: If you haven't done something important by age 30, you never will. Obviously, they were unfamiliar with the history of George Green, the miller of Nottingham." Julian Schwinger (1918-1994)

The wave equation, heat equation, and Laplace's equation are typical homogeneous partial differential equations. They can be written in the form

$$\mathcal{L}u(x) = 0,$$

where $\mathcal{L}$ is a differential operator. For example, these equations can be written as

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u &= 0, \\ \left(\frac{\partial}{\partial t} - k\nabla^2\right) u &= 0, \\ \nabla^2 u &= 0. \end{aligned} \tag{7.1}$$

In this chapter we will explore solutions of nonhomogeneous partial differential equations,

$$\mathcal{L}u(x) = f(x),$$

by seeking out the so-called Green's function. The history of the Green's function dates back to 1828, when George Green published work in which he sought solutions of Poisson's equation $\nabla^2 u = f$ for the electric potential u defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined the "Green's function". In this chapter we will derive the initial value Green's function for ordinary differential equations. Later in the chapter we will return to boundary value Green's functions and Green's functions for partial differential equations.

George Green (1793-1841), a British mathematical physicist who had little formal education and worked as a miller and a baker, published *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* in which he not only introduced what is now known as Green's function, but he also introduced potential theory and Green's Theorem in his studies of electricity and magnetism. Recently his paper was posted at arXiv.org, arXiv:0807.0088.

As a simple example, consider Poisson's equation,

$$\nabla^2 u(\mathbf{r}) = f(\mathbf{r}).$$

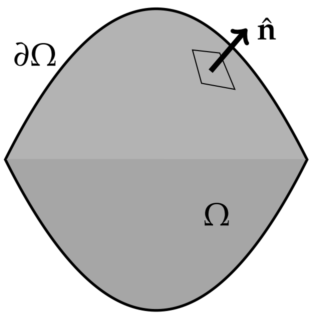

Figure 7.1: Let Poisson's equation hold inside region Ω bounded by surface $\partial\Omega$.

Let Poisson's equation hold inside a region Ω bounded by the surface $\partial\Omega$ as shown in Figure 7.1. This is the nonhomogeneous form of Laplace's equation. The nonhomogeneous term, $f(\mathbf{r})$, could represent a heat source in a steady-state problem or a charge distribution (source) in an electrostatic problem.

Now think of the source as a point source in which we are interested in the response of the system to this point source. If the point source is located at a point $\mathbf{r}'$, then the response to the point source could be felt at points $\mathbf{r}$. We will call this response $G(\mathbf{r}, \mathbf{r}')$. The response function would satisfy a point source equation of the form

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}').$$

The Dirac delta function satisfies

$$\delta(\mathbf{r}) = 0, \quad \mathbf{r} \neq \mathbf{0},$$

$$\int_\Omega \delta(\mathbf{r})\, dV = 1.$$

Here $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac delta function, which we will consider in more detail in Section 9.4. A key property of this generalized function is the sifting property,

$$\int_\Omega \delta(\mathbf{r} - \mathbf{r}') f(\mathbf{r})\, dV = f(\mathbf{r}').$$

The connection between the Green's function and the solution to Poisson's equation can be found from Green's second identity:

$$\int_{\partial\Omega} [\phi\nabla\psi - \psi\nabla\phi] \cdot \mathbf{n}\, dS = \int_\Omega [\phi\nabla^2\psi - \psi\nabla^2\phi]\, dV.$$

Letting $\phi = u(\mathbf{r})$ and $\psi = G(\mathbf{r}, \mathbf{r}')$, we have[1]

[1] We note that in the following the volume and surface integrals and differentiation using ∇ are performed using the $\mathbf{r}$-coordinates.

$$\begin{aligned}
&\int_{\partial\Omega} [u(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla u(\mathbf{r})] \cdot \mathbf{n}\, dS \\
&= \int_\Omega \left[u(\mathbf{r})\nabla^2 G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla^2 u(\mathbf{r})\right] dV \\
&= \int_\Omega \left[u(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') - G(\mathbf{r}, \mathbf{r}') f(\mathbf{r})\right] dV \\
&= u(\mathbf{r}') - \int_\Omega G(\mathbf{r}, \mathbf{r}') f(\mathbf{r})\, dV. \qquad (7.2)
\end{aligned}$$

Solving for $u(\mathbf{r}')$, we have

$$\begin{aligned}
u(\mathbf{r}') &= \int_\Omega G(\mathbf{r}, \mathbf{r}') f(\mathbf{r})\, dV \\
&\quad + \int_{\partial\Omega} [u(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla u(\mathbf{r})] \cdot \mathbf{n}\, dS. \qquad (7.3)
\end{aligned}$$

If both $u(\mathbf{r})$ and $G(\mathbf{r}, \mathbf{r}')$ satisfied Dirichlet conditions, $u = 0$ on $\partial\Omega$, then the last integral vanishes and we are left with[2]

[2] In many applications there is a symmetry,

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r}).$$

Then, the result can be written as

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}')\, dV'.$$

$$u(\mathbf{r}') = \int_\Omega G(\mathbf{r}, \mathbf{r}') f(\mathbf{r})\, dV.$$

So, if we know the Green's function, we can solve the nonhomogeneous differential equation. In fact, we can use the Green's function to solve nonhomogenous boundary value and initial value problems. That is what we will see develop in this chapter as we explore nonhomogeneous problems in more detail. We will begin with the search for Green's functions for ordinary differential equations.

7.1 Initial Value Green's Functions

In this section we will investigate the solution of initial value problems involving nonhomogeneous differential equations using Green's functions. Our goal is to solve the nonhomogeneous differential equation

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \tag{7.4}$$

subject to the initial conditions

$$y(0) = y_0 \quad y'(0) = v_0.$$

Since we are interested in initial value problems, we will denote the independent variable as a time variable, t.

Equation (7.4) can be written compactly as

$$L[y] = f,$$

where L is the differential operator

$$L = a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t).$$

The solution is formally given by

$$y = L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y(t) = \int G(t,\tau)f(\tau)\,d\tau.$$

The function $G(t,\tau)$ is referred to as the kernel of the integral operator and is called the Green's function.

$G(t,\tau)$ is called a Green's function.

In the last section we solved nonhomogeneous equations like (7.4) using the Method of Variation of Parameters. Letting,

$$y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t), \tag{7.5}$$

we found that we have to solve the system of equations

$$\begin{aligned} c_1'(t)y_1(t) + c_2'(t)y_2(t) &= 0. \\ c_1'(t)y_1'(t) + c_2'(t)y_2'(t) &= \frac{f(t)}{q(t)}. \end{aligned} \tag{7.6}$$

This system is easily solved to give

$$\begin{aligned} c_1'(t) &= -\frac{f(t)y_2(t)}{a(t)\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]} \\ c_2'(t) &= \frac{f(t)y_1(t)}{a(t)\left[y_1(t)y_2'(t) - y_1'(t)y_2(t)\right]}. \end{aligned} \tag{7.7}$$

We note that the denominator in these expressions involves the Wronskian of the solutions to the homogeneous problem, which is given by the determinant

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

When $y_1(t)$ and $y_2(t)$ are linearly independent, then the Wronskian is not zero and we are guaranteed a solution to the above system.

So, after an integration, we find the parameters as

$$\begin{aligned} c_1(t) &= -\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau \\ c_2(t) &= \int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau, \end{aligned} \tag{7.8}$$

where t_0 and t_1 are arbitrary constants to be determined from the initial conditions.

Therefore, the particular solution of (7.4) can be written as

$$y_p(t) = y_2(t)\int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t)\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{7.9}$$

We begin with the particular solution (7.9) of the nonhomogeneous differential equation (7.4). This can be combined with the general solution of the homogeneous problem to give the general solution of the nonhomogeneous differential equation:

$$y_p(t) = c_1y_1(t) + c_2y_2(t) + y_2(t)\int_{t_1}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t)\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{7.10}$$

However, an appropriate choice of t_0 and t_1 can be found so that we need not explicitly write out the solution to the homogeneous problem, $c_1y_1(t) + c_2y_2(t)$. However, setting up the solution in this form will allow us to use t_0 and t_1 to determine particular solutions which satisfies certain homogeneous conditions. In particular, we will show that Equation (7.10) can be written in the form

$$y(t) = c_1y_1(t) + c_2y_2(t) + \int_0^t G(t,\tau)f(\tau)\, d\tau, \tag{7.11}$$

where the function $G(t,\tau)$ will be identified as the Green's function.

The goal is to develop the Green's function technique to solve the initial value problem

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0. \tag{7.12}$$

We first note that we can solve this initial value problem by solving two separate initial value problems. We assume that the solution of the homogeneous problem satisfies the original initial conditions:

$$a(t)y_h''(t) + b(t)y_h'(t) + c(t)y_h(t) = 0, \quad y_h(0) = y_0, \quad y_h'(0) = v_0. \tag{7.13}$$

We then assume that the particular solution satisfies the problem

$$a(t)y_p''(t) + b(t)y_p'(t) + c(t)y_p(t) = f(t), \quad y_p(0) = 0, \quad y_p'(0) = 0. \tag{7.14}$$

Since the differential equation is linear, then we know that

$$y(t) = y_h(t) + y_p(t)$$

is a solution of the nonhomogeneous equation. Also, this solution satisfies the initial conditions:

$$y(0) = y_h(0) + y_p(0) = y_0 + 0 = y_0,$$
$$y'(0) = y_h'(0) + y_p'(0) = v_0 + 0 = v_0.$$

Therefore, we need only focus on finding a particular solution that satisfies homogeneous initial conditions. This will be done by finding values for t_0 and t_1 in Equation (7.9) which satisfy the homogeneous initial conditions, $y_p(0) = 0$ and $y_p'(0) = 0$.

First, we consider $y_p(0) = 0$. We have

$$y_p(0) = y_2(0)\int_{t_1}^{0} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(0)\int_{t_0}^{0} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{7.15}$$

Here, $y_1(t)$ and $y_2(t)$ are taken to be any solutions of the homogeneous differential equation. Let's assume that $y_1(0) = 0$ and $y_2 \neq (0) = 0$. Then, we have

$$y_p(0) = y_2(0)\int_{t_1}^{0} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau \tag{7.16}$$

We can force $y_p(0) = 0$ if we set $t_1 = 0$.

Now, we consider $y_p'(0) = 0$. First we differentiate the solution and find that

$$y_p'(t) = y_2'(t)\int_{0}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1'(t)\int_{t_0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau, \tag{7.17}$$

since the contributions from differentiating the integrals will cancel. Evaluating this result at $t = 0$, we have

$$y_p'(0) = -y_1'(0)\int_{t_0}^{0} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau. \tag{7.18}$$

Assuming that $y_1'(0) \neq 0$, we can set $t_0 = 0$.

Thus, we have found that

$$\begin{aligned} y_p(x) &= y_2(t)\int_{0}^{t} \frac{f(\tau)y_1(\tau)}{a(\tau)W(\tau)}\, d\tau - y_1(t)\int_{0}^{t} \frac{f(\tau)y_2(\tau)}{a(\tau)W(\tau)}\, d\tau \\ &= \int_0^t \left[\frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}\right] f(\tau)\, d\tau. \end{aligned} \tag{7.19}$$

This result is in the correct form and we can identify the temporal, or initial value, Green's function. So, the particular solution is given as

$$y_p(t) = \int_0^t G(t,\tau) f(\tau)\, d\tau, \tag{7.20}$$

where the initial value Green's function is defined as

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)}.$$

We summarize

Solution of IVP Using the Green's Function

The solution of the initial value problem,

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t), \quad y(0) = y_0, \quad y'(0) = v_0,$$

takes the form

$$y(t) = y_h(t) + \int_0^t G(t,\tau)f(\tau)\,d\tau, \tag{7.21}$$

where

$$G(t,\tau) = \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \tag{7.22}$$

is the Green's function and y_1, y_2, y_h are solutions of the homogeneous equation satisfying

$$y_1(0) = 0, y_2(0) \neq 0, y_1'(0) \neq 0, y_2'(0) = 0, y_h(0) = y_0, y_h'(0) = v_0.$$

Example 7.1. *Solve the forced oscillator problem*

$$x'' + x = 2\cos t, \quad x(0) = 4, \quad x'(0) = 0.$$

We first solve the homogeneous problem with nonhomogeneous initial conditions:

$$x_h'' + x_h = 0, \quad x_h(0) = 4, \quad x_h'(0) = 0.$$

The solution is easily seen to be $x_h(t) = 4\cos t$.

Next, we construct the Green's function. We need two linearly independent solutions, $y_1(x)$, $y_2(x)$, *to the homogeneous differential equation satisfying different homogeneous conditions,* $y_1(0) = 0$ *and* $y_2'(0) = 0$. *The simplest solutions are* $y_1(t) = \sin t$ *and* $y_2(t) = \cos t$. *The Wronskian is found as*

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = -\sin^2 t - \cos^2 t = -1.$$

Since $a(t) = 1$ *in this problem, we compute the Green's function,*

$$\begin{aligned} G(t,\tau) &= \frac{y_1(\tau)y_2(t) - y_1(t)y_2(\tau)}{a(\tau)W(\tau)} \\ &= \sin t\cos\tau - \sin\tau\cos t \\ &= \sin(t-\tau). \end{aligned} \tag{7.23}$$

Note that the Green's function depends on $t - \tau$. *While this is useful in some contexts, we will use the expanded form when carrying out the integration.*

We can now determine the particular solution of the nonhomogeneous differential equation. We have

$$x_p(t) = \int_0^t G(t,\tau)f(\tau)\,d\tau$$

$$\begin{aligned}
&= \int_0^t (\sin t \cos\tau - \sin\tau \cos t)\,(2\cos\tau)\; d\tau \\
&= 2\sin t \int_0^t \cos^2\tau d\tau - 2\cos t \int_0^t \sin\tau\cos\tau d\tau \\
&= 2\sin t \left[\frac{\tau}{2} + \frac{1}{2}\sin 2\tau\right]_0^t - 2\cos t\left[\frac{1}{2}\sin^2\tau\right]_0^t \\
&= t\sin t. \qquad (7.24)
\end{aligned}$$

Therefore, the solution of the nonhomogeneous problem is the sum of the solution of the homogeneous problem and this particular solution: $x(t) = 4\cos t + t\sin t$.

7.2 Boundary Value Green's Functions

We solved nonhomogeneous initial value problems in Section 7.1 using a Green's function. In this section we will extend this method to the solution of nonhomogeneous boundary value problems using a boundary value Green's function. Recall that the goal is to solve the nonhomogeneous differential equation

$$L[y] = f, \quad a \le x \le b,$$

where L is a differential operator and $y(x)$ satisfies boundary conditions at $x = a$ and $x = b$.. The solution is formally given by

$$y = L^{-1}[f].$$

The inverse of a differential operator is an integral operator, which we seek to write in the form

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi.$$

The function $G(x,\xi)$ is referred to as the kernel of the integral operator and is called the *Green's function*.

We will consider boundary value problems in Sturm-Liouville form,

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a < x < b, \qquad (7.25)$$

with fixed values of $y(x)$ at the boundary, $y(a) = 0$ and $y(b) = 0$. However, the general theory works for other forms of homogeneous boundary conditions.

We seek the Green's function by first solving the nonhomogeneous differential equation using the Method of Variation of Parameters. Recall this method from Section B.3.3. We assume a particular solution of the form

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x),$$

which is formed from two linearly independent solution of the homogeneous problem, $y_i(x)$, $i = 1, 2$. We had found that the coefficient functions satisfy the equations

$$\begin{aligned}
c_1'(x)y_1(x) + c_2'(x)y_2(x) &= 0 \\
c_1'(x)y_1'(x) + c_2'(x)y_2'(x) &= \frac{f(x)}{p(x)}. \qquad (7.26)
\end{aligned}$$

Solving this system, we obtain

$$c_1'(x) = -\frac{fy_2}{pW(y_1, y_2)},$$

$$c_1'(x) = \frac{fy_1}{pW(y_1, y_2)},$$

where $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ is the Wronskian. Integrating these forms and inserting the results back into the particular solution, we find

$$y(x) = y_2(x) \int_{x_1}^{x} \frac{f(\xi) y_1(\xi)}{p(\xi) W(\xi)}\, d\xi - y_1(x) \int_{x_0}^{x} \frac{f(\xi) y_2(\xi)}{p(\xi) W(\xi)}\, d\xi,$$

where x_0 and x_1 are to be determined using the boundary values. In particular, we will seek x_0 and x_1 so that the solution to the boundary value problem can be written as a single integral involving a Green's function. Note that we can absorb the solution to the homogeneous problem, $y_h(x)$, into the integrals with an appropriate choice of limits on the integrals.

We now look to satisfy the conditions $y(a) = 0$ and $y(b) = 0$. First we use solutions of the homogeneous differential equation that satisfy $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$. Evaluating $y(x)$ at $x = 0$, we have

$$\begin{aligned} y(a) &= y_2(a) \int_{x_1}^{a} \frac{f(\xi) y_1(\xi)}{p(\xi) W(\xi)}\, d\xi - y_1(a) \int_{x_0}^{a} \frac{f(\xi) y_2(\xi)}{p(\xi) W(\xi)}\, d\xi \\ &= y_2(a) \int_{x_1}^{a} \frac{f(\xi) y_1(\xi)}{p(\xi) W(\xi)}\, d\xi. \end{aligned} \tag{7.27}$$

We can satisfy the condition at $x = a$ if we choose $x_1 = a$.

Similarly, at $x = b$ we find that

$$\begin{aligned} y(b) &= y_2(b) \int_{x_1}^{b} \frac{f(\xi) y_1(\xi)}{p(\xi) W(\xi)}\, d\xi - y_1(b) \int_{x_0}^{b} \frac{f(\xi) y_2(\xi)}{p(\xi) W(\xi)}\, d\xi \\ &= -y_1(b) \int_{x_0}^{b} \frac{f(\xi) y_2(\xi)}{p(\xi) W(\xi)}\, d\xi. \end{aligned} \tag{7.28}$$

The general solution of the boundary value problem.

This expression vanishes for $x_0 = b$.

So, we have found that the solution takes the form

$$y(x) = y_2(x) \int_{a}^{x} \frac{f(\xi) y_1(\xi)}{p(\xi) W(\xi)}\, d\xi - y_1(x) \int_{b}^{x} \frac{f(\xi) y_2(\xi)}{p(\xi) W(\xi)}\, d\xi. \tag{7.29}$$

This solution can be written in a compact form just like we had done for the initial value problem in Section 7.1. We seek a Green's function so that the solution can be written as a single integral. We can move the functions of x under the integral. Also, since $a < x < b$, we can flip the limits in the second integral. This gives

$$y(x) = \int_{a}^{x} \frac{f(\xi) y_1(\xi) y_2(x)}{p(\xi) W(\xi)}\, d\xi + \int_{x}^{b} \frac{f(\xi) y_1(x) y_2(\xi)}{p(\xi) W(\xi)}\, d\xi. \tag{7.30}$$

This result can now be written in a compact form:

Boundary Value Green's Function

The solution of the boundary value problem

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x), \quad a < x < b,$$
$$y(a) = 0, \quad y(b) = 0. \tag{7.31}$$

takes the form

$$y(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi, \tag{7.32}$$

where the Green's function is the piecewise defined function

$$G(x,\xi) = \begin{cases} \dfrac{y_1(\xi)y_2(x)}{pW}, & a \le \xi \le x, \\ \dfrac{y_1(x)y_2(\xi)}{pW}, & x \le \xi \le b, \end{cases} \tag{7.33}$$

where $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous problem satisfying $y_1(a) = 0$, $y_2(b) = 0$ and $y_1(b) \neq 0$, $y_2(a) \neq 0$.

