This series arises in an attempt to describe the magnetic field created by and enclosed by a current loop.

8.3.8 Show that

$$\lim_{x\to\infty} x^{b-a}\frac{(x+a)!}{(x+b)!} = 1.$$

8.3.9 Show that

$$\lim_{n\to\infty}\frac{(2n-1)!!}{(2n)!!}n^{1/2} = \pi^{-1/2}.$$

8.3.10 Calculate the binomial coefficient $\binom{2n}{n}$ to six significant figures for $n = 10$, 20, and 30. Check your values by

(a) a Stirling series approximation through terms in n^{-1},
(b) a double precision calculation.

ANS. $\binom{20}{10} = 1.84756\times 10^5$, $\binom{40}{20} = 1.37846\times 10^{11}$,
$\binom{60}{30} = 1.18264\times 10^{17}$.

8.3.11 Write a program (or subprogram) that will calculate $\log_{10}(x!)$ directly from Stirling's series. Assume that $x \geq 10$. (Smaller values could be calculated via the factorial recurrence relation.) Tabulate $\log_{10}(x!)$ versus x for $x = 10(10)300$. Check your results against AMS-55 (see Additional Readings for this reference) or by direct multiplication (for $n = 10$, 20, and 30).

Check value. $\log_{10}(100!) = 157.97$.

8.3.12 Using the complex arithmetic capability of FORTRAN, write a subroutine that will calculate $\ln(z!)$ for complex z based on Stirling's series. Include a test and an appropriate error message if z is too close to a negative real integer. Check your subroutine against alternate calculations for z real, z pure imaginary, and $z = 1 + ib$ (Exercise 8.2.23).

Check values. $|(i0.5)!| = 0.82618$
phase $(i0.5)! = -0.24406$.

8.4 THE BETA FUNCTION

Using the integral definition (Eq. (8.25)), we write the product of two factorials as the product of two integrals. To facilitate a change in variables, we take the integrals over a finite range:

$$m!n! = \lim_{a^2\to\infty}\int_0^{a^2} e^{-u}u^m\,du\int_0^{a^2} e^{-v}v^n\,dv, \qquad \begin{aligned}\Re(m) &> -1,\\ \Re(n) &> -1.\end{aligned} \tag{8.56a}$$

Replacing u with x^2 and v with y^2, we obtain

$$m!n! = \lim_{a\to\infty} 4\int_0^a e^{-x^2}x^{2m+1}\,dx\int_0^a e^{-y^2}y^{2n+1}\,dy. \tag{8.56b}$$

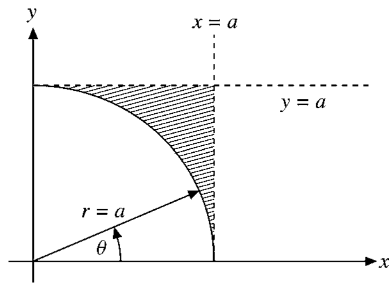

FIGURE 8.6 Transformation from Cartesian to polar coordinates.

Transforming to polar coordinates gives us

$$\begin{aligned} m!n! &= \lim_{a\to\infty} 4\int_0^a e^{-r^2} r^{2m+2n+3}\,dr \int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta\, d\theta \\ &= (m+n+1)!2\int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta\, d\theta. \end{aligned} \tag{8.57}$$

Here the Cartesian area element $dx\,dy$ has been replaced by $r\,dr\,d\theta$ (Fig. 8.6). The last equality in Eq. (8.57) follows from Exercise 8.1.11.

The definite integral, together with the factor 2, has been named the beta function:

$$\begin{aligned} B(m+1,n+1) &\equiv 2\int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta\, d\theta \\ &= \frac{m!n!}{(m+n+1)!}. \end{aligned} \tag{8.58a}$$

Equivalently, in terms of the gamma function and noting its symmetry,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad B(q,p) = B(p,q). \tag{8.58b}$$

The only reason for choosing $m+1$ and $n+1$, rather than m and n, as the arguments of B is to be in agreement with the conventional, historical beta function.

Definite Integrals, Alternate Forms

The beta function is useful in the evaluation of a wide variety of definite integrals. The substitution $t = \cos^2\theta$ converts Eq. (8.58a) to[7]

$$B(m+1,n+1) = \frac{m!n!}{(m+n+1)!} = \int_0^1 t^m (1-t)^n\, dt. \tag{8.59a}$$

[7]The Laplace transform convolution theorem provides an alternate derivation of Eq. (8.58a), compare Exercise 15.11.2.

Replacing t by x^2, we obtain

$$\frac{m!n!}{2(m+n+1)!} = \int_0^1 x^{2m+1}\bigl(1-x^2\bigr)^n\,dx. \tag{8.59b}$$

The substitution $t = u/(1+u)$ in Eq. (8.59a) yields still another useful form,

$$\frac{m!n!}{(m+n+1)!} = \int_0^\infty \frac{u^m}{(1+u)^{m+n+2}}du. \tag{8.60}$$

The beta function as a definite integral is useful in establishing integral representations of the Bessel function (Exercise 11.1.18) and the hypergeometric function (Exercise 13.4.10).