The Green's function satisfies several properties, which we will explore further in the next section. For example, the Green's function satisfies the boundary conditions at $x = a$ and $x = b$. Thus,

$$G(a,\xi) = \frac{y_1(a)y_2(\xi)}{pW} = 0,$$

$$G(b,\xi) = \frac{y_1(\xi)y_2(b)}{pW} = 0.$$

Also, the Green's function is symmetric in its arguments. Interchanging the arguments gives

$$G(\xi,x) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{pW}, & a \le x \le \xi, \\ \dfrac{y_1(\xi)y_2(x)}{pW}. & \xi \le x \le b, \end{cases} \tag{7.34}$$

But a careful look at the original form shows that

$$G(x,\xi) = G(\xi,x).$$

We will make use of these properties in the next section to quickly determine the Green's functions for other boundary value problems.

Example 7.2. *Solve the boundary value problem $y'' = x^2$, $\quad y(0) = 0 = y(1)$ using the boundary value Green's function.*

We first solve the homogeneous equation, $y'' = 0$. After two integrations, we have $y(x) = Ax + B$, for A and B constants to be determined.

We need one solution satisfying $y_1(0) = 0$ Thus,

$$0 = y_1(0) = B.$$

So, we can pick $y_1(x) = x$, since A is arbitrary.

The other solution has to satisfy $y_2(1) = 0$. So,

$$0 = y_2(1) = A + B.$$

This can be solved for $B = -A$. Again, A is arbitrary and we will choose $A = -1$. Thus, $y_2(x) = 1 - x$.

For this problem $p(x) = 1$. Thus, for $y_1(x) = x$ and $y_2(x) = 1 - x$,

$$p(x)W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) = x(-1) - 1(1-x) = -1.$$

Note that $p(x)W(x)$ is a constant, as it should be.

Now we construct the Green's function. We have

$$G(x,\xi) = \begin{cases} -\xi(1-x), & 0 \le \xi \le x, \\ -x(1-\xi), & x \le \xi \le 1. \end{cases} \tag{7.35}$$

Notice the symmetry between the two branches of the Green's function. Also, the Green's function satisfies homogeneous boundary conditions: $G(0,\xi) = 0$, from the lower branch, and $G(1,\xi) = 0$, from the upper branch.

Finally, we insert the Green's function into the integral form of the solution and evaluate the integral.

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi) f(\xi)\, d\xi \\ &= \int_0^1 G(x,\xi)\xi^2\, d\xi \\ &= -\int_0^x \xi(1-x)\xi^2\, d\xi - \int_x^1 x(1-\xi)\xi^2\, d\xi \\ &= -(1-x)\int_0^x \xi^3\, d\xi - x\int_x^1 (\xi^2 - \xi^3)\, d\xi \\ &= -(1-x)\left[\frac{\xi^4}{4}\right]_0^x - x\left[\frac{\xi^3}{3} - \frac{\xi^4}{4}\right]_x^1 \\ &= -\frac{1}{4}(1-x)x^4 - \frac{1}{12}x(4-3) + \frac{1}{12}x(4x^3 - 3x^4) \\ &= \frac{1}{12}(x^4 - x). \end{aligned} \tag{7.36}$$

Checking the answer, we can easily verify that $y'' = x^2$, $y(0) = 0$, and $y(1) = 0$.

7.2.1 Properties of Green's Functions

WE HAVE NOTED SOME PROPERTIES OF GREEN'S FUNCTIONS in the last section. In this section we will elaborate on some of these properties as a tool for quickly constructing Green's functions for boundary value problems. We list five basic properties:

1. **Differential Equation:**

   The boundary value Green's function satisfies the differential equation $\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0$, $x \ne \xi$.

This is easily established. For $x < \xi$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, since $y_1(x)$ is a solution of the homogeneous equation, then so is $G(x,\xi)$. For $x > \xi$ we are on the first branch and $G(x,\xi)$ is proportional to $y_2(x)$. So, once again $G(x,\xi)$ is a solution of the homogeneous problem.

2. **Boundary Conditions:**

   In the example in the last section we had seen that $G(a,\xi) = 0$ and $G(b,\xi) = 0$. For example, for $x = a$ we are on the second branch and $G(x,\xi)$ is proportional to $y_1(x)$. Thus, whatever condition $y_1(x)$ satisfies, $G(x,\xi)$ will satisfy. A similar statement can be made for $x = b$.

3. **Symmetry or Reciprocity:** $G(x,\xi) = G(\xi,x)$

   We had shown this reciprocity property in the last section.

4. **Continuity of G at** $x = \xi$**:** $G(\xi^+,\xi) = G(\xi^-,\xi)$

   Here we define $\xi^\pm$ through the limits of a function as x approaches ξ from above or below. In particular,

$$G(\xi^+,x) = \lim_{x\downarrow\xi} G(x,\xi), \quad x > \xi,$$

$$G(\xi^-,x) = \lim_{x\uparrow\xi} G(x,\xi), \quad x < \xi.$$

   Setting $x = \xi$ in both branches, we have

$$\frac{y_1(\xi)y_2(\xi)}{pW} = \frac{y_1(\xi)y_2(\xi)}{pW}.$$

   Therefore, we have established the continuity of $G(x,\xi)$ between the two branches at $x = \xi$.

5. **Jump Discontinuity of** $\frac{\partial G}{\partial x}$ **at** $x = \xi$**:**

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}$$

   This case is not as obvious. We first compute the derivatives by noting which branch is involved and then evaluate the derivatives and subtract them. Thus, we have

$$\begin{aligned}
\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} &= -\frac{1}{pW}y_1(\xi)y_2'(\xi) + \frac{1}{pW}y_1'(\xi)y_2(\xi) \\
&= -\frac{y_1'(\xi)y_2(\xi) - y_1(\xi)y_2'(\xi)}{p(\xi)(y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi))} \\
&= \frac{1}{p(\xi)}. \qquad (7.37)
\end{aligned}$$

Here is a summary of the properties of the boundary value Green's function based upon the previous solution.

Properties of the Green's Function

1. **Differential Equation:**
   $\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, x \neq \xi$
2. **Boundary Conditions:** Whatever conditions $y_1(x)$ and $y_2(x)$ satisfy, $G(x,\xi)$ will satisfy.
3. **Symmetry or Reciprocity:** $G(x,\xi) = G(\xi, x)$
4. **Continuity of G at** $x = \xi$**:** $G(\xi^+,\xi) = G(\xi^-,\xi)$
5. **Jump Discontinuity of** $\frac{\partial G}{\partial x}$ **at** $x = \xi$**:**

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}$$

We now show how a knowledge of these properties allows one to quickly construct a Green's function with an example.

Example 7.3. *Construct the Green's function for the problem*

$$y'' + \omega^2 y = f(x), \quad 0 < x < 1,$$

$$y(0) = 0 = y(1),$$

with $\omega \neq 0$.

I. **Find solutions to the homogeneous equation.**
A general solution to the homogeneous equation is given as

$$y_h(x) = c_1 \sin\omega x + c_2 \cos\omega x.$$

Thus, for $x \neq \xi$,

$$G(x,\xi) = c_1(\xi)\sin\omega x + c_2(\xi)\cos\omega x.$$

II. **Boundary Conditions.**
First, we have $G(0,\xi) = 0$ *for* $0 \leq x \leq \xi$. *So,*

$$G(0,\xi) = c_2(\xi)\cos\omega x = 0.$$

So,

$$G(x,\xi) = c_1(\xi)\sin\omega x, \quad 0 \leq x \leq \xi.$$

Second, we have $G(1,\xi) = 0$ *for* $\xi \leq x \leq 1$. *So,*

$$G(1,\xi) = c_1(\xi)\sin\omega + c_2(\xi)\cos\omega. = 0$$

A solution can be chosen with

$$c_2(\xi) = -c_1(\xi)\tan\omega.$$

This gives

$$G(x,\xi) = c_1(\xi)\sin\omega x - c_1(\xi)\tan\omega\cos\omega x.$$

This can be simplified by factoring out the $c_1(\xi)$ and placing the remaining terms over a common denominator. The result is

$$\begin{aligned} G(x,\xi) &= \frac{c_1(\xi)}{\cos\omega}\left[\sin\omega x\cos\omega - \sin\omega\cos\omega x\right] \\ &= -\frac{c_1(\xi)}{\cos\omega}\sin\omega(1-x). \end{aligned} \tag{7.38}$$

Since the coefficient is arbitrary at this point, as can write the result as

$$G(x,\xi) = d_1(\xi)\sin\omega(1-x), \quad \xi \le x \le 1.$$

We note that we could have started with $y_2(x) = \sin\omega(1-x)$ as one of the linearly independent solutions of the homogeneous problem in anticipation that $y_2(x)$ satisfies the second boundary condition.

III. **Symmetry or Reciprocity**

We now impose that $G(x,\xi) = G(\xi,x)$. To this point we have that

$$G(x,\xi) = \begin{cases} c_1(\xi)\sin\omega x, & 0 \le x \le \xi, \\ d_1(\xi)\sin\omega(1-x), & \xi \le x \le 1. \end{cases}$$

We can make the branches symmetric by picking the right forms for $c_1(\xi)$ and $d_1(\xi)$. We choose $c_1(\xi) = C\sin\omega(1-\xi)$ and $d_1(\xi) = C\sin\omega\xi$. Then,

$$G(x,\xi) = \begin{cases} C\sin\omega(1-\xi)\sin\omega x, & 0 \le x \le \xi, \\ C\sin\omega(1-x)\sin\omega\xi, & \xi \le x \le 1. \end{cases}$$

Now the Green's function is symmetric and we still have to determine the constant C. We note that we could have gotten to this point using the Method of Variation of Parameters result where $C = \frac{1}{pW}$.

IV. **Continuity of** $G(x,\xi)$

We already have continuity by virtue of the symmetry imposed in the last step.

V. **Jump Discontinuity in** $\frac{\partial}{\partial x}G(x,\xi)$**.**

We still need to determine C. We can do this using the jump discontinuity in the derivative:

$$\frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} = \frac{1}{p(\xi)}.$$

For this problem $p(x) = 1$. Inserting the Green's function, we have

$$\begin{aligned} 1 &= \frac{\partial G(\xi^+,\xi)}{\partial x} - \frac{\partial G(\xi^-,\xi)}{\partial x} \\ &= \frac{\partial}{\partial x}\left[C\sin\omega(1-x)\sin\omega\xi\right]_{x=\xi} - \frac{\partial}{\partial x}\left[C\sin\omega(1-\xi)\sin\omega x\right]_{x=\xi} \\ &= -\omega C\cos\omega(1-\xi)\sin\omega\xi - \omega C\sin\omega(1-\xi)\cos\omega\xi \\ &= -\omega C\sin\omega(\xi+1-\xi) \\ &= -\omega C\sin\omega. \end{aligned} \tag{7.39}$$

Therefore,

$$C = -\frac{1}{\omega \sin \omega}.$$

Finally, we have the Green's function:

$$G(x,\xi) = \begin{cases} -\dfrac{\sin \omega(1-\xi) \sin \omega x}{\omega \sin \omega}, & 0 \leq x \leq \xi, \\ -\dfrac{\sin \omega(1-x) \sin \omega \xi}{\omega \sin \omega}, & \xi \leq x \leq 1. \end{cases} \tag{7.40}$$

It is instructive to compare this result to the Variation of Parameters result.

Example 7.4. *Use the Method of Variation of Parameters to solve*

$$y'' + \omega^2 y = f(x), \quad 0 < x < 1,$$

$$y(0) = 0 = y(1), \quad \omega \neq 0.$$

We have the functions $y_1(x) = \sin \omega x$ *and* $y_2(x) = \sin \omega(1-x)$ *as the solutions of the homogeneous equation satisfying* $y_1(0) = 0$ *and* $y_2(1) = 0$*. We need to compute* pW*:*

$$\begin{aligned} p(x)W(x) &= y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= -\omega \sin \omega x \cos \omega(1-x) - \omega \cos \omega x \sin \omega(1-x) \\ &= -\omega \sin \omega \end{aligned} \tag{7.41}$$

Inserting this result into the Variation of Parameters result for the Green's function leads to the same Green's function as above.

7.2.2 *The Differential Equation for the Green's Function*

As we progress in the book we will develop a more general theory of Green's functions for ordinary and partial differential equations. Much of this theory relies on understanding that the Green's function really is the system response function to a point source. This begins with recalling that the boundary value Green's function satisfies a homogeneous differential equation for $x \neq \xi$,

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = 0, \quad x \neq \xi. \tag{7.42}$$

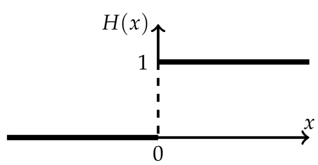

Figure 7.2: The Heaviside step function, $H(x)$.

For $x = \xi$, we have seen that the derivative has a jump in its value. This is similar to the step, or Heaviside, function,

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

This function is shown in Figure 7.2 and we see that the derivative of the step function is zero everywhere except at the jump, or discontinuity . At the jump, there is an infinite slope, though technically, we have learned that there is no derivative at this point. We will try to remedy this situation by introducing the Dirac delta function,

$$\delta(x) = \frac{d}{dx}H(x).$$

We will show that the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{7.43}$$

However, we will first indicate why this knowledge is useful for the general theory of solving differential equations using Green's functions.

As noted, the Green's function satisfies the differential equation

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi) \tag{7.44}$$

and satisfies homogeneous conditions. We will use the Green's function to solve the nonhomogeneous equation

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = f(x). \tag{7.45}$$

The Dirac delta function is described in more detail in Section 9.4. The key property we will need here is the sifting property,

$$\int_a^b f(x)\delta(x-\xi)\,dx = f(\xi)$$

for $a < \xi < b$.

These equations can be written in the more compact forms

$$\begin{aligned} \mathcal{L}[y] &= f(x) \\ \mathcal{L}[G] &= \delta(x-\xi). \end{aligned} \tag{7.46}$$

Using these equations, we can determine the solution, $y(x)$, in terms of the Green's function. Multiplying the first equation by $G(x,\xi)$, the second equation by $y(x)$, and then subtracting, we have

$$G\mathcal{L}[y] - y\mathcal{L}[G] = f(x)G(x,\xi) - \delta(x-\xi)y(x).$$

Now, integrate both sides from $x = a$ to $x = b$. The left hand side becomes

$$\int_a^b [f(x)G(x,\xi) - \delta(x-\xi)y(x)]\,dx = \int_a^b f(x)G(x,\xi)\,dx - y(\xi).$$

Using Green's Identity from Section 4.2.2, the right side is

$$\int_a^b (G\mathcal{L}[y] - y\mathcal{L}[G])\,dx = \left[p(x)\left(G(x,\xi)y'(x) - y(x)\frac{\partial G}{\partial x}(x,\xi)\right)\right]_{x=a}^{x=b}.$$

Recall that Green's identity is given by

$$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx = [p(uv' - vu')]_a^b.$$

Combining these results and rearranging, we obtain

The general solution in terms of the boundary value Green's function with corresponding surface terms.

$$\boxed{\begin{aligned} y(\xi) &= \int_a^b f(x)G(x,\xi)\,dx \\ &\quad - \left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\xi) - G(x,\xi)y'(x)\right)\right]_{x=a}^{x=b}. \end{aligned}} \tag{7.47}$$

We will refer to the extra terms in the solution,

$$S(b,\xi) - S(a,\xi) = \left[p(x)\left(y(x)\frac{\partial G}{\partial x}(x,\xi) - G(x,\xi)y'(x) \right) \right]_{x=a}^{x=b},$$

as the boundary, or surface, terms. Thus,

$$y(\xi) = \int_a^b f(x)G(x,\xi)\,dx - [S(b,\xi) - S(a,\xi)].$$

The result in Equation (7.47) is the key equation in determining the solution of a nonhomogeneous boundary value problem. The particular set of boundary conditions in the problem will dictate what conditions $G(x,\xi)$ has to satisfy. For example, if we have the boundary conditions $y(a) = 0$ and $y(b) = 0$, then the boundary terms yield

$$\begin{aligned} y(\xi) &= \int_a^b f(x)G(x,\xi)\,dx - \left[p(b)\left(y(b)\frac{\partial G}{\partial x}(b,\xi) - G(b,\xi)y'(b) \right) \right] \\ &\quad + \left[p(a)\left(y(a)\frac{\partial G}{\partial x}(a,\xi) - G(a,\xi)y'(a) \right) \right] \\ &= \int_a^b f(x)G(x,\xi)\,dx + p(b)G(b,\xi)y'(b) - p(a)G(a,\xi)y'(a). \end{aligned} \tag{7.48}$$

The right hand side will only vanish if $G(x,\xi)$ also satisfies these homogeneous boundary conditions. This then leaves us with the solution

$$y(\xi) = \int_a^b f(x)G(x,\xi)\,dx.$$

We should rewrite this as a function of x. So, we replace ξ with x and x with ξ. This gives

$$y(x) = \int_a^b f(\xi)G(\xi,x)\,d\xi.$$

However, this is not yet in the desirable form. The arguments of the Green's function are reversed. But, in this case $G(x,\xi)$ is symmetric in its arguments. So, we can simply switch the arguments getting the desired result.

We can now see that the theory works for other boundary conditions. If we had $y'(a) = 0$, then the $y(a)\frac{\partial G}{\partial x}(a,\xi)$ term in the boundary terms could be made to vanish if we set $\frac{\partial G}{\partial x}(a,\xi) = 0$. So, this confirms that other boundary value problems can be posed besides the one elaborated upon in the chapter so far.

We can even adapt this theory to nonhomogeneous boundary conditions. We first rewrite Equation (7.47) as

$$y(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi - \left[p(\xi)\left(y(\xi)\frac{\partial G}{\partial \xi}(x,\xi) - G(x,\xi)y'(\xi) \right) \right]_{\xi=a}^{\xi=b}. \tag{7.49}$$

Let's consider the boundary conditions $y(a) = \alpha$ and $y'(b) = \beta$. We also assume that $G(x,\xi)$ satisfies homogeneous boundary conditions,

$$G(a,\xi) = 0, \quad \frac{\partial G}{\partial \xi}(b,\xi) = 0.$$

in both x and ξ since the Green's function is symmetric in its variables. Then, we need only focus on the boundary terms to examine the effect on the solution. We have

$$\begin{aligned} S(b,x) - S(a,x) &= \left[p(b)\left(y(b)\frac{\partial G}{\partial \xi}(x,b) - G(x,b)y'(b)\right)\right] \\ &\quad - \left[p(a)\left(y(a)\frac{\partial G}{\partial \xi}(x,a) - G(x,a)y'(a)\right)\right] \\ &= -\beta p(b)G(x,b) - \alpha p(a)\frac{\partial G}{\partial \xi}(x,a). \end{aligned} \tag{7.50}$$

Therefore, we have the solution

General solution satisfying the nonhomogeneous boundary conditions $y(a) = \alpha$ and $y'(b) = \beta$. Here the Green's function satisfies homogeneous boundary conditions, $G(a,\xi) = 0$, $\frac{\partial G}{\partial \xi}(b,\xi) = 0$.

$$y(x) = \int_a^b G(x,\xi)f(\xi)\,d\xi + \beta p(b)G(x,b) + \alpha p(a)\frac{\partial G}{\partial \xi}(x,a). \tag{7.51}$$

This solution satisfies the nonhomogeneous boundary conditions.

Example 7.5. *Solve* $y'' = x^2$, $y(0) = 1$, $y(1) = 2$ *using the boundary value Green's function.*

This is a modification of Example 7.2. We can use the boundary value Green's function that we found in that problem,

$$G(x,\xi) = \begin{cases} -\xi(1-x), & 0 \le \xi \le x, \\ -x(1-\xi), & x \le \xi \le 1. \end{cases} \tag{7.52}$$

We insert the Green's function into the general solution (7.51) and use the given boundary conditions to obtain

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi)\xi^2\,d\xi - \left[y(\xi)\frac{\partial G}{\partial \xi}(x,\xi) - G(x,\xi)y'(\xi)\right]_{\xi=0}^{\xi=1} \\ &= \int_0^x (x-1)\xi^3\,d\xi + \int_x^1 x(\xi-1)\xi^2\,d\xi + y(0)\frac{\partial G}{\partial \xi}(x,0) - y(1)\frac{\partial G}{\partial \xi}(x,1) \\ &= \frac{(x-1)x^4}{4} + \frac{x(1-x^4)}{4} - \frac{x(1-x^3)}{3} + (x-1) - 2x \\ &= \frac{x^4}{12} + \frac{35}{12}x - 1. \end{aligned} \tag{7.53}$$

Of course, this problem can be solved by direct integration. The general solution is

$$y(x) = \frac{x^4}{12} + c_1 x + c_2.$$

Inserting this solution into each boundary condition yields the same result.

The Green's function satisfies a delta function forced differential equation.

We have seen how the introduction of the Dirac delta function in the differential equation satisfied by the Green's function, Equation (7.44), can lead to the solution of boundary value problems. The Dirac delta function also aids in the interpretation of the Green's function. We note that the Green's function is a solution of an equation in which the nonhomogeneous function is $\delta(x-\xi)$. Note that if we multiply the delta function by $f(\xi)$ and integrate, we obtain

$$\int_{-\infty}^{\infty} \delta(x-\xi)f(\xi)\,d\xi = f(x).$$

We can view the delta function as a unit impulse at $x = \xi$ which can be used to build $f(x)$ as a sum of impulses of different strengths, $f(\xi)$. Thus, the Green's function is the response to the impulse as governed by the differential equation and given boundary conditions.

Derivation of the jump condition for the Green's function.

In particular, the delta function forced equation can be used to derive the jump condition. We begin with the equation in the form

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi) = \delta(x-\xi). \tag{7.54}$$

Now, integrate both sides from $\xi - \epsilon$ to $\xi + \epsilon$ and take the limit as $\epsilon \to 0$. Then,

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\left[\frac{\partial}{\partial x}\left(p(x)\frac{\partial G(x,\xi)}{\partial x}\right) + q(x)G(x,\xi)\right] dx &= \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon}\delta(x-\xi)\, dx \\ &= 1. \end{aligned} \tag{7.55}$$

Since the $q(x)$ term is continuous, the limit as $\epsilon \to 0$ of that term vanishes. Using the Fundamental Theorem of Calculus, we then have

$$\lim_{\epsilon\to 0}\left[p(x)\frac{\partial G(x,\xi)}{\partial x}\right]_{\xi-\epsilon}^{\xi+\epsilon} = 1. \tag{7.56}$$

This is the jump condition that we have been using!

7.2.3 *Series Representations of Green's Functions*

There are times that it might not be so simple to find the Green's function in the simple closed form that we have seen so far. However, there is a method for determining the Green's functions of Sturm-Liouville boundary value problems in the form of an eigenfunction expansion. We will finish our discussion of Green's functions for ordinary differential equations by showing how one obtains such series representations. (Note that we are really just repeating the steps towards developing eigenfunction expansion which we had seen in Section 4.3.)

We will make use of the complete set of eigenfunctions of the differential operator, $\mathcal{L}$, satisfying the homogeneous boundary conditions:

$$\mathcal{L}[\phi_n] = -\lambda_n\sigma\phi_n, \quad n = 1, 2, \ldots$$

We want to find the particular solution y satisfying $\mathcal{L}[y] = f$ and homogeneous boundary conditions. We assume that

$$y(x) = \sum_{n=1}^{\infty} a_n\phi_n(x).$$

Inserting this into the differential equation, we obtain

$$\mathcal{L}[y] = \sum_{n=1}^{\infty} a_n\mathcal{L}[\phi_n] = -\sum_{n=1}^{\infty}\lambda_n a_n\sigma\phi_n = f.$$

This has resulted in the generalized Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \sigma \phi_n(x)$$

with coefficients

$$c_n = -\lambda_n a_n.$$

We have seen how to compute these coefficients earlier in section 4.3. We multiply both sides by $\phi_k(x)$ and integrate. Using the orthogonality of the eigenfunctions,

$$\int_a^b \phi_n(x)\phi_k(x)\sigma(x)\,dx = N_k \delta_{nk},$$

one obtains the expansion coefficients (if $\lambda_k \neq 0$)

$$a_k = -\frac{(f, \phi_k)}{N_k \lambda_k},$$

where $(f, \phi_k) \equiv \int_a^b f(x)\phi_k(x)\,dx$.

As before, we can rearrange the solution to obtain the Green's function. Namely, we have

$$y(x) = \sum_{n=1}^{\infty} \frac{(f, \phi_n)}{-N_n \lambda_n} \phi_n(x) = \int_a^b \underbrace{\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-N_n \lambda_n}}_{G(x,\xi)} f(\xi)\,d\xi$$

Therefore, we have found the Green's function as an expansion in the eigenfunctions:

Green's function as an expansion in the eigenfunctions.

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n N_n}. \tag{7.57}$$

We will conclude this discussion with an example. We will solve this problem three different ways in order to summarize the methods we have used in the text.

Example 7.6. *Solve*

$$y'' + 4y = x^2, \quad x \in (0, 1), \quad y(0) = y(1) = 0$$

using the Green's function eigenfunction expansion.

Example using the Green's function eigenfunction expansion.

The Green's function for this problem can be constructed fairly quickly for this problem once the eigenvalue problem is solved. The eigenvalue problem is

$$\phi''(x) + 4\phi(x) = -\lambda\phi(x),$$

where $\phi(0) = 0$ *and* $\phi(1) = 0$. *The general solution is obtained by rewriting the equation as*

$$\phi''(x) + k^2\phi(x) = 0,$$

where

$$k^2 = 4 + \lambda.$$

Solutions satisfying the boundary condition at $x = 0$ are of the form

$$\phi(x) = A \sin kx.$$

Forcing $\phi(1) = 0$ gives

$$0 = A \sin k \Rightarrow k = n\pi, \quad k = 1, 2, 3 \ldots.$$

So, the eigenvalues are

$$\lambda_n = n^2\pi^2 - 4, \quad n = 1, 2, \ldots$$

and the eigenfunctions are

$$\phi_n = \sin n\pi x, \quad n = 1, 2, \ldots.$$

We also need the normalization constant, N_n. We have that

$$N_n = \|\phi_n\|^2 = \int_0^1 \sin^2 n\pi x = \frac{1}{2}.$$

We can now construct the Green's function for this problem using Equation (7.57).

$$G(x, \xi) = 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{(4 - n^2\pi^2)}. \tag{7.58}$$

Using this Green's function, the solution of the boundary value problem becomes

$$\begin{aligned} y(x) &= \int_0^1 G(x, \xi) f(\xi)\, d\xi \\ &= \int_0^1 \left(2 \sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{(4 - n^2\pi^2)} \right) \xi^2\, d\xi \\ &= 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4 - n^2\pi^2)} \int_0^1 \xi^2 \sin n\pi\xi\, d\xi \\ &= 2 \sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4 - n^2\pi^2)} \left[\frac{(2 - n^2\pi^2)(-1)^n - 2}{n^3\pi^3} \right] \end{aligned} \tag{7.59}$$

We can compare this solution to the one we would obtain if we did not employ Green's functions directly. The eigenfunction expansion method for solving boundary value problems, which we saw earlier is demonstrated in the next example.

Example 7.7. *Solve*

$$y'' + 4y = x^2, \quad x \in (0, 1), \quad y(0) = y(1) = 0$$

using the eigenfunction expansion method.

Example using the eigenfunction expansion method.

We assume that the solution of this problem is in the form

$$y(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

Inserting this solution into the differential equation $\mathcal{L}[y] = x^2$, gives

$$\begin{aligned} x^2 &= \mathcal{L}\left[\sum_{n=1}^{\infty} c_n \sin n\pi x\right] \\ &= \sum_{n=1}^{\infty} c_n \left[\frac{d^2}{dx^2}\sin n\pi x + 4\sin n\pi x\right] \\ &= \sum_{n=1}^{\infty} c_n [4 - n^2\pi^2] \sin n\pi x \end{aligned} \tag{7.60}$$

This is a Fourier sine series expansion of $f(x) = x^2$ on $[0,1]$. Namely,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

In order to determine the c_n's in Equation (7.60), we will need the Fourier sine series expansion of x^2 on $[0,1]$. Thus, we need to compute

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 x^2 \sin n\pi x \\ &= 2\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right], \quad n = 1, 2, \ldots. \end{aligned} \tag{7.61}$$

The resulting Fourier sine series is

$$x^2 = 2\sum_{n=1}^{\infty}\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right] \sin n\pi x.$$

Inserting this expansion in Equation (7.60), we find

$$2\sum_{n=1}^{\infty}\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right] \sin n\pi x = \sum_{n=1}^{\infty} c_n [4 - n^2\pi^2] \sin n\pi x.$$

Due to the linear independence of the eigenfunctions, we can solve for the unknown coefficients to obtain

$$c_n = 2\frac{(2-n^2\pi^2)(-1)^n - 2}{(4-n^2\pi^2)n^3\pi^3}.$$

Therefore, the solution using the eigenfunction expansion method is

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x) \\ &= 2\sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4-n^2\pi^2)}\left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right]. \end{aligned} \tag{7.62}$$

We note that the solution in this example is the same solution as we had obtained using the Green's function obtained in series form in the previous example.

One remaining question is the following: Is there a closed form for the Green's function and the solution to this problem? The answer is yes!

Example 7.8. *Find the closed form Green's function for the problem*

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0$$

and use it to obtain a closed form solution to this boundary value problem.

We note that the differential operator is a special case of the example done in section 7.2. Namely, we pick $\omega = 2$. The Green's function was already found in that section. For this special case, we have

Using the closed form Green's function.

$$G(x,\xi) = \begin{cases} -\dfrac{\sin 2(1-\xi)\sin 2x}{2\sin 2}, & 0 \le x \le \xi, \\ -\dfrac{\sin 2(1-x)\sin 2\xi}{2\sin 2}, & \xi \le x \le 1. \end{cases} \tag{7.63}$$

Using this Green's function, the solution to the boundary value problem is readily computed

$$\begin{aligned} y(x) &= \int_0^1 G(x,\xi) f(\xi)\,d\xi \\ &= -\int_0^x \frac{\sin 2(1-x)\sin 2\xi}{2\sin 2}\xi^2\,d\xi + \int_x^1 \frac{\sin 2(\xi-1)\sin 2x}{2\sin 2}\xi^2\,d\xi \\ &= -\frac{1}{4\sin 2}\left[-x^2\sin 2 + (1-\cos^2 x)\sin 2 + \sin x\cos x(1+\cos 2)\right]. \\ &= -\frac{1}{4\sin 2}\left[-x^2\sin 2 + 2\sin^2 x\sin 1\cos 1 + 2\sin x\cos x\cos^2 1)\right]. \\ &= -\frac{1}{8\sin 1\cos 1}\left[-x^2\sin 2 + 2\sin x\cos 1(\sin x\sin 1 + \cos x\cos 1)\right]. \\ &= \frac{x^2}{4} - \frac{\sin x\cos(1-x)}{4\sin 1}. \end{aligned} \tag{7.64}$$

In Figure 7.3 we show a plot of this solution along with the first five terms of the series solution. The series solution converges quickly to the closed form solution.

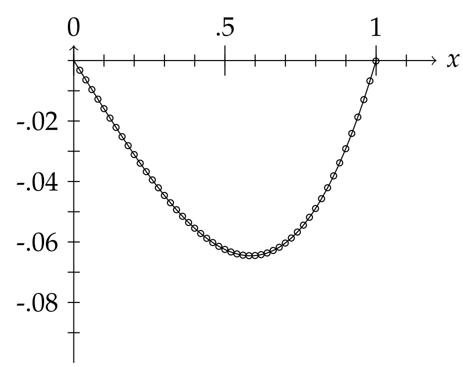

Figure 7.3: Plots of the exact solution to Example 7.6 with the first five terms of the series solution.

As one last check, we solve the boundary value problem directly, as we had done in the last chapter.

Example 7.9. *Solve directly:*

$$y'' + 4y = x^2, \quad x \in (0,1), \quad y(0) = y(1) = 0.$$

Direct solution of the boundary value problem.

The problem has the general solution

$$y(x) = c_1\cos 2x + c_2\sin 2x + y_p(x),$$

where y_p is a particular solution of the nonhomogeneous differential equation. Using the Method of Undetermined Coefficients, we assume a solution of the form

$$y_p(x) = Ax^2 + Bx + C.$$

Inserting this guess into the nonhomogeneous equation, we have

$$2A + 4(Ax^2 + Bx + C) = x^2,$$

Thus, $B = 0$, $4A = 1$ and $2A + 4C = 0$. The solution of this system is

$$A = \frac{1}{4}, \quad B = 0, \quad C = -\frac{1}{8}.$$

So, the general solution of the nonhomogeneous differential equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{x^2}{4} - \frac{1}{8}.$$

We next determine the arbitrary constants using the boundary conditions. We have

$$\begin{aligned} 0 &= y(0) \\ &= c_1 - \frac{1}{8} \\ 0 &= y(1) \\ &= c_1 \cos 2 + c_2 \sin 2 + \frac{1}{8} \end{aligned} \tag{7.65}$$

Thus, $c_1 = \frac{1}{8}$ and

$$c_2 = -\frac{\frac{1}{8} + \frac{1}{8}\cos 2}{\sin 2}.$$

Inserting these constants into the solution we find the same solution as before.

$$\begin{aligned} y(x) &= \frac{1}{8}\cos 2x - \left[\frac{\frac{1}{8} + \frac{1}{8}\cos 2}{\sin 2}\right] \sin 2x + \frac{x^2}{4} - \frac{1}{8} \\ &= \frac{(\cos 2x - 1)\sin 2 - \sin 2x(1 + \cos 2)}{8 \sin 2} + \frac{x^2}{4} \\ &= \frac{(-2\sin^2 x) 2 \sin 1 \cos 1 - \sin 2x (2\cos^2 1)}{16 \sin 1 \cos 1} + \frac{x^2}{4} \\ &= -\frac{(\sin^2 x)\sin 1 + \sin x \cos x (\cos 1)}{4 \sin 1} + \frac{x^2}{4} \\ &= \frac{x^2}{4} - \frac{\sin x \cos(1 - x)}{4 \sin 1}. \end{aligned} \tag{7.66}$$

7.2.4 *The Generalized Green's Function*

When solving $Lu = f$ using eigenfuction expansions, there can be a problem when there are zero eigenvalues. Recall from Section 4.3 the solution of this problem is given by

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x), \\ c_n &= -\frac{\int_a^b f(x)\phi_n(x)\,dx}{\lambda_m \int_a^b \phi_n^2(x)\sigma(x)\,dx}. \end{aligned} \tag{7.67}$$

Here the eigenfunctions, $\phi_n(x)$, satisfy the eigenvalue problem

$$\mathcal{L}\phi_n(x) = -\lambda_n \sigma(x)\phi_n(x), \quad x \in [a, b]$$

subject to given homogeneous boundary conditions.

Note that if $\lambda_m = 0$ for some value of $n = m$, then c_m is undefined. However, if we require

$$(f, \phi_m) = \int_a^b f(x)\phi_n(x)\,dx = 0,$$

The Fredholm Alternative.

then there is no problem. This is a form of the Fredholm Alternative. Namely, if $\lambda_n = 0$ for some n, then there is no solution unless $f, \phi_m) = 0$; i.e., f is orthogonal to ϕ_n. In this case, a_n will be arbitrary and there are an infinite number of solutions.

Example 7.10. $u'' = f(x)$, $u'(0) = 0$, $u'(L) = 0$.

The eigenfunctions satisfy $\phi_n''(x) = -\lambda_n \phi_n(x)$, $\phi_n'(0) = 0$, $\phi_n'(L) = 0$. *There are the usual solutions,*

$$\phi_n(x) = \cos\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots.$$

However, when $\lambda_n = 0$, $\phi_0''(x) = 0$. *So,* $\phi_0(x) = Ax + B$. *The boundary conditions are satisfied if* $A = 0$. *So, we can take* $\phi_0(x) = 1$. *Therefore, there exists an eigenfunction corresponding to a zero eigenvalue. Thus, in order to have a solution, we have to require*

$$\int_0^L f(x)\,dx = 0.$$

Example 7.11. $u'' + \pi^2 u = \beta + 2x$, $u(0) = 0$, $u(1) = 0$.

In this problem we check to see if there is an eigenfunctions with a zero eigenvalue. The eigenvalue problem is

$$\phi'' + \pi^2\phi = 0, \quad \phi(0) = 0, \quad \phi(1) = 0.$$

A solution satisfying this problem is easily founds as

$$\phi(x) = \sin \pi x.$$

Therefore, there is a zero eigenvalue. For a solution to exist, we need to require

$$\begin{aligned} 0 &= \int_0^1 (\beta + 2x)\sin \pi x\,dx \\ &= -\frac{\beta}{\pi}\cos \pi x\Big|_0^1 + 2\left[\frac{1}{\pi}x\cos \pi x - \frac{1}{\pi^2}\sin \pi x\right]_0^1 \\ &= -\frac{2}{\pi}(\beta + 1). \end{aligned} \tag{7.68}$$

Thus, either $\beta = -1$ *or there are no solutions.*

Recall the series representation of the Green's function for a Sturm-Liouville problem in Equation (7.57),

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{-\lambda_n N_n}. \tag{7.69}$$

We see that if there is a zero eigenvalue, then we also can run into trouble as one of the terms in the series is undefined.

Recall that the Green's function satisfies the differential equation $LG(x,\xi) = \delta(x-\xi)$, $x,\xi \in [a,b]$ and satisfies some appropriate set of boundary conditions. Using the above analysis, if there is a zero eigenvalue, then $L\phi_h(x) = 0$. In order for a solution to exist to the Green's function differential equation, then $f(x) = \delta(x-\xi)$ and we have to require

$$0 = (f,\phi_h) = \int_a^b \phi_h(x)\delta(x-\xi)\,dx = \phi_h(\xi),$$

for and $\xi \in [a,b]$. Therefore, the Green's function does not exist.

We can fix this problem by introducing a modified Green's function. Let's consider a modified differential equation,

$$LG_M(x,\xi) = \delta(x-\xi) + c\phi_h(x)$$

for some constant c. Now, the orthogonality condition becomes

$$\begin{aligned} 0 = (f,\phi_h) &= \int_a^b \phi_h(x)[\delta(x-\xi) + c\phi_h(x)]\,dx \\ &= \phi_h(\xi) + c\int_a^b \phi_h^2(x)\,dx. \end{aligned} \tag{7.70}$$

Thus, we can choose

$$c = -\frac{\phi_h(\xi)}{\int_a^b \phi_h^2(x)\,dx}$$

Using the modified Green's function, we can obtain solutions to $Lu = f$. We begin with Green's identity from Section 4.2.2, given by

$$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx = [p(uv' - vu')]_a^b.$$

Letting $v = G_M$, we have

$$\int_a^b \left(G_M\mathcal{L}[u] - u\mathcal{L}[G_M]\right)\,dx = \left[p(x)\left(G_M(x,\xi)u'(x) - u(x)\frac{\partial G_M}{\partial x}(x,\xi)\right)\right]_{x=a}^{x=b}.$$

Applying homogeneous boundary conditions, the right hand side vanishes. Then we have

$$\begin{aligned} 0 &= \int_a^b \left(G_M(x,\xi)\mathcal{L}[u(x)] - u(x)\mathcal{L}[G_M(x,\xi)]\right)\,dx \\ &= \int_a^b \left(G_M(x,\xi)f(x) - u(x)[\delta(x-\xi) + c\phi_h(x)]\right)\,dx \\ u(\xi) &= \int_a^b G_M(x,\xi)f(x)\,dx - c\int_a^b u(x)\phi_h(x)\,dx. \end{aligned} \tag{7.71}$$

Noting that $u(x,t) = c_1\phi_h(x) + u_p(x),$, the last integral gives

$$-c\int_a^b u(x)\phi_h(x)\,dx = \frac{\phi_h(\xi)}{\int_a^b \phi_h^2(x)\,dx}\int_a^b \phi_h^2(x)\,dx = c_1\phi_h(\xi).$$

Therefore, the solution can be written as

$$u(x) = \int_a^b f(\xi)G_M(x,\xi)\,d\xi + c_1\phi_h(x).$$

Here we see that there are an infinite number of solutions when solutions exist.

Example 7.12. *Use the modified Green's function to solve* $u'' + \pi^2 u = 2x - 1$, $u(0) = 0$, $u(1) = 0$.

We have already seen that a solution exists for this problem, where we have set $\beta = -1$ *in Example 7.11.*

We construct the modified Green's function from the solutions of

$$\phi_n'' + \pi^2\phi_n = -\lambda_n\phi_n, \quad \phi(0) = 0, \quad \phi(1) = 0.$$

The general solutions of this equation are

$$\phi_n(x) = c_1\cos\sqrt{\pi^2 + \lambda_n}x + c_2\sin\sqrt{\pi^2 + \lambda_n}x.$$

Applying the boundary conditions, we have $c_1 = 0$ *and* $\sqrt{\pi^2 + \lambda_n} = n\pi$. *Thus, the eigenfunctions and eigenvalues are*

$$\phi_n(x) = \sin n\pi x, \quad \lambda_n = (n^2 - 1)\pi^2, \quad n = 1, 2, 3, \ldots.$$

Note that $\lambda_1 = 0$.

The modified Green's function satisfies

$$\frac{d^2}{dx^2}G_M(x,\xi) + \pi^2 G_M(x,\xi) = \delta(x - \xi) + c\phi_h(x),$$

where

$$\begin{aligned} c &= -\frac{\phi_1(\xi)}{\int_0^1 \phi_1^2(x)\,dx} \\ &= -\frac{\sin\pi\xi}{\int_0^1 \sin^2\pi\xi,dx} \\ &= -2\sin\pi\xi. \end{aligned} \tag{7.72}$$

We need to solve for $G_M(x,\xi)$. *The modified Green's function satisfies*

$$\frac{d^2}{dx^2}G_M(x,\xi) + \pi^2 G_M(x,\xi) = \delta(x - \xi) - 2\sin\pi\xi\sin\pi x,$$

and the boundary conditions $G_M(0,\xi) = 0$ *and* $G_M(1,\xi) = 0$. *We assume an eigenfunction expansion,*

$$G_M(x,\xi) = \sum_{n=1}^{\infty} c_n(\xi)\sin n\pi x.$$

Then,

$$\begin{aligned} \delta(x-\xi) - 2\sin\pi\xi\sin\pi x &= \frac{d^2}{dx^2}G_M(x,\xi) + \pi^2 G_M(x,\xi) \\ &= -\sum_{n=1}^{\infty} \lambda_n c_n(\xi)\sin n\pi x \end{aligned} \tag{7.73}$$

The coefficients are found as

$$\begin{aligned} -\lambda_n c_n &= 2\int_0^1 [\delta(x-\xi) - 2\sin\pi\xi\sin\pi x]\sin n\pi x\, dx \\ &= 2\sin n\pi\xi - 2\sin\pi\xi\delta_{n1}. \end{aligned} \tag{7.74}$$

Therefore, $c_1 = 0$ *and* $c_n = 2\sin n\pi\xi$, *for* $n > 1$.

We have found the modified Green's function as

$$G_M(x,\xi) = -2\sum_{n=2}^{\infty} \frac{\sin n\pi x \sin n\pi\xi}{\lambda_n}.$$

We can use this to find the solution. Namely, we have (for $c_1 = 0$*)*

$$\begin{aligned} u(x) &= \int_0^1 (2\xi - 1) G_M(x,\xi)\, d\xi \\ &= -2\sum_{n=2}^{\infty} \frac{\sin n\pi x}{\lambda_n} \int_0^1 (2\xi-1)\sin n\pi\xi\, dx \\ &= -2\sum_{n=2}^{\infty} \frac{\sin n\pi x}{(n^2-1)\pi^2}\left[-\frac{1}{n\pi}(2\xi-1)\cos n\pi\xi + \frac{1}{n^2\pi^2}\sin n\pi\xi\right]_0^1 \\ &= 2\sum_{n=2}^{\infty} \frac{1+\cos n\pi}{n(n^2-1)\pi^3}\sin n\pi x. \end{aligned} \tag{7.75}$$

We can also solve this problem exactly. The general solution is given by

$$u(x) = c_1 \sin\pi x + c_2\cos\pi x + \frac{2x-1}{\pi^2}.$$

Imposing the boundary conditions, we obtain

$$u(x) = c_1 \sin\pi x + \frac{1}{\pi^2}\cos\pi x + \frac{2x-1}{\pi^2}.$$

Notice that there are an infinite number of solutions. Choosing $c_1 = 0$, *we have the particular solution*

$$u(x) = \frac{1}{\pi^2}\cos\pi x + \frac{2x-1}{\pi^2}.$$

In Figure 7.4 we plot this solution and that obtained using the modified Green's function. The result is that they are in complete agreement.

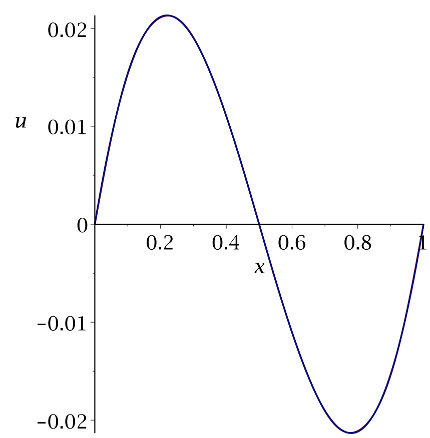

Figure 7.4: The solution for Example 7.12.

7.3 *The Nonhomogeneous Heat Equation*

Boundary value Green's functions do not only arise in the solution of nonhomogeneous ordinary differential equations. They are also important in arriving at the solution of nonhomogeneous partial differential equations. In this section we will show that this is the case by turning to the nonhomogeneous heat equation.

7.3.1 Nonhomogeneous Time Independent Boundary Conditions

Consider the nonhomogeneous heat equation with nonhomogeneous boundary conditions:

$$\begin{aligned} u_t - ku_{xx} &= h(x), \qquad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= a, \quad u(L,t) = b, \\ u(x,0) &= f(x). \end{aligned} \tag{7.76}$$

We are interested in finding a particular solution to this initial-boundary value problem. In fact, we can represent the solution to the general nonhomogeneous heat equation as the sum of two solutions that solve different problems.

First, we let $v(x,t)$ satisfy the homogeneous problem

$$\begin{aligned} v_t - kv_{xx} &= 0, \qquad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \\ v(x,0) &= g(x), \end{aligned} \tag{7.77}$$

which has homogeneous boundary conditions.

The steady state solution, $w(t)$, satisfies a nonhomogeneous differential equation with nonhomogeneous boundary conditions. The transient solution, $v(t)$, satisfies the homogeneous heat equation with homogeneous boundary conditions and satisfies a modified initial condition.

We will also need a steady state solution to the original problem. A steady state solution is one that satisfies $u_t = 0$. Let $w(x)$ be the steady state solution. It satisfies the problem

$$\begin{aligned} -kw_{xx} &= h(x), \qquad 0 \le x \le L. \\ w(0,t) &= a, \quad w(L,t) = b. \end{aligned} \tag{7.78}$$

Now consider $u(x,t) = w(x) + v(x,t)$, the sum of the steady state solution, $w(x)$, and the transient solution, $v(x,t)$. We first note that $u(x,t)$ satisfies the nonhomogeneous heat equation,

$$\begin{aligned} u_t - ku_{xx} &= (w+v)_t - (w+v)_{xx} \\ &= v_t - kv_{xx} - kw_{xx} \equiv h(x). \end{aligned} \tag{7.79}$$

The boundary conditions are also satisfied. Evaluating, $u(x,t)$ at $x = 0$ and $x = L$, we have

$$\begin{aligned} u(0,t) &= w(0) + v(0,t) = a, \\ u(L,t) &= w(L) + v(L,t) = b. \end{aligned} \tag{7.80}$$

The transient solution satisfies

$$v(x,0) = f(x) - w(x).$$

Finally, the initial condition gives

$$u(x,0) = w(x) + v(x,0) = w(x) + g(x).$$

Thus, if we set $g(x) = f(x) - w(x)$, then $u(x,t) = w(x) + v(x,t)$ will be the solution of the nonhomogeneous boundary value problem. We all ready know how to solve the homogeneous problem to obtain $v(x,t)$. So, we only need to find the steady state solution, $w(x)$.

There are several methods we could use to solve Equation (7.78) for the steady state solution. One is the Method of Variation of Parameters, which is closely related to the Green's function method for boundary value problems which we described in the last several sections. However, we will just integrate the differential equation for the steady state solution directly to find the solution. From this solution we will be able to read off the Green's function.

Integrating the steady state equation (7.78) once, yields

$$\frac{dw}{dx} = -\frac{1}{k}\int_0^x h(z)\,dz + A,$$

where we have been careful to include the integration constant, $A = w'(0)$. Integrating again, we obtain

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)\,dz\right)\,dy + Ax + B,$$

where a second integration constant has been introduced. This gives the general solution for Equation (7.78).

The boundary conditions can now be used to determine the constants. It is clear that $B = a$ for the condition at $x = 0$ to be satisfied. The second condition gives

$$b = w(L) = -\frac{1}{k}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + AL + a.$$

Solving for A, we have

$$A = \frac{1}{kL}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + \frac{b-a}{L}.$$

Inserting the integration constants, the solution of the boundary value problem for the steady state solution is then

The steady state solution.

$$w(x) = -\frac{1}{k}\int_0^x \left(\int_0^y h(z)\,dz\right)\,dy + \frac{x}{kL}\int_0^L \left(\int_0^y h(z)\,dz\right)\,dy + \frac{b-a}{L}x + a.$$

This is sufficient for an answer, but it can be written in a more compact form. In fact, we will show that the solution can be written in a way that a Green's function can be identified.

First, we rewrite the double integrals as single integrals. We can do this using integration by parts. Consider integral in the first term of the solution,

$$I = \int_0^x \left(\int_0^y h(z)\,dz\right)\,dy.$$

Setting $u = \int_0^y h(z)\,dz$ and $dv = dy$ in the standard integration by parts formula, we obtain

$$\begin{aligned} I &= \int_0^x \left(\int_0^y h(z)\,dz\right)\,dy \\ &= y\int_0^y h(z)\,dz\Big|_0^x - \int_0^x yh(y)\,dy \\ &= \int_0^x (x-y)h(y)\,dy. \end{aligned} \tag{7.81}$$

Thus, the double integral has now collapsed to a single integral. Replacing the integral in the solution, the steady state solution becomes

$$w(x) = -\frac{1}{k}\int_0^x (x-y)h(y)\,dy + \frac{x}{kL}\int_0^L (L-y)h(y)\,dy + \frac{b-a}{L}x + a.$$

We can make a further simplification by combining these integrals. This can be done if the integration range, $[0, L]$, in the second integral is split into two pieces, $[0, x]$ and $[x, L]$. Writing the second integral as two integrals over these subintervals, we obtain

$$\begin{aligned} w(x) &= -\frac{1}{k}\int_0^x (x-y)h(y)\,dy + \frac{x}{kL}\int_0^x (L-y)h(y)\,dy \\ &\quad + \frac{x}{kL}\int_x^L (L-y)h(y)\,dy + \frac{b-a}{L}x + a. \end{aligned} \tag{7.82}$$

Next, we rewrite the integrands,

$$\begin{aligned} w(x) &= -\frac{1}{k}\int_0^x \frac{L(x-y)}{L}h(y)\,dy + \frac{1}{k}\int_0^x \frac{x(L-y)}{L}h(y)\,dy \\ &\quad + \frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)\,dy + \frac{b-a}{L}x + a. \end{aligned} \tag{7.83}$$

It can now be seen how we can combine the first two integrals:

$$w(x) = -\frac{1}{k}\int_0^x \frac{y(L-x)}{L}h(y)\,dy + \frac{1}{k}\int_x^L \frac{x(L-y)}{L}h(y)\,dy + \frac{b-a}{L}x + a.$$

The resulting integrals now take on a similar form and this solution can be written compactly as

$$w(x) = -\int_0^L G(x,y)[-\frac{1}{k}h(y)]\,dy + \frac{b-a}{L}x + a,$$

where

$$G(x,y) = \begin{cases} \dfrac{x(L-y)}{L}, & 0 \le x \le y, \\ \dfrac{y(L-x)}{L}, & y \le x \le L, \end{cases}$$

is the Green's function for this problem.

The Green's function for the steady state problem.

The full solution to the original problem can be found by adding to this steady state solution a solution of the homogeneous problem,

$$\begin{aligned} u_t - ku_{xx} &= 0, \qquad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \\ u(x,0) &= f(x) - w(x). \end{aligned} \tag{7.84}$$

Example 7.13. *Solve the nonhomogeneous problem,*

$$\begin{aligned} u_t - u_{xx} &= 10, \qquad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 20, \quad u(1,t) = 0, \\ u(x,0) &= 2x(1-x). \end{aligned} \tag{7.85}$$

In this problem we have a rod initially at a temperature of $u(x,0) = 2x(1-x)$. *The ends of the rod are maintained at fixed temperatures and the bar is continually heated at a constant temperature, represented by the source term,* 10.

First, we find the steady state temperature, $w(x)$, *satisfying*

$$\begin{aligned} -w_{xx} &= 10, \qquad 0 \le x \le 1. \\ w(0,t) &= 20, \quad w(1,t) = 0. \end{aligned} \tag{7.86}$$

Using the general solution, we have

$$w(x) = \int_0^1 10G(x,y)\,dy - 20x + 20,$$

where

$$G(x,y) = \begin{cases} x(1-y), & 0 \le x \le y, \\ y(1-x), & y \le x \le 1, \end{cases}$$

we compute the solution

$$\begin{aligned} w(x) &= \int_0^x 10y(1-x)\,dy + \int_x^1 10x(1-y)\,dy - 20x + 20 \\ &= 5(x - x^2) - 20x + 20, \\ &= 20 - 15x - 5x^2. \end{aligned} \tag{7.87}$$

Checking this solution, it satisfies both the steady state equation and boundary conditions.

The transient solution satisfies

$$\begin{aligned} v_t - v_{xx} &= 0, \qquad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(1,t) = 0, \\ v(x,0) &= x(1-x) - 10. \end{aligned} \tag{7.88}$$

Recall, that we have determined the solution of this problem as

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x,$$

where the Fourier sine coefficients are given in terms of the initial temperature distribution,

$$b_n = 2\int_0^1 [x(1-x) - 10] \sin n\pi x\,dx, \quad n = 1, 2, \ldots.$$

Therefore, the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x + 20 - 15x - 5x^2.$$

Note that for large t, *the transient solution tends to zero and we are left with the steady state solution as expected.*

7.3.2 Time Dependent Boundary Conditions

In the last section we solved problems with time independent boundary conditions using equilibrium solutions satisfying the steady state heat equation sand nonhomogeneous boundary conditions. When the boundary conditions are time dependent, we can also convert the problem to an auxiliary problem with homogeneous boundary conditions.

Consider the problem

$$\begin{aligned} u_t - ku_{xx} &= h(x), \qquad 0 \le x \le L, \quad t > 0, \\ u(0,t) &= a(t), \quad u(L,t) = b(t), \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{7.89}$$

We define $u(x,t) = v(x,t) + w(x,t)$, where $w(x,t)$ is a modified form of the steady state solution from the last section,

$$w(x,t) = a(t) + \frac{b(t) - a(t)}{L} x.$$

Noting that

$$\begin{aligned} u_t &= v_t + \dot{a} + \frac{\dot{b} - \dot{a}}{L} x, \\ u_{xx} &= v_{xx}, \end{aligned} \tag{7.90}$$

we find that $v(x,t)$ is a solution of the problem

$$\begin{aligned} v_t - kv_{xx} &= h(x) - \left[\dot{a}(t) + \frac{\dot{b}(t) - \dot{a}(t)}{L} x \right], \qquad 0 \le x \le L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= f(x) - \left[a(0) + \frac{b(0) - a(0)}{L} x \right], \quad 0 \le x \le L. \end{aligned} \tag{7.91}$$

Thus, we have converted the original problem into a nonhomogeneous heat equation with homogeneous boundary conditions and a new source term and new initial condition.

Example 7.14. *Solve the problem*

$$\begin{aligned} u_t - u_{xx} &= x, \qquad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 2, \quad u(L,t) = t, \quad t > 0 \\ u(x,0) &= 3 \sin 2\pi x + 2(1-x), \quad 0 \le x \le 1. \end{aligned} \tag{7.92}$$

We first define

$$u(x,t) = v(x,t) + 2 + (t-2)x.$$

Then, $v(x,t)$ satisfies the problem

$$\begin{aligned} v_t - v_{xx} &= 0, \qquad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= 3 \sin 2\pi x, \quad 0 \le x \le 1. \end{aligned} \tag{7.93}$$

This problem is easily solved. The general solution is given by

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x e^{-n^2\pi^2 t}.$$

We can see that the Fourier coefficients all vanish except for b_2. This gives $v(x,t) = 3\sin 2\pi x e^{-4\pi^2 t}$ and, therefore, we have found the solution

$$u(x,t) = 3\sin 2\pi x e^{-4\pi^2 t} + 2 + (t-2)x.$$

7.4 Green's Functions for 1D Partial Differential Equations

In Section 7.1 we encountered the initial value Green's function for initial value problems for ordinary differential equations. In that case we were able to express the solution of the differential equation $L[y] = f$ in the form

$$y(t) = \int G(t,\tau) f(\tau)\, d\tau,$$

where the Green's function $G(t,\tau)$ was used to handle the nonhomogeneous term in the differential equation. In a similar spirit, we can introduce Green's functions of different types to handle nonhomogeneous terms, nonhomogeneous boundary conditions, or nonhomogeneous initial conditions. Occasionally, we will stop and rearrange the solutions of different problems and recast the solution and identify the Green's function for the problem.

In this section we will rewrite the solutions of the heat equation and wave equation on a finite interval to obtain an initial value Green;s function. Assuming homogeneous boundary conditions and a homogeneous differential operator, we can write the solution of the heat equation in the form

$$u(x,t) = \int_0^L G(x,\xi;t,t_0) f(\xi)\, d\xi.$$

where $u(x,t_0) = f(x)$, and the solution of the wave equation as

$$u(x,t) = \int_0^L G_c(x,\xi,t,t_0) f(\xi)\, d\xi + \int_0^L G_s(x,\xi,t,t_0) g(\xi)\, d\xi.$$

where $u(x,t_0) = f(x)$ and $u_t(x,t_0) = g(x)$. The functions $G(x,\xi;t,t_0)$, $G(x,\xi;t,t_0)$, and $G(x,\xi;t,t_0)$ are initial value Green's functions and we will need to explore some more methods before we can discuss the properties of these functions. [For example, see Section.]

We will now turn to showing that for the solutions of the one dimensional heat and wave equations with fixed, homogeneous boundary conditions, we can construct the particular Green's functions.

7.4.1 Heat Equation

In Section 3.5 we obtained the solution to the one dimensional heat equation on a finite interval satisfying homogeneous Dirichlet conditions,

$$u_t = k u_{xx}, \quad 0 < t, \quad 0 \le x \le L,$$

$$\begin{aligned} u(x,0) &= f(x), \quad 0 < x < L, \\ u(0,t) &= 0, \quad t > 0, \\ u(L,t) &= 0, \quad t > 0. \end{aligned} \tag{7.94}$$

The solution we found was the Fourier sine series

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\lambda_n kt} \sin \frac{n\pi x}{L},$$

where

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2$$

and the Fourier sine coefficients are given in terms of the initial temperature distribution,

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots.$$

Inserting the coefficients b_n into the solution, we have

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{L}\int_0^L f(\xi) \sin \frac{n\pi \xi}{L}\, d\xi \right) e^{\lambda_n kt} \sin \frac{n\pi x}{L}.$$

Interchanging the sum and integration, we obtain

$$u(x,t) = \int_0^L \left(\frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi \xi}{L} e^{\lambda_n kt} \right) f(\xi)\, d\xi.$$

This solution is of the form

$$u(x,t) = \int_0^L G(x,\xi;t,0) f(\xi)\, d\xi.$$

Here the function $G(x,\xi;t,0)$ is the initial value Green's function for the heat equation in the form

$$G(x,\xi;t,0) = \frac{2}{L}\sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi \xi}{L} e^{\lambda_n kt}.$$

which involves a sum over eigenfunctions of the spatial eigenvalue problem, $X_n(x) = \sin \frac{n\pi x}{L}$.

7.4.2 *Wave Equation*

THE SOLUTION OF THE ONE DIMENSIONAL WAVE EQUATION (2.2),

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < t, \quad 0 \le x \le L, \\ u(0,t) &= 0, \quad u(L,0) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L, \end{aligned} \tag{7.95}$$

was found as

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L} \right] \sin \frac{n\pi x}{L}.$$

The Fourier coefficients were determined from the initial conditions,

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}, \\ g(x) &= \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin \frac{n\pi x}{L}, \end{aligned} \tag{7.96}$$

as

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^L f(\xi) \sin \frac{n\pi \xi}{L}\, d\xi, \\ B_n &= \frac{L}{n\pi c} \frac{2}{L} \int_0^L f(\xi) \sin \frac{n\pi \xi}{L}\, d\xi. \end{aligned} \tag{7.97}$$

Inserting these coefficients into the solution and interchanging integration with summation, we have

$$\begin{aligned} u(x,t) &= \int_0^\infty \left[\frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi \xi}{L} \cos \frac{n\pi ct}{L} \right] f(\xi)\, d\xi \\ &\quad + \int_0^\infty \left[\frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi \xi}{L} \frac{\sin \frac{n\pi ct}{L}}{n\pi c/L} \right] g(\xi)\, d\xi \\ &= \int_0^L G_c(x,\xi,t,0) f(\xi)\, d\xi + \int_0^L G_s(x,\xi,t,0) g(\xi)\, d\xi. \end{aligned} \tag{7.98}$$

In this case, we have defined two Green's functions,

$$\begin{aligned} G_c(x,\xi,t,0) &= \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi \xi}{L} \cos \frac{n\pi ct}{L}, \\ G_s(x,\xi,t,0) &= \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi \xi}{L} \frac{\sin \frac{n\pi ct}{L}}{n\pi c/L}. \end{aligned} \tag{7.99}$$

The first, G_c, provides the response to the initial profile and the second, G_s, to the initial velocity.

7.5 *Green's Functions for the 2D Poisson Equation*

In this section we consider the two dimensional Poisson equation with Dirichlet boundary conditions. We consider the problem

$$\begin{aligned} \nabla^2 u &= f, \quad \text{in } D, \\ u &= g, \quad \text{on } C, \end{aligned} \tag{7.100}$$

for the domain in Figure 7.5

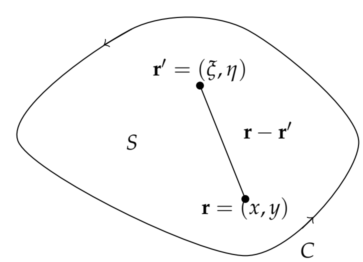

Figure 7.5: Domain for solving Poisson's equation.

We seek to solve this problem using a Green's function. As in earlier discussions, the Green's function satisfies the differential equation and homogeneous boundary conditions. The associated problem is given by

$$\begin{aligned} \nabla^2 G &= \delta(\xi - x, \eta - y), \quad \text{in } D, \\ G &\equiv 0, \quad \text{on } C. \end{aligned} \tag{7.101}$$

However, we need to be careful as to which variables appear in the differentiation. Many times we just make the adjustment after the derivation of the solution, assuming that the Green's function is symmetric in its arguments. However, this is not always the case and depends on things such as the self-adjointedness of the problem. Thus, we will assume that the Green's function satisfies

$$\nabla_{r'}^2 G = \delta(\xi - x, \eta - y),$$

where the notation $\nabla_{r'}$ means differentiation with respect to the variables ξ and η. Thus,

$$\nabla_{r'}^2 G = \frac{\partial^2 G}{\partial \xi^2} + \frac{\partial^2 G}{\partial \eta^2}.$$

With this notation in mind, we now apply Green's second identity for two dimensions from Problem 8 in Chapter 9. We have

$$\int_D (u\nabla_{r'}^2 G - G\nabla_{r'}^2 u)\, dA' = \int_C (u\nabla_{r'} G - G\nabla_{r'} u) \cdot d\mathbf{s}'. \tag{7.102}$$

Inserting the differential equations, the left hand side of the equation becomes

$$\begin{aligned} & \int_D \left[u\nabla_{r'}^2 G - G\nabla_{r'}^2 u\right]\, dA' \\ = \quad & \int_D \left[u(\xi,\eta)\delta(\xi - x, \eta - y) - G(x,y;\xi,\eta) f(\xi,\eta)\right]\, d\xi d\eta \\ = \quad & u(x,y) - \int_D G(x,y;\xi,\eta) f(\xi,\eta)\, d\xi d\eta. \end{aligned} \tag{7.103}$$

Using the boundary conditions, $u(\xi,\eta) = g(\xi,\eta)$ on C and $G(x,y;\xi,\eta) = 0$ on C, the right hand side of the equation becomes

$$\int_C (u\nabla_{r'} G - G\nabla_{r'} u) \cdot d\mathbf{s}' = \int_C g(\xi,\eta)\nabla_{r'} G \cdot d\mathbf{s}'. \tag{7.104}$$

Solving for $u(x,y)$, we have the solution written in terms of the Green's function,

$$u(x,y) = \int_D G(x,y;\xi,\eta) f(\xi,\eta)\, d\xi d\eta + \int_C g(\xi,\eta)\nabla_{r'} G \cdot d\mathbf{s}'.$$

Now we need to find the Green's function. We find the Green's functions for several examples.

Example 7.15. *Find the two dimensional Green's function for the antisymmetric Poisson equation; that is, we seek solutions that are θ-independent.*

The problem we need to solve in order to find the Green's function involves writing the Laplacian in polar coordinates,

$$v_{rr} + \frac{1}{r} v_r = \delta(r).$$

For $r \neq 0$, this is a Cauchy-Euler type of differential equation. The general solution is $v(r) = A \ln r + B$.

Due to the singularity at $r = 0$, we integrate over a domain in which a small circle of radius ϵ is cut form the plane and apply the two dimensional Divergence Theorem. In particular, we have

$$\begin{aligned} 1 &= \int_{D_\epsilon} \delta(r)\, dA \\ &= \int_{D_\epsilon} \nabla^2 v\, dA \\ &= \int_{C_\epsilon} \nabla v \cdot \cdot d\mathbf{s} \\ &= \int_{C_\epsilon} \frac{\partial v}{\partial r}\, dS = 2\pi A. \end{aligned} \tag{7.105}$$

Therefore, $A = 1/2\pi$. We note that B is arbitrary, so we will take $B = 0$ in the remaining discussion.

Using this solution for a source of the form $\delta(\mathbf{r} - \mathbf{r}')$, we obtain the Green's function for Poisson's equation as

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi} \ln |\mathbf{r} - \mathbf{r}'|.$$

Green's function for the infinite plane.

Example 7.16. *Find the Green's function for the infinite plane.*

From Figure 7.5 we have $|\mathbf{r} - \mathbf{r}'| = \sqrt{(x-\xi)^2 + (y-\eta)^2}$. Therefore, the Green's function from the last example gives

$$G(x, y, \xi, \eta) = \frac{1}{4\pi} \ln((\xi - x)^2 + (\eta - y)^2).$$

Green's function for the half plane using the Method of Images.

Example 7.17. *Find the Green's function for the half plane, $\{(x, y) | y > 0\}$, using the Method of Images*

This problem can be solved using the result for the Green's function for the infinite plane. We use the Method of Images to construct a function such that $G = 0$ on the boundary, $y = 0$. Namely, we use the image of the point (x, y) with respect to the x-axis, $(x, -y)$.

Imagine that the Green's function $G(x, y, \xi, \eta)$ represents a point charge at (x, y) and $G(x, y, \xi, \eta)$ provides the electric potential, or response, at (ξ, η). This single charge cannot yield a zero potential along the x-axis (y=0). One needs an additional charge to yield a zero equipotential line. This is shown in Figure 7.6.

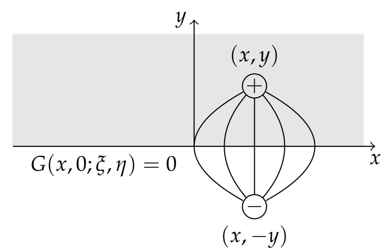

Figure 7.6: The Method of Images: The source and image source for the Green's function for the half plane. Imagine two opposite charges forming a dipole. The electric field lines are depicted indicating that the electric potential, or Green's function, is constant along $y = 0$.

The positive charge has a source of $\delta(\mathbf{r} - \mathbf{r}')$ at $\mathbf{r} = (x, y)$ and the negative charge is represented by the source $-\delta(\mathbf{r}^* - \mathbf{r}')$ at $\mathbf{r}^* = (x, -y)$. We construct the Green's functions at these two points and introduce a negative sign for the negative image source. Thus, we have

$$G(x, y, \xi, \eta) = \frac{1}{4\pi} \ln((\xi - x)^2 + (\eta - y)^2) - \frac{1}{4\pi} \ln((\xi - x)^2 + (\eta + y)^2).$$

These functions satisfy the differential equation and the boundary condition

$$G(x, 0, \xi, \eta) = \frac{1}{4\pi} \ln((\xi - x)^2 + (\eta)^2) - \frac{1}{4\pi} \ln((\xi - x)^2 + (\eta)^2) = 0.$$

Example 7.18. *Solve the homogeneous version of the problem; i.e., solve Laplace's equation on the half plane with a specified value on the boundary.*

We want to solve the problem

$$\nabla^2 u = 0, \quad \text{in } D,$$
$$u = f, \quad \text{on } C, \tag{7.106}$$

This is displayed in Figure 7.7.

From the previous analysis, the solution takes the form

$$u(x,y) = \int_C f \nabla G \cdot \mathbf{n}\, ds = \int_C f \frac{\partial G}{\partial n}\, ds.$$

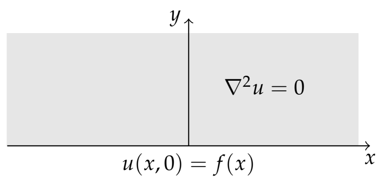

Figure 7.7: This is the domain for a semi-infinite slab with boundary value $u(x,0) = f(x)$ and governed by Laplace's equation.

Since

$$G(x,y,\xi,\eta) = \frac{1}{4\pi}\ln((\xi - x)^2 + (\eta - y)^2) - \frac{1}{4\pi}\ln((\xi - x)^2 + (\eta + y)^2),$$
$$\frac{\partial G}{\partial n} = \frac{\partial G(x,y,\xi,\eta)}{\partial \eta}\Big|_{\eta=0} = \frac{1}{\pi}\frac{y}{(\xi - x)^2 + y^2}.$$

We have arrived at the same surface Green's function as we had found in Example 9.11.2 and the solution is

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} f(\xi)\, d\xi.$$

7.6 *Method of Eigenfunction Expansions*

We have seen that the use of eigenfunction expansions is another technique for finding solutions of differential equations. In this section we will show how we can use eigenfunction expansions to find the solutions to nonhomogeneous partial differential equations. In particular, we will apply this technique to solving nonhomogeneous versions of the heat and wave equations.

7.6.1 *The Nonhomogeneous Heat Equation*

In this section we solve the one dimensional heat equation with a source using an eigenfunction expansion. Consider the problem

$$\begin{aligned} u_t &= k u_{xx} + Q(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < L. \end{aligned} \tag{7.107}$$

The homogeneous version of this problem is given by

$$\begin{aligned} v_t &= k v_{xx}, \quad 0 < x < L, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0. \end{aligned} \tag{7.108}$$

We know that a separation of variables leads to the eigenvalue problem

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = 0, \quad \phi(L) = 0.$$

The eigenfunctions and eigenvalues are given by

$$\phi_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots.$$

We can use these eigenfunctions to obtain a solution of the nonhomogeneous problem (7.107). We begin by assuming the solution is given by the eigenfunction expansion

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x). \tag{7.109}$$

In general, we assume that $v(x,t)$ and $\phi_n(x)$ satisfy the same boundary conditions and that $v(x,t)$ and $v_x(x,t)$ are continuous functions.Note that the difference between this eigenfunction expansion and that in Section 4.3 is that the expansion coefficients are functions of time.

In order to carry out the full process, we will also need to expand the initial profile, $f(x)$, and the source term, $Q(x,t)$, in the basis of eigenfunctions. Thus, we assume the forms

$$\begin{aligned} f(x) &= u(x,0) \\ &= \sum_{n=1}^{\infty} a_n(0)\phi_n(x), \qquad (7.110) \\ Q(x,t) &= \sum_{n=1}^{\infty} q_n(t)\phi_n(x). \qquad (7.111) \end{aligned}$$

Recalling from Chapter 4, the generalized Fourier coefficients are given by

$$\begin{aligned} a_n(0) &= \frac{\langle f, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_0^L f(x)\phi_n(x)\,dx, \qquad (7.112) \\ q_n(t) &= \frac{\langle Q, \phi_n\rangle}{\|\phi_n\|^2} = \frac{1}{\|\phi_n\|^2}\int_0^L Q(x,t)\phi_n(x)\,dx. \qquad (7.113) \end{aligned}$$

The next step is to insert the expansions (7.109) and (7.111) into the nonhomogeneous heat equation (7.107). We first note that

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty} \dot{a}_n(t)\phi_n(x), \\ u_{xx}(x,t) &= -\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x). \qquad (7.114) \end{aligned}$$

Inserting these expansions into the heat equation (7.107), we have

$$\begin{aligned} u_t &= ku_{xx} + Q(x,t), \\ \sum_{n=1}^{\infty} \dot{a}_n(t)\phi_n(x) &= -k\sum_{n=1}^{\infty} a_n(t)\lambda_n\phi_n(x) + \sum_{n=1}^{\infty} q_n(t)\phi_n(x). \qquad (7.115) \end{aligned}$$

Collecting like terms, we have

$$\sum_{n=1}^{\infty}[\dot{a}_n(t) + k\lambda_n a_n(t) - q_n(t)]\phi_n(x) = 0, \quad \forall x \in [0, L].$$

Due to the linear independence of the eigenfunctions, we can conclude that

$$\dot{a}_n(t) + k\lambda_n a_n(t) = q_n(t), \quad n = 1, 2, 3, \ldots.$$

This is a linear first order ordinary differential equation for the unknown expansion coefficients.

We further note that the initial condition can be used to specify the initial condition for this first order ODE. In particular,

$$f(x) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x).$$

The coefficients can be found as generalized Fourier coefficients in an expansion of $f(x)$ in the basis $\phi_n(x)$. These are given by Equation (7.112).

Recall from Appendix B that the solution of a first order ordinary differential equation of the form

$$y'(t) + a(t)y(t) = p(t)$$

is found using the integrating factor

$$\mu(t) = \exp \int^t a(\tau)\, d\tau.$$

Multiplying the ODE by the integrating factor, one has

$$\frac{d}{dt}\left[y(t) \exp \int^t a(\tau)\, d\tau\right] = p(t) \exp \int^t a(\tau)\, d\tau.$$

After integrating, the solution can be found providing the integral is doable.

For the current problem, we have

$$\dot{a}_n(t) + k\lambda_n a_n(t) = q_n(t), \quad n = 1, 2, 3, \ldots.$$

Then, the integrating factor is

$$\mu(t) = \exp \int^t k\lambda_n\, d\tau = e^{k\lambda_n t}.$$

Multiplying the differential equation by the integrating factor, we find

$$\begin{aligned} [\dot{a}_n(t) + k\lambda_n a_n(t)]e^{k\lambda_n t} &= q_n(t)e^{k\lambda_n t} \\ \frac{d}{dt}\left(a_n(t)e^{k\lambda_n t}\right) &= q_n(t)e^{k\lambda_n t}. \end{aligned} \tag{7.116}$$

Integrating, we have

$$a_n(t)e^{k\lambda_n t} - a_n(0) = \int_0^t q_n(\tau)e^{k\lambda_n \tau}\, d\tau,$$

or

$$a_n(t) = a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\, d\tau.$$

Using these coefficients, we can write out the general solution.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \\ &= \sum_{n=1}^{\infty} \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\, d\tau\right]\phi_n(x). \end{aligned} \tag{7.117}$$

We will apply this theory to a more specific problem which not only has a heat source but also has nonhomogeneous boundary conditions.

Example 7.19. *Solve the following nonhomogeneous heat problem using eigenfunction expansions:*

$$\begin{aligned} u_t - u_{xx} &= x + t\sin 3\pi x, \qquad 0 \le x \le 1, \quad t > 0, \\ u(0,t) &= 2, \quad u(L,t) = t, \quad t > 0 \\ u(x,0) &= 3\sin 2\pi x + 2(1-x), \quad 0 \le x \le 1. \end{aligned} \tag{7.118}$$

This problem has the same nonhomogeneous boundary conditions as those in Example 7.14. Recall that we can define

$$u(x,t) = v(x,t) + 2 + (t-2)x$$

to obtain a new problem for $v(x,t)$. The new problem is

$$\begin{aligned} v_t - v_{xx} &= t\sin 3\pi x, \qquad 0 \le x \le 1, \quad t > 0, \\ v(0,t) &= 0, \quad v(L,t) = 0, \quad t > 0, \\ v(x,0) &= 3\sin 2\pi x, \quad 0 \le x \le 1. \end{aligned} \tag{7.119}$$

We can now apply the method of eigenfunction expansions to find $v(x,t)$. The eigenfunctions satisfy the homogeneous problem

$$\phi_n'' + \lambda_n \phi_n = 0, \quad \phi_n(0) = 0, \quad \phi_n(1) = 0.$$

The solutions are

$$\phi_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,3,\ldots.$$

Now, let

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin n\pi x.$$

Inserting $v(x,t)$ into the PDE, we have

$$\sum_{n=1}^{\infty} [\dot{a}_n(t) + n^2\pi^2 a_n(t)]\sin n\pi x = t\sin 3\pi x.$$

Due to the linear independence of the eigenfunctions, we can equate the coefficients of the $\sin n\pi x$ terms. This gives

$$\begin{aligned} \dot{a}_n(t) + n^2\pi^2 a_n(t) &= 0, \quad n \ne 3, \\ \dot{a}_3(t) + 9\pi^2 a_3(t) &= t, \quad n = 3. \end{aligned} \tag{7.120}$$

This is a system of first order ordinary differential equations. The first set of equations are separable and are easily solved. For $n \ne 3$, we seek solutions of

$$\frac{d}{dt}a_n = -n^2\pi^2 a_n(t).$$

These are given by

$$a_n(t) = a_n(0)e^{-n^2\pi^2 t}, \quad n \ne 3.$$

In the case $n = 3$, we seek solutions of

$$\frac{d}{dt}a_3 + 9\pi^2 a_3(t) = t.$$

The integrating factor for this first order equation is given by

$$\mu(t) = e^{9\pi^2 t}.$$

Multiplying the differential equation by the integrating factor, we have

$$\frac{d}{dt}\left(a_3(t)e^{9\pi^2 t}\right) = te^{9\pi^2 t}.$$

Integrating, we obtain the solution

$$\begin{aligned} a_3(t) &= a_3(0)e^{-9\pi^2 t} + e^{-9\pi^2 t}\int_0^t \tau e^{9\pi^2 \tau}\, d\tau, \\ &= a_3(0)e^{-9\pi^2 t} + e^{-9\pi^2 t}\left[\frac{1}{9\pi^2}\tau e^{9\pi^2\tau} - \frac{1}{(9\pi^2)^2}e^{9\pi^2\tau}\right]_0^t, \\ &= a_3(0)e^{-9\pi^2 t} + \frac{1}{9\pi^2}t - \frac{1}{(9\pi^2)^2}\left[1 - e^{-9\pi^2\tau}\right]. \end{aligned} \tag{7.121}$$

Up to this point, we have the solution

$$\begin{aligned} u(x,t) &= v(x,t) + w(x,t) \\ &= \sum_{n=1}^{\infty} a_n(t)\sin n\pi x + 2 + (t-2)x, \end{aligned} \tag{7.122}$$

where

$$\begin{aligned} a_n(t) &= a_n(0)e^{-n^2\pi^2 t}, \quad n \neq 3 \\ a_3(t) &= a_3(0)e^{-9\pi^2 t} + \frac{1}{9\pi^2}t - \frac{1}{(9\pi^2)^2}\left[1 - e^{-9\pi^2\tau}\right]. \end{aligned} \tag{7.123}$$

We still need to find $a_n(0)$, $n = 1, 2, 3, \ldots$.

The initial values of the expansion coefficients are found using the initial condition

$$v(x,0) = 3\sin 2\pi x = \sum_{n=1}^{\infty} a_n(0)\sin n\pi x.$$

It is clear that we have $a_n(0) = 0$ for $n \neq 2$ and $a_2(0) = 3$. Thus, the series for $v(x,t)$ has two nonvanishing coefficients,

$$\begin{aligned} a_2(t) &= 3e^{-4\pi^2 t}, \\ a_3(t) &= \frac{1}{9\pi^2}t - \frac{1}{(9\pi^2)^2}\left[1 - e^{-9\pi^2\tau}\right]. \end{aligned} \tag{7.124}$$

Therefore, the final solution is given by

$$u(x,t) = 2 + (t-2)x + 3e^{-4\pi^2 t}\sin 2\pi x + \frac{9\pi^2 t - (1 - e^{-9\pi^2\tau})}{81\pi^4}\sin 3\pi x.$$

7.6.2 The Forced Vibrating Membrane

We now consider the forced vibrating membrane. A two-dimensional membrane is stretched over some domain D. We assume Dirichlet conditions on the boundary, $u = 0$ on ∂D. The forced membrane can be modeled as

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u + Q(\mathbf{r},t), \quad \mathbf{r}\in D, \quad t>0, \\ u(\mathbf{r},t) &= 0, \quad \mathbf{r}\in\partial D, \quad t>0, \\ u(\mathbf{r},0) &= f(\mathbf{r}), \quad u_t(\mathbf{r},0) = g(\mathbf{r}), \quad \mathbf{r}\in D. \end{aligned} \tag{7.125}$$

The method of eigenfunction expansions relies on the use of eigenfunctions, $\phi_\alpha(\mathbf{r})$, for $\alpha \in J \subset Z^2$ a set of indices typically of the form (i, j) in some lattice grid of integers. The eigenfunctions satisfy the eigenvalue equation

$$\nabla^2\phi_\alpha(\mathbf{r}) = -\lambda_\alpha\phi_\alpha(\mathbf{r}), \quad \phi_\alpha(\mathbf{r}) = 0, \text{ on } \partial D.$$

We assume that the solution and forcing function can be expanded in the basis of eigenfunctions,

$$\begin{aligned} u(\mathbf{r},t) &= \sum_{\alpha\in J} a_\alpha(t)\phi_\alpha(\mathbf{r}), \\ Q(\mathbf{r},t) &= \sum_{\alpha\in J} q_\alpha(t)\phi_\alpha(\mathbf{r}). \end{aligned} \tag{7.126}$$

Inserting this form into the forced wave equation (7.125), we have

$$\begin{aligned} u_{tt} &= c^2\nabla^2 u + Q(\mathbf{r},t) \\ \sum_{\alpha\in J} \ddot{a}_\alpha(t)\phi_\alpha(\mathbf{r}) &= -c^2\sum_{\alpha\in J}\lambda_\alpha a_\alpha(t)\phi_\alpha(\mathbf{r}) + \sum_{\alpha\in J} q_\alpha(t)\phi_\alpha(\mathbf{r}) \\ 0 &= \sum_{\alpha\in J}[\ddot{a}_\alpha(t) + c^2\lambda_\alpha a_\alpha(t) - q_\alpha(t)]\phi_\alpha(\mathbf{r}). \end{aligned} \tag{7.127}$$

The linear independence of the eigenfunctions then gives the ordinary differential equation

$$\ddot{a}_\alpha(t) + c^2\lambda_\alpha a_\alpha(t) = q_\alpha(t).$$

We can solve this equation with initial conditions $a_\alpha(0)$ and $\dot{a}_\alpha(0)$ found from

$$\begin{aligned} f(\mathbf{r}) &= u(\mathbf{r},0) = \sum_{\alpha\in J} a_\alpha(0)\phi_\alpha(\mathbf{r}), \\ g(\mathbf{r}) &= u_t(\mathbf{r},0) = \sum_{\alpha\in J} \dot{a}_\alpha(0)\phi_\alpha(\mathbf{r}). \end{aligned} \tag{7.128}$$

Example 7.20. *Periodic Forcing, $Q(\mathbf{r},t) = G(\mathbf{r})\cos\omega t$.*

It is enough to specify $Q(\mathbf{r},t)$ in order to solve for the time dependence of the expansion coefficients. A simple example is the case of periodic forcing, $Q(\mathbf{r},t) = h(\mathbf{r})\cos\omega t$. In this case, we expand Q in the basis of eigenfunctions,

$$\begin{aligned} Q(\mathbf{r},t) &= \sum_{\alpha\in J} q_\alpha(t)\phi_\alpha(\mathbf{r}), \\ G(\mathbf{r})\cos\omega t &= \sum_{\alpha\in J} \gamma_\alpha \cos\omega t\phi_\alpha(\mathbf{r}). \end{aligned} \tag{7.129}$$

Inserting these expressions into the forced wave equation (7.125), we obtain a system of differential equations for the expansion coefficients,

$$\ddot{a}_\alpha(t) + c^2\lambda_\alpha a_\alpha(t) = \gamma_\alpha \cos \omega t.$$

In order to solve this equation we borrow the methods from a course on ordinary differential equations for solving nonhomogeneous equations. In particular we can use the Method of Undetermined Coefficients as reviewed in Section B.3.1. The solution of these equations are of the form

$$a_\alpha(t) = a_{\alpha h}(t) + a_{\alpha p}(t),$$

where $a_{\alpha h}(t)$ satisfies the homogeneous equation,

$$\ddot{a}_{\alpha h}(t) + c^2\lambda_\alpha a_{\alpha h}(t) = 0, \tag{7.130}$$

and $a_{\alpha p}(t)$ is a particular solution of the nonhomogeneous equation,

$$\ddot{a}_{\alpha p}(t) + c^2\lambda_\alpha a_{\alpha p}(t) = \gamma_\alpha \cos \omega t. \tag{7.131}$$

The solution of the homogeneous problem (7.130) is easily founds as

$$a_{\alpha h}(t) = c_{1\alpha} \cos(\omega_{0\alpha} t) + c_{2\alpha} \sin(\omega_{0\alpha} t),$$

where $\omega_{0\alpha} = c\sqrt{\lambda_\alpha}$.

The particular solution is found by making the guess $a_{\alpha p}(t) = A_\alpha \cos \omega t$. Inserting this guess into Equation (ceqn2), we have

$$[-\omega^2 + c^2\lambda_\alpha] A_\alpha \cos \omega t = \gamma_\alpha \cos \omega t.$$

Solving for A_α, we obtain

$$A_\alpha = \frac{\gamma_\alpha}{-\omega^2 + c^2\lambda_\alpha}, \quad \omega^2 \neq c^2\lambda_\alpha.$$

Then, the general solution is given by

$$a_\alpha(t) = c_{1\alpha} \cos(\omega_{0\alpha} t) + c_{2\alpha} \sin(\omega_{0\alpha} t) + \frac{\gamma_\alpha}{-\omega^2 + c^2\lambda_\alpha} \cos \omega t,$$

where $\omega_{0\alpha} = c\sqrt{\lambda_\alpha}$ and $\omega^2 \neq c^2\lambda_\alpha$.

In the case where $\omega^2 = c^2\lambda_\alpha$, we have a resonant solution. This is discussed in Section FO on forced oscillations. In this case the Method of Undetermined Coefficients fails and we need the Modified Method of Undetermined Coefficients. This is because the driving term, $\gamma_\alpha \cos \omega t$, is a solution of the homogeneous problem. So, we make a different guess for the particular solution. We let

$$a_{\alpha p}(t) = t(A_\alpha \cos \omega t + B_\alpha \sin \omega t).$$

Then, the needed derivatives are

$$\begin{aligned} a_{\alpha p}(t) &= \omega t(-A_\alpha \sin \omega t + B_\alpha \cos \omega t) + A_\alpha \cos \omega t + B_\alpha \sin \omega t, \\ a_{\alpha p}(t) &= -\omega^2 t(A_\alpha \cos \omega t + B_\alpha \sin \omega t) - 2\omega A_\alpha \sin \omega t + 2\omega B_\alpha \cos \omega t, \\ &= -\omega^2 a_{\alpha p}(t) - 2\omega A_\alpha \sin \omega t + 2\omega B_\alpha \cos \omega t. \end{aligned} \tag{7.132}$$

Inserting this guess into Equation (ceqn2) and noting that $\omega^2 = c^2\lambda_\alpha$, *we have*

$$-2\omega A_\alpha \sin \omega t + 2\omega B_\alpha \cos \omega t = \gamma_\alpha \cos \omega t.$$

Therefore, $A_\alpha = 0$ *and*

$$B_\alpha = \frac{\gamma_\alpha}{2\omega}.$$

So, the particular solution becomes

$$a_{\alpha p}(t) = \frac{\gamma_\alpha}{2\omega} t \sin \omega t.$$

The full general solution is then

$$a_\alpha(t) = c_{1\alpha} \cos(\omega t) + c_{2\alpha} \sin(\omega t) + \frac{\gamma_\alpha}{2\omega} t \sin \omega t,$$

where $\omega = c\sqrt{\lambda_\alpha}$.

We see from this result that the solution tends to grow as t gets large. This is what is called a resonance. Essentially, one is driving the system at its natural frequency for one of the frequencies in the system. A typical plot of such a solution is given in Figure 7.8.

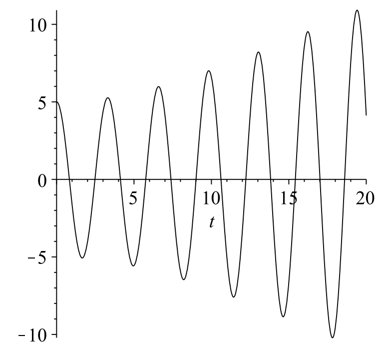

Figure 7.8: Plot of a solution showing resonance.

7.7 Green's Function Solution of Nonhomogeneous Heat Equation

We solved the one dimensional heat equation with a source using an eigenfunction expansion. In this section we rewrite the solution and identify the Green's function form of the solution. Recall that the solution of the nonhomogeneous problem,

$$\begin{aligned} u_t &= k u_{xx} + Q(x,t), \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < L, \end{aligned} \tag{7.133}$$

is given by Equation (7.117)

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \\ &= \sum_{n=1}^{\infty} \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\, d\tau \right] \phi_n(x). \end{aligned} \tag{7.134}$$

The generalized Fourier coefficients for $a_n(0)$ and $q_n(t)$ are given by

$$a_n(0) = \frac{1}{\|\phi_n\|^2} \int_0^L f(x)\phi_n(x)\, dx, \tag{7.135}$$

$$q_n(t) = \frac{1}{\|\phi_n\|^2} \int_0^L Q(x,t)\phi_n(x)\, dx. \tag{7.136}$$

The solution in Equation (7.134) can be rewritten using the Fourier coefficients in Equations (7.135) and (7.136).

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\, d\tau \right] \phi_n(x)$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} a_n(0)e^{-k\lambda_n t}\phi_n(x) + \int_0^t \sum_{n=1}^{\infty} \left(q_n(\tau)e^{-k\lambda_n(t-\tau)}\phi_n(x)\right)\, d\tau \\
&= \sum_{n=1}^{\infty} \frac{1}{\|\phi_n\|^2}\left(\int_0^L f(\xi)\phi_n(\xi)\, d\xi\right) e^{-k\lambda_n t}\phi_n(x) \\
&\quad + \int_0^t \sum_{n=1}^{\infty} \frac{1}{\|\phi_n\|^2}\left(\int_0^L Q(\xi,\tau)\phi_n(\xi)\, d\xi\right) e^{-k\lambda_n(t-\tau)}\phi_n(x)\, d\tau \\
&= \int_0^L \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n t}}{\|\phi_n\|^2}\right) f(\xi)\, d\xi \\
&\quad + \int_0^t \int_0^L \left(\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right) Q(\xi,\tau)\, d\xi\, d\tau. \quad (7.137)
\end{aligned}
$$

Defining

$$G(x,t;\xi,\tau) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2},$$

we see that the solution can be written in the form

The solution can be written in terms of the initial value Green's function, $G(x,t;\xi,0)$, and the general Green's function, $G(x,t;\xi,\tau)$.

$$u(x,t) = \int_0^L G(x,t;\xi,0)f(\xi)\, d\xi + \int_0^t \int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)\, d\xi\, d\tau.$$

Thus, we see that $G(x,t;\xi,0)$ is the initial value Green's function and $G(x,t;\xi,\tau)$ is the general Green's function for this problem.

The only thing left is to introduce nonhomogeneous boundary conditions into this solution. So, we modify the original problem to the fully nonhomogeneous heat equation:

$$
\begin{aligned}
u_t &= ku_{xx} + Q(x,t), \quad 0 < x < L, \quad t > 0, \\
u(0,t) &= \alpha(t), \quad u(L,t) = \beta(t), \quad t > 0, \\
u(x,0) &= f(x), \quad 0 < x < L, \qquad (7.138)
\end{aligned}
$$

As before, we begin with the expansion of the solution in the basis of eigenfunctions,

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x).$$

However, due to potential convergence problems, we cannot expect that u_{xx} can be obtained by simply differentiating the series twice and expecting the resulting series to converge to u_{xx}. So, we need to be a little more careful.

We first note that

$$u_t = \sum_{n=1}^{\infty} \dot{a}_n(t)\phi_n(x) = ku_{xx} + Q(x,t).$$

Solving for the expansion coefficients, we have

$$\dot{a}(t) = \frac{\int_0^L (ku_{xx} + Q(x,t))\phi_n(x)\, dx}{\|\phi_n\|^2}.$$

In order to proceed, we need an expression for $\int_a^b u_{xx}\phi_n(x)\, dx$. We can find this using Green's identity from Section 4.2.2.

We start with

$$\int_a^b (u\mathcal{L}v - v\mathcal{L}u)\,dx = [p(uv' - vu')]_a^b$$

and let $v = \phi_n$. Then,

$$\begin{aligned}
\int_0^L (u(x,t)\phi_n''(x) - \phi_n(x)u_{xx}(x,t))\,dx &= [u(x,t)\phi_n'(x) - \phi_n(x)u_x(x,t)]_0^L \\
\int_0^L (-\lambda_n u(x,t) + u_{xx}(x,t))\phi_n(x)\,dx &= [u(L,t)\phi_n'(L) - \phi_n(L)u_x(L,t))] \\
&\quad -[u(0,t)\phi_n'(0) - \phi_n(0)u_x(0,t))] \\
-\lambda_n a_n\|\phi_n\|^2 - \int_0^L u_{xx}(x,t)\phi_n(x)\,dx &= \beta(t)\phi_n'(L) - \alpha(t)\phi_n'(0). \qquad (7.139)
\end{aligned}$$

Thus,

$$\int_0^L u_{xx}(x,t)\phi_n(x)\,dx = -\lambda_n a_n\|\phi_n\|^2 + \alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L).$$

Inserting this result into the equation for $\dot{a}_n(t)$, we have

$$\dot{a}(t) = -k\lambda_n a_n(t) + q_n(t) + k\frac{\alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L)}{\|\phi_n\|^2}.$$

As we had seen before, this first order equation can be solved using the integrating factor

$$\mu(t) = \exp\int^t k\lambda_n\,d\tau = e^{k\lambda_n t}.$$

Multiplying the differential equation by the integrating factor, we find

$$\begin{aligned}
[\dot{a}_n(t) + k\lambda_n a_n(t)]e^{k\lambda_n t} &= \left[q_n(t) + k\frac{\alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{k\lambda_n t} \\
\frac{d}{dt}\left(a_n(t)e^{k\lambda_n t}\right) &= \left[q_n(t) + k\frac{\alpha(t)\phi_n'(0) - \beta(t)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{k\lambda_n t}.
\end{aligned} \qquad (7.140)$$

Integrating, we have

$$a_n(t)e^{k\lambda_n t} - a_n(0) = \int_0^t \left[q_n(\tau) + k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{k\lambda_n \tau}\,d\tau,$$

or

$$a_n(t) = a_n(0)e^{-k\lambda_n t} + \int_0^t \left[q_n(\tau) + k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\,d\tau.$$

We can now insert these coefficients into the solution and see how to extract the Green's function contributions. Inserting the coefficients, we have

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^\infty a_n(t)\phi_n(x) \\
&= \sum_{n=1}^\infty \left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\,d\tau\right]\phi_n(x) \\
&\quad + \sum_{n=1}^\infty \left(\int_0^t \left[k\frac{\alpha(\tau)\phi_n'(0) - \beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\,d\tau\right)\phi_n(x).
\end{aligned} \qquad (7.141)$$

Recall that the generalized Fourier coefficients for $a_n(0)$ and $q_n(t)$ are given by

$$a_n(0) = \frac{1}{\|\phi_n\|^2}\int_0^L f(x)\phi_n(x)\,dx, \tag{7.142}$$

$$q_n(t) = \frac{1}{\|\phi_n\|^2}\int_0^L Q(x,t)\phi_n(x)\,dx. \tag{7.143}$$

The solution in Equation (7.141) can be rewritten using the Fourier coefficients in Equations (7.142) and (7.143).

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}\left[a_n(0)e^{-k\lambda_n t} + \int_0^t q_n(\tau)e^{-k\lambda_n(t-\tau)}\,d\tau\right]\phi_n(x) \\
&\quad + \sum_{n=1}^{\infty}\left(\int_0^t \left[k\frac{\alpha(\tau)\phi_n'(0)-\beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\,d\tau\right)\phi_n(x) \\
&= \sum_{n=1}^{\infty}a_n(0)e^{-k\lambda_n t}\phi_n(x) + \int_0^t\sum_{n=1}^{\infty}\left(q_n(\tau)e^{-k\lambda_n(t-\tau)}\phi_n(x)\right)d\tau \\
&\quad + \int_0^t\sum_{n=1}^{\infty}\left(\left[k\frac{\alpha(\tau)\phi_n'(0)-\beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\right)\phi_n(x)d\tau \\
&= \sum_{n=1}^{\infty}\frac{1}{\|\phi_n\|^2}\left(\int_0^L f(\xi)\phi_n(\xi)\,d\xi\right)e^{-k\lambda_n t}\phi_n(x) \\
&\quad + \int_0^t\sum_{n=1}^{\infty}\frac{1}{\|\phi_n\|^2}\left(\int_0^L Q(\xi,\tau)\phi_n(\xi)\,d\xi\right)e^{-k\lambda_n(t-\tau)}\phi_n(x)\,d\tau \\
&\quad + \int_0^t\sum_{n=1}^{\infty}\left(\left[k\frac{\alpha(\tau)\phi_n'(0)-\beta(\tau)\phi_n'(L)}{\|\phi_n\|^2}\right]e^{-k\lambda_n(t-\tau)}\right)\phi_n(x)d\tau \\
&= \int_0^L\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n t}}{\|\phi_n\|^2}\right)f(\xi)\,d\xi \\
&\quad + \int_0^t\int_0^L\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right)Q(\xi,\tau)\,d\xi\,d\tau. \\
&\quad + k\int_0^t\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n'(0)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right)\alpha(\tau)\,d\tau \\
&\quad - k\int_0^t\left(\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n'(L)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}\right)\beta(\tau)\,d\tau.
\end{aligned} \tag{7.144}$$

As before, we can define the general Green's function as

$$G(x,t;\xi,\tau) = \sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)e^{-k\lambda_n(t-\tau)}}{\|\phi_n\|^2}.$$

Then, we can write the solution to the fully homogeneous problem as

$$\begin{aligned}
u(x,t) &= \int_0^t\int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)\,d\xi\,d\tau + \int_0^L G(x,t;\xi,0)f(\xi)\,d\xi \\
&\quad + k\int_0^t\left[\alpha(\tau)\frac{\partial G}{\partial\xi}(x,0;t,\tau) - \beta(\tau)\frac{\partial G}{\partial\xi}(x,L;t,\tau)\right]d\tau.
\end{aligned} \tag{7.145}$$

The first integral handles the source term, the second integral handles the initial condition, and the third term handles the fixed boundary conditions.

This general form can be deduced from the differential equation for the Green's function and original differential equation by using a more general form of Green's identity. Let the heat equation operator be defined as $\mathcal{L} = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$. The differential equations for $u(x,t)$ and $G(x,t;\xi,\tau)$ for $0 \leq x, \xi \leq L$ and $t, \tau \geq 0$, are taken to be

$$\begin{aligned} \mathcal{L}u(x,t) &= Q(x,t), \\ \mathcal{L}G(x,t;\xi,\tau) &= \delta(x-\xi)\delta(t-\tau). \end{aligned} \tag{7.146}$$

Multiplying the first equation by $G(x,t;\xi,\tau)$ and the second by $u(x,t)$, we obtain

$$\begin{aligned} G(x,t;\xi,\tau)\mathcal{L}u(x,t) &= G(x,t;\xi,\tau)Q(x,t), \\ u(x,t)\mathcal{L}G(x,t;\xi,\tau) &= \delta(x-\xi)\delta(t-\tau)u(x,t). \end{aligned} \tag{7.147}$$

Now, we subtract the equations and integrate with respect to x and t. This gives

$$\begin{aligned} &\int_0^\infty \int_0^L \left[G(x,t;\xi,\tau)\mathcal{L}u(x,t) - u(x,t)\mathcal{L}G(x,t;\xi,\tau)\right]\, dxdt \\ = &\int_0^\infty \int_0^L \left[G(x,t;\xi,\tau)Q(x,t) - \delta(x-\xi)\delta(t-\tau)u(x,t)\right]\, dxdt \\ = &\int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(x,t)\, dxdt - u(\xi,\tau). \end{aligned} \tag{7.148}$$

and

$$\begin{aligned} &\int_0^\infty \int_0^L \left[G(x,t;\xi,\tau)\mathcal{L}u(x,t) - u(x,t)\mathcal{L}G(x,t;\xi,\tau)\right]\, dxdt \\ = &\int_0^L \int_0^\infty \left[G(x,t;\xi,\tau)u_t - u(x,t)G_t(x,t;\xi,\tau)\right]\, dtdx \\ &-k\int_0^\infty \int_0^L \left[G(x,t;\xi,\tau)u_{xx}(x,t) - u(x,t)G_{xx}(x,t;\xi,\tau)\right]\, dxdt \\ = &\int_0^L \left[G(x,t;\xi,\tau)u_t\Big|_0^\infty - 2\int_0^\infty u(x,t)G_t(x,t;\xi,\tau)\, dt\right] dx \\ &-k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial x}(x,t) - u(x,t)\frac{\partial G}{\partial x}(x,t;\xi,\tau)\right]_0^L dxdt \end{aligned} \tag{7.149}$$

Equating these two results and solving for $u(\xi,\tau)$, we have

$$\begin{aligned} u(\xi,\tau) &= \int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(x,t)\, dxdt \\ &\quad +k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial x}(x,t) - u(x,t)\frac{\partial G}{\partial x}(x,t;\xi,\tau)\right]_0^L dxdt \\ &\quad +\int_0^L \left[G(x,0;\xi,\tau)u(x,0) + 2\int_0^\infty u(x,t)G_t(x,t;\xi,\tau)\, dt\right] dx. \end{aligned} \tag{7.150}$$

Exchanging (ξ,τ) with (x,t) and assuming that the Green's function is symmetric in these arguments, we have

$$u(x,t) = \int_0^\infty \int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)\, d\xi d\tau$$

$$+k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial \xi}(\xi,\tau) - u(\xi,\tau)\frac{\partial G}{\partial \xi}(x,t;\xi,\tau)\right]_0^L dxdt$$
$$+\int_0^L G(x,t;\xi,0)u(\xi,0)\,d\xi + 2\int_0^L\int_0^\infty u(\xi,\tau)G_\tau(x,t;\xi,\tau)\,d\tau d\xi. \tag{7.151}$$

This result is almost in the desired form except for the last integral. Thus, if

$$\int_0^L\int_0^\infty u(\xi,\tau)G_\tau(x,t;\xi,\tau)\,d\tau d\xi = 0,$$

then we have

$$\begin{aligned} u(x,t) &= \int_0^\infty\int_0^L G(x,t;\xi,\tau)Q(\xi,\tau)\,d\xi d\tau + \int_0^L G(x,t;\xi,0)u(\xi,0)\,d\xi \\ &\quad +k\int_0^\infty \left[G(x,t;\xi,\tau)\frac{\partial u}{\partial \xi}(\xi,\tau) - u(\xi,\tau)\frac{\partial G}{\partial \xi}(x,t;\xi,\tau)\right]_0^L dxdt. \end{aligned} \tag{7.152}$$

7.8 Summary

We have seen throughout the chapter that Green's functions are the solutions of a differential equation representing the effect of a point impulse on either source terms, or initial and boundary conditions. The Green's function is obtained from transform methods or as an eigenfunction expansion. In the text we have occasionally rewritten solutions of differential equations in term's of Green's functions. We will first provide a few of these examples and then present a compilation of Green's Functions for generic partial differential equations.

For example, in section 7.4 we wrote the solution of the one dimensional heat equation as

$$u(x,t) = \int_0^L G(x,\xi;t,0)f(\xi)\,d\xi,$$

where

$$G(x,\xi;t,0) = \frac{2}{L}\sum_{n=1}^\infty \sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{\lambda_n kt},$$

and the solution of the wave equation as

$$u(x,t) = \int_0^L G_c(x,\xi,t,0)f(\xi)\,d\xi + \int_0^L G_s(x,\xi,t,0)g(\xi)\,d\xi,$$

where

$$\begin{aligned} G_c(x,\xi,t,0) &= \frac{2}{L}\sum_{n=1}^\infty \sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\cos\frac{n\pi ct}{L}, \\ G_s(x,\xi,t,0) &= \frac{2}{L}\sum_{n=1}^\infty \sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\frac{\sin\frac{n\pi ct}{L}}{n\pi c/L}. \end{aligned}$$

We note that setting $t = 0$ in $G_c(x, \xi; t, 0)$, we obtain

$$G_c(x, \xi, 0, 0) = \frac{2}{L} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi\xi}{L}.$$

This is the Fourier sine series representation of the Dirac delta function, $\delta(x - \xi)$. Similarly, if we differentiate $G_s(x, \xi, t, 0)$ with repsect to t and set $t = 0$, we once again obtain the Fourier sine series representation of the Dirac delta function.

It is also possible to find closed form expression for Green's functions, which we had done for the heat equation on the infinite interval,

$$u(x, t) = \int_{-\infty}^{\infty} G(x, t; \xi, 0) f(\xi)\, d\xi,$$

where

$$G(x, t; \xi, 0) = \frac{e^{-(x-\xi)^2/4t}}{\sqrt{4\pi t}},$$

and for Poisson's equation,

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}')\, d^3 r',$$

where the three dimensional Green's function is given by

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}.$$

We can construct Green's functions for other problems which we have seen in the book. For example, the solution of the two dimensional wave equation on a rectangular membrane was found in Equation (6.37) as

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \quad (7.153)$$

where

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L f(x, y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy, \quad (7.154)$$

$$b_{nm} = \frac{4}{\omega_{nm} LH} \int_0^H \int_0^L g(x, y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}\, dxdy, \quad (7.155)$$

where the angular frequencies are given by

$$\omega_{nm} = c \sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}. \quad (7.156)$$

Rearranging the solution, we have

$$u(x, y, t) = \int_0^H \int_0^L \left[G_c(x, y; \xi, \eta; t, 0) f(\xi, \eta) + G_s(x, y; \xi, \eta; t, 0) g(\xi, \eta)\right]\, d\xi d\eta,$$

where

$$G_c(x, y; \xi, \eta; t, 0) = \frac{4}{LH} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sin \frac{n\pi x}{L} \sin \frac{n\pi\xi}{L} \sin \frac{m\pi y}{H} \sin \frac{m\pi\eta}{H} \cos \omega_{nm} t$$

and

$$G_s(x,y;\xi,\eta;t,0) = \frac{4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\sin\frac{m\pi y}{H}\sin\frac{m\pi\eta}{H}\frac{\sin\omega_{nm}t}{\omega_{nm}}.$$

Once again, we note that setting $t = 0$ in $G_c(x,\xi;t,0)$ and setting $t = 0$ in $\frac{\partial G_c(x,\xi;t,0)}{\partial t}$, we obtain a Fourier series representation of the Dirac delta function in two dimensions,

$$\delta(x-\xi)\delta(y-\eta) = \frac{4}{LH}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\sin\frac{m\pi y}{H}\sin\frac{m\pi\eta}{H}.$$

Another example was the solution of the two dimensional Laplace equation on a disk given by Equation 6.87. We found that

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos n\theta + b_n\sin n\theta\right)r^n. \qquad (7.157)$$

$$a_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta, \quad n = 0,1,\ldots, \qquad (7.158)$$

$$b_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta \quad n = 1,2\ldots. \qquad (7.159)$$

We saw that this solution can be written as

$$u(r,\theta) = \int_{-\pi}^{\pi} G(\theta,\phi;r,a)f(\phi)\,d\phi,$$

where the Green's function could be summed giving the Poisson kernel

$$G(\theta,\phi;r,a) = \frac{1}{2\pi}\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}.$$

We had also investigated the nonhomogeneous heat equation in section 9.11.4,

$$\begin{aligned} u_t - ku_{xx} &= h(x,t), \quad 0 \le x \le L, \quad t > 0. \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le . \end{aligned} \qquad (7.160)$$

We found that the solution of the heat equation is given by

$$u(x,t) = \int_0^L f(\xi)G(x,\xi;t,0)d\xi + \int_0^t\int_0^L h(\xi,\tau)G(x,\xi;t,\tau)\,d\xi d\tau,$$

where

$$G(x,\xi;t,\tau) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}e^{-\omega_n^2(t-\tau)}.$$

Note that setting $t = \tau$, we again get a Fourier sine series representation of the Dirac delta function.

In general, Green's functions based on eigenfunction expansions over eigenfunctions of Sturm-Liouville eigenvalue problems are a common way to construct Green's functions. For example, surface and initial value Green's functions are constructed in terms of a modification of delta function representations modified by factors which make the Green's function a solution of the given differential equations and a factor taking into account the boundary or initial condition plus a restoration of the delta function when applied to the condition. Examples with an indication of these factors are shown below.

1. Surface Green's Function: Cube $[0,a]\times[0,b]\times[0,c]$

$$g(x,y,z;x',y',c) = \underbrace{\sum_{\ell,n} \frac{2}{a}\sin\frac{\ell\pi x}{a}\sin\frac{\ell\pi x'}{a}\frac{2}{b}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b}}_{\delta-\text{function}} \left[\underbrace{\sinh\gamma_{\ell n} z}_{\text{D.E.}} / \underbrace{\sinh\gamma_{\ell n} c}_{\text{restore }\delta}\right].$$

2. Surface Green's Function: Sphere $[0,a]\times[0,\pi]\times[0,2\pi]$

$$g(r,\phi,\theta;a,\phi',\theta') = \sum_{\ell,m}\underbrace{Y_\ell^{m*}(\psi'\,\theta')Y_\ell^{m*}(\psi\,\theta)}_{\delta-\text{function}}\left[\underbrace{r^\ell}_{\text{D.E.}} / \underbrace{a^\ell}_{\text{restore }\delta}\right].$$

3. Initial Value Green's Function: 1D Heat Equation on $[0,L]$, $k_n = \frac{n\pi}{L}$

$$g(x,t;x',t_0) = \sum_n \underbrace{\frac{2}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi x'}{L}}_{\delta-\text{function}}\left[\underbrace{e^{-a^2k_n^2 t}}_{\text{D.E.}} / \underbrace{e^{-a^2k_n^2 t_0}}_{\text{restore }\delta}\right].$$

4. Initial Value Green's Function: 1D Heat Equation on infinite domain

$$g(x,t;x',0) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ik(x-x')}}_{\delta-\text{function}}\underbrace{e^{-a^2k^2t}}_{\text{D.E.}} = \frac{e^{-(x-x')^2/4a^2t}}{\sqrt{4\pi a^2 t}}.$$

We can extend this analysis to a more general theory of Green's functions. This theory is based upon Green's Theorems, or identities.

1. Green's First Theorem

$$\oint_S \varphi\nabla\chi\cdot\hat{\mathbf{n}}\,dS = \int_V (\nabla\varphi\cdot\nabla\chi + \varphi\nabla^2\chi)\,dV.$$

   This is easily proven starting with the identity

$$\nabla\cdot(\varphi\nabla\chi) = \nabla\varphi\cdot\nabla\chi + \varphi\nabla^2\chi,$$

   integrating over a volume of space and using Gauss' Integral Theorem.

2. Green's Second Theorem

$$\int_V (\varphi\nabla^2\chi - \chi\nabla^2\varphi)\,dV = \oint_S (\varphi\nabla\chi - \chi\nabla\varphi)\cdot\hat{\mathbf{n}}\,dS.$$

   This is proven by interchanging φ and χ in the first theorem and subtracting the two versions of the theorem.

The next step is to let $\varphi = u$ and $\chi = G$. Then,

$$\int_V (u\nabla^2 G - G\nabla^2 u)\, dV = \oint_S (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, dS.$$

As we had seen earlier for Poisson's equation, inserting the differential equation yields

$$u(x,y) = \int_V Gf\, dV + \oint_S (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}}\, dS.$$

If we have the Green's function, we only need to know the source term and boundary conditions in order to obtain the solution to a given problem.

In the next sections we provide a summary of these ideas as applied to some generic partial differential equations.[3]

[3] This is an adaptation of notes from J. Franklin's course on mathematical physics.

7.8.1 *Laplace's Equation:* $\nabla^2\psi = 0$.

1. **Boundary Conditions**

   (a) *Dirichlet* - ψ is given on the surface.

   (b) *Neumann* - $\hat{\mathbf{n}} \cdot \nabla\psi = \frac{\partial\psi}{\partial n}$ is given on the surface.

   Note: Boundary conditions can be Dirichlet on part of the surface and Neumann on part. If they are Neumann on the whole surface, then the Divergence Theorem requires the constraint

   $$\int \frac{\partial\psi}{\partial n}\, dS = 0.$$

2. **Solution by Surface Green's Function,** $g(\vec{\mathbf{r}}, \vec{\mathbf{r}}')$.

   (a) Dirichlet conditions

   $$\nabla^2 g_D(\vec{\mathbf{r}}, \vec{\mathbf{r}}') = 0,$$

   $$g_D(\vec{\mathbf{r}}_s, \vec{\mathbf{r}}'_s) = \delta^{(2)}(\vec{\mathbf{r}}_s - \vec{\mathbf{r}}'_s),$$

   $$\psi(\vec{\mathbf{r}}) = \int g_D(\vec{\mathbf{r}}, \vec{\mathbf{r}}'_{\mathbf{s}})\psi(\vec{\mathbf{r}}'_s)\, dS'.$$

   (b) Neumann conditions

   $$\nabla^2 g_N(\vec{\mathbf{r}}, \vec{\mathbf{r}}') = 0,$$

   $$\frac{\partial g_N}{\partial n}(\vec{\mathbf{r}}_s, \vec{\mathbf{r}}'_s) = \delta^{(2)}(\vec{\mathbf{r}}_s - \vec{\mathbf{r}}'_s),$$

   $$\psi(\vec{\mathbf{r}}) = \int g_N(\vec{\mathbf{r}}, \vec{\mathbf{r}}'_s)\frac{\partial\psi}{\partial n}(\vec{\mathbf{r}}'_s)\, dS'.$$

   Note: Use of g is readily generalized to any number of dimensions.

7.8.2 Homogeneous Time Dependent Equations

1. **Typical Equations**
    (a) Diffusion/Heat Equation $\nabla^2\Psi = \frac{1}{a^2}\frac{\partial}{\partial t}\Psi$.
    (b) Schrödinger Equation $-\nabla^2\Psi + U\Psi = i\frac{\partial}{\partial t}\Psi$.
    (c) Wave Equation $\nabla^2\Psi = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi$.
    (d) General form: $\mathcal{D}\Psi = \mathcal{T}\Psi$.
2. **Initial Value Green's Function,** $g(\vec{\mathbf{r}},\vec{\mathbf{r}}';t,t')$.
    (a) **Homogeneous Boundary Conditions**
        i. Diffusion, or Schrödinger Equation (1st order in time), $\mathcal{D}g = \mathcal{T}g$.

$$\Psi(\vec{\mathbf{r}},t) = \int g(\vec{\mathbf{r}},\vec{\mathbf{r}}';t,t_0)\Psi(\mathbf{r}',t_0)\,d^3\mathbf{r}',$$

        where

$$g(\mathbf{r},\mathbf{r}';t_0,t_0) = \delta(\mathbf{r}-\mathbf{r}'),$$

$g(\mathbf{r}_s)$ satisfies homogeneous boundary conditions.

        ii. Wave Equation

$$\Psi(\mathbf{r},t) = \int [g_c(\mathbf{r},\mathbf{r}';t,t_0)\Psi(\mathbf{r}',t_0) + g_s(\mathbf{r},\mathbf{r}';t,t_0)\dot{\Psi}(\mathbf{r}',t_0)]\,d^3\mathbf{r}'.$$

        The first two properties in (a) above hold, but

$$g_c(\mathbf{r},\mathbf{r}';t_0,t_0) = \delta(\mathbf{r}-\mathbf{r}')$$
$$\dot{g}_s(\mathbf{r},\mathbf{r}';t_0,t_0) = \delta(\mathbf{r}-\mathbf{r}')$$

        Note: For the diffusion and Schrödinger equations the initial condition is Dirichlet in time. For the wave equation the initial condition is Cauchy, where Ψ and $\dot{\Psi}$ are given.
    (b) **Inhomogeneous, Time Independent (steady) Boundary Conditions**
        i. Solve Laplace's equation, $\nabla^2\psi_s = 0$, for inhomogeneous B.C.'s
        ii. Solve homogeneous, time-dependent equation for

$$\Psi_t(\mathbf{r},t) \text{ satisfying } \Psi_t(\mathbf{r},t_0) = \Psi(\mathbf{r},t_0) - \psi_s(\mathbf{r}).$$

        iii. Then $\Psi(\mathbf{r},t) = \Psi_t(\mathbf{r},t) + \psi_s(\mathbf{r})$.
        Note: Ψ_t is the *transient part* and ψ_s is the *steady state part*.
3. **Time Dependent Boundary Conditions with Homogeneous Initial Conditions**
    (a) Use the Boundary Value Green's Function, $h(\mathbf{r},\mathbf{r}_s';t,t')$, which is similar to the surface Green's function in an earlier section.

$$\Psi(\mathbf{r},t) = \int_{t_0}^{\infty} h_D(\mathbf{r},\mathbf{r}_s';t,t')\Psi(\mathbf{r}_s',t')\,dt',$$

    or

$$\Psi(\mathbf{r},t) = \int_{t_0}^{\infty} \frac{\partial h_N}{\partial n}(\mathbf{r},\mathbf{r}_s';t,t')\Psi(\mathbf{r}_s',t')\,dt'.$$

(b) Properties of $h(\mathbf{r}, \mathbf{r}'_s; t, t')$:

$$\mathcal{D}h = \mathcal{T}h$$

$$h_D(\mathbf{r}_s, \mathbf{r}'_s; t, t') = \delta(t - t'), \text{ or} \frac{\partial h_N}{\partial n}(\mathbf{r}_s, \mathbf{r}'_s; t, t') = \delta(t - t'),$$

$$h(\mathbf{r}, \mathbf{r}'_s; t, t') = 0, \quad t' > t, \text{ (causality)}.$$

(c) **Note:** For inhomogeneous I.C.,

$$\Psi = \int g \Psi(\mathbf{r}', t_0) + \int dt' h_D \Psi(\mathbf{r}'_s, t')\, d^3\mathbf{r}'.$$

7.8.3 Inhomogeneous Steady State Equation

1. **Poisson's Equation**

$$\nabla^2 \psi(\mathbf{r}, t) = f(\mathbf{r}), \qquad \psi(\mathbf{r}_s) \quad \text{or} \quad \frac{\partial \psi}{\partial n}(\mathbf{r}_s) \quad \text{given.}$$

(a) Green's Theorem:

$$\int [\psi(\mathbf{r}')\nabla'^2 G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla'^2 \psi(\mathbf{r}')]\, d^3\mathbf{r}'$$
$$= \int [\psi(\mathbf{r}')\nabla' G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla' \psi(\mathbf{r}')] \cdot \vec{dS}',$$

where ∇' denotes differentiation with respect to r'.

(b) Properties of $G(\mathbf{r}, \mathbf{r}')$:

i. $\nabla'^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$.

ii. $G|_s = 0$ or $\frac{\partial G}{\partial n'}|_s = 0$.

iii. Solution

$$\psi(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}') f(\mathbf{r}')\, d^3\mathbf{r}' + \int [\psi(\mathbf{r}')\nabla' G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla' \psi(\mathbf{r}')] \cdot \vec{dS}'. \tag{7.161}$$

(c) For the case of pure Neumann B.C.'s, the Divergence Theorem leads to the constraint

$$\int \nabla \psi \cdot \vec{dS} = \int f\, d^3\mathbf{r}.$$

If there are pure Neumann conditions and S is finite and $\int f\, d^3\mathbf{r} \neq 0$ by symmetry, then $\vec{n}' \cdot \nabla' G|_s \neq 0$ and the Green's function method is much more complicated to solve.

(d) From the above result:

$$\vec{n}' \cdot \nabla' G(\mathbf{r}, \mathbf{r}'_s) = g_D(\mathbf{r}, \mathbf{r}'_s)$$

or

$$G_N(\mathbf{r}, \mathbf{r}'_s) = -g_N(\mathbf{r}, \mathbf{r}'_s).$$

It is often simpler to use G for $\int d^3\mathbf{r}'$ and g for $\int \vec{dS}'$, separately.

(e) G satisfies a reciprocity property, $G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$ for either Dirichlet or Neumann boundary conditions.

(f) $G(\mathbf{r},\mathbf{r}')$ can be considered as a potential at $\mathbf{r}$ due to a point charge $q = -1/4\pi$ at $\mathbf{r}'$, with all surfaces being grounded conductors.

7.8.4 Inhomogeneous, Time Dependent Equations

1. **Diffusion/Heat Flow** $\nabla^2\Psi - \frac{1}{a^2}\dot{\Psi} = f(\mathbf{r},t)$.

(a)

$$\begin{aligned} [\nabla^2 - \frac{1}{a^2}\frac{\partial}{\partial t}]G(\mathbf{r},\mathbf{r}';t,t') &= [\nabla'^2 + \frac{1}{a^2}\frac{\partial}{\partial t'}]G(\mathbf{r},\mathbf{r}';t,t') \\ &= \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \end{aligned} \tag{7.162}$$

(b) Green's Theorem in 4 dimensions $(\mathbf{r},t)$ yields

$$\begin{aligned} \Psi(\mathbf{r},t) &= \int\int_{t_0}^{\infty} G(\mathbf{r},\mathbf{r}';t,t')f(\mathbf{r}',t')\,dt'd^3\mathbf{r}' - \frac{1}{a^2}\int G(\mathbf{r},\mathbf{r}';t,t_0)\Psi(\mathbf{r}',t_0)\,d^3\mathbf{r}' \\ &+ \int_{t_0}^{\infty}\int [\Psi(\mathbf{r}'_s,t)\nabla' G_D(\mathbf{r},\mathbf{r}'_s;t,t') - G_N(\mathbf{r},\mathbf{r}'_s;t,t')\nabla'\Psi(\mathbf{r}'_s,t')]\cdot \vec{dS}'dt'. \end{aligned}$$

(c) Either $G_D(\mathbf{r}'_s) = 0$ or $G_N(\mathbf{r}'_s) = 0$ on S at any point $\mathbf{r}'_s$.

(d) $\hat{\mathbf{n}}'\cdot\nabla' G_D(\mathbf{r}'_s) = h_D(\mathbf{r}'_s)$, $G_N(\mathbf{r}'_s) = -h_N(\mathbf{r}'_s)$, and $-\frac{1}{a^2}G(\mathbf{r},\mathbf{r}';t,t_0) = g(\mathbf{r},\mathbf{r}';t,t_0)$.

2. **Wave Equation** $\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial^2 t} = f(\mathbf{r},t)$.

(a)

$$\begin{aligned} [\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]G(\mathbf{r},\mathbf{r}';t,t') &= [\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]G(\mathbf{r},\mathbf{r}';t,t') \\ &= \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \end{aligned} \tag{7.163}$$

(b) Green's Theorem in 4 dimensions $(\mathbf{r},t)$ yields

$$\begin{aligned} \Psi(\mathbf{r},t) &= \int\int_{t_0}^{\infty} G(\mathbf{r},\mathbf{r}';t,t')f(\mathbf{r}',t')\,dt'd^3\mathbf{r}' \\ &- \frac{1}{c^2}\int [G(\mathbf{r},\mathbf{r}';t,t_0)\frac{\partial}{\partial t'}\Psi(\mathbf{r}',t_0) - \Psi(\mathbf{r}',t_0)\frac{\partial}{\partial t'}G(\mathbf{r},\mathbf{r}';t,t_0)]\,d^3\mathbf{r}' \\ &+ \int_{t_0}^{\infty}\int [\Psi(\mathbf{r}'_s,t)\nabla' G_D(\mathbf{r},\mathbf{r}'_s;t,t') - G_N(\mathbf{r},\mathbf{r}'_s;t,t')\nabla'\psi(\mathbf{r}'_s,t')]\cdot \vec{dS}'dt'. \end{aligned}$$

(c) Cauchy initial conditions are given: $\Psi(t_0)$ and $\dot{\Psi}(t_0)$.

(d) The wave and diffusion equations satisfy a causality condition $G(t,t') = 0, \quad t' > t$.

Problems

1. Find the solution of each initial value problem using the appropriate initial value Green's function.

a. $y'' - 3y' + 2y = 20e^{-2x}, \quad y(0) = 0, \quad y'(0) = 6.$

b. $y'' + y = 2\sin 3x, \quad y(0) = 5, \quad y'(0) = 0.$

c. $y'' + y = 1 + 2\cos x, \quad y(0) = 2, \quad y'(0) = 0.$

d. $x^2 y'' - 2xy' + 2y = 3x^2 - x, \quad y(1) = \pi, \quad y'(1) = 0.$

2. Use the initial value Green's function for $x'' + x = f(t)$, $x(0) = 4$, $x'(0) = 0$, to solve the following problems.

a. $x'' + x = 5t^2$.

b. $x'' + x = 2\tan t$.

3. For the problem $y'' - k^2 y = f(x)$, $y(0) = 0$, $y'(0) = 1$,

a. Find the initial value Green's function.

b. Use the Green's function to solve $y'' - y = e^{-x}$.

c. Use the Green's function to solve $y'' - 4y = e^{2x}$.

4. Find and use the initial value Green's function to solve

$$x^2 y'' + 3xy' - 15y = x^4 e^x, \quad y(1) = 1, y'(1) = 0.$$

5. Consider the problem $y'' = \sin x$, $y'(0) = 0$, $y(\pi) = 0$.

a. Solve by direct integration.

b. Determine the Green's function.

c. Solve the boundary value problem using the Green's function.

d. Change the boundary conditions to $y'(0) = 5$, $y(\pi) = -3$.

i. Solve by direct integration.

ii. Solve using the Green's function.

6. Let C be a closed curve and D the enclosed region. Prove the identity

$$\int_C \phi \nabla \phi \cdot \mathbf{n}\, ds = \int_D (\phi \nabla^2 \phi + \nabla \phi \cdot \nabla \phi)\, dA.$$

7. Let S be a closed surface and V the enclosed volume. Prove Green's first and second identities, respectively.

a. $\int_S \phi \nabla \psi \cdot \mathbf{n}\, dS = \int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi)\, dV.$

b. $\int_S [\phi \nabla \psi - \psi \nabla \phi] \cdot \mathbf{n}\, dS = \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV.$

8. Let C be a closed curve and D the enclosed region. Prove Green's identities in two dimensions.

a. First prove

$$\int_D (v \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla v)\, dA = \int_C (v\mathbf{F}) \cdot d\mathbf{s}.$$

b. Let $\mathbf{F} = \nabla u$ and obtain Green's first identity,

$$\int_D (v \nabla^2 u + \nabla u \cdot \nabla v)\, dA = \int_C (v \nabla u) \cdot d\mathbf{s}.$$

c. Use Green's first identity to prove Green's second identity,

$$\int_D (u\nabla^2 v - v\nabla^2 u)\, dA = \int_C (u\nabla v - v\nabla u) \cdot d\mathbf{s}.$$

9. Consider the problem:

$$\frac{\partial^2 G}{\partial x^2} = \delta(x - x_0), \quad \frac{\partial G}{\partial x}(0, x_0) = 0, \quad G(\pi, x_0) = 0.$$

a. Solve by direct integration.

b. Compare this result to the Green's function in part b of the last problem.

c. Verify that G is symmetric in its arguments.

10. Consider the boundary value problem: $y'' - y = x$, $x \in (0, 1)$, with boundary conditions $y(0) = y(1) = 0$.

a. Find a closed form solution without using Green's functions.

b. Determine the closed form Green's function using the properties of Green's functions. Use this Green's function to obtain a solution of the boundary value problem.

c. Determine a series representation of the Green's function. Use this Green's function to obtain a solution of the boundary value problem.

d. Confirm that all of the solutions obtained give the same results.

11. Rewrite the solution to Problem 15 and identify the initial value Green's function.

12. Rewrite the solution to Problem 16 and identify the initial value Green's functions.

13. Find the Green's function for the homogeneous fixed values on the boundary of the quarter plane $x > 0$, $y > 0$, for Poisson's equation using the infinite plane Green's function for Poisson's equation. Use the method of images.

14. Find the Green's function for the one dimensional heat equation with boundary conditions $u(0, t) = 0$ $u_x(L, t)$, $t > 0$.

15. Consider Laplace's equation on the rectangular plate in Figure 6.8. Construct the Green's function for this problem.

16. Construct the Green's function for Laplace's equation in the spherical domain in Figure 6.18.