Verification of $\pi\alpha/\sin\pi\alpha$ Relation

If we take $m = a$, $n = -a$, $-1 < a < 1$, then

$$\int_0^\infty \frac{u^a}{(1+u)^2}du = a!(-a)!. \tag{8.61}$$

By contour integration this integral may be shown to be equal to $\pi a/\sin\pi a$ (Exercise 7.1.18), thus providing another method of obtaining Eq. (8.32).

Derivation of Legendre Duplication Formula

The form of Eq. (8.58a) suggests that the beta function may be useful in deriving the doubling formula used in the preceding section. From Eq. (8.59a) with $m = n = z$ and $\Re(z) > -1$,

$$\frac{z!z!}{(2z+1)!} = \int_0^1 t^z(1-t)^z\,dt. \tag{8.62}$$

By substituting $t = (1+s)/2$, we have

$$\frac{z!z!}{(2z+1)!} = 2^{-2z-1}\int_{-1}^1 \bigl(1-s^2\bigr)^z\,ds = 2^{-2z}\int_0^1 \bigl(1-s^2\bigr)^z\,ds. \tag{8.63}$$

The last equality holds because the integrand is even. Evaluating this integral as a beta function (Eq. (8.59b)), we obtain

$$\frac{z!z!}{(2z+1)!} = 2^{-2z-1}\frac{z!(-\frac{1}{2})!}{(z+\frac{1}{2})!}. \tag{8.64}$$

Rearranging terms and recalling that $(-\frac{1}{2})! = \pi^{1/2}$, we reduce this equation to one form of the Legendre duplication formula,

$$z!\bigl(z+\tfrac{1}{2}\bigr)! = 2^{-2z-1}\pi^{1/2}(2z+1)!. \tag{8.65a}$$

Dividing by $(z+\frac{1}{2})$, we obtain an alternate form of the duplication formula:

$$z!\bigl(z-\tfrac{1}{2}\bigr)! = 2^{-2z}\pi^{1/2}(2z)!. \tag{8.65b}$$

Although the integrals used in this derivation are defined only for $\Re(z) > -1$, the results (Eqs. (8.65a) and (8.65b) hold for all regular points z by analytic continuation.[8]

Using the double factorial notation (Section 8.1), we may rewrite Eq. (8.65a) (with $z = n$, an integer) as

$$\left(n+\tfrac{1}{2}\right)! = \pi^{1/2}(2n+1)!!/2^{n+1}. \tag{8.65c}$$

This is often convenient for eliminating factorials of fractions.

Incomplete Beta Function

Just as there is an incomplete gamma function (Section 8.5), there is also an incomplete beta function,

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt, \qquad 0 \le x \le 1,\ p > 0,\ q > 0 \qquad (\text{if } x = 1). \tag{8.66}$$

Clearly, $B_{x=1}(p,q)$ becomes the regular (complete) beta function, Eq. (8.59a). A power-series expansion of $B_x(p,q)$ is the subject of Exercises 5.2.18 and 5.7.8. The relation to hypergeometric functions appears in Section 13.4.

The incomplete beta function makes an appearance in probability theory in calculating the probability of at most k successes in n independent trials.[9]

Exercises

8.4.1 Derive the doubling formula for the factorial function by integrating $(\sin 2\theta)^{2n+1} = (2\sin\theta\cos\theta)^{2n+1}$ (and using the beta function).

8.4.2 Verify the following beta function identities:

(a) $B(a,b) = B(a+1,b) + B(a,b+1)$,

(b) $B(a,b) = \dfrac{a+b}{b}B(a,b+1)$,

(c) $B(a,b) = \dfrac{b-1}{a}B(a+1,b-1)$,

(d) $B(a,b)B(a+b,c) = B(b,c)B(a,b+c)$.

8.4.3 (a) Show that

$$\int_{-1}^{1}\left(1-x^2\right)^{1/2}x^{2n}\,dx = \begin{cases} \pi/2, & n = 0 \\ \pi\dfrac{(2n-1)!!}{(2n+2)!!}, & n = 1,2,3,\dots. \end{cases}$$

[8] If $2z$ is a negative integer, we get the valid but unilluminating result $\infty = \infty$.

[9] W. Feller, *An Introduction to Probability Theory and Its Applications*, 3rd ed. New York: Wiley (1968), Section VI.10.

(b) Show that

$$\int_{-1}^{1}\left(1-x^2\right)^{-1/2}x^{2n}\,dx=\begin{cases}\pi, & n=0\\ \pi\,\dfrac{(2n-1)!!}{(2n)!!}, & n=1,2,3,\ldots.\end{cases}$$

8.4.4 Show that

$$\int_{-1}^{1}\left(1-x^2\right)^{n}\,dx=\begin{cases}2^{2n+1}\dfrac{n!n!}{(2n+1)!}, & n>-1\\ 2\dfrac{(2n)!!}{(2n+1)!!}, & n=0,1,2,\ldots.\end{cases}$$

8.4.5 Evaluate $\int_{-1}^{1}(1+x)^a(1-x)^b\,dx$ in terms of the beta function.

ANS. $2^{a+b+1}B(a+1,b+1)$.

8.4.6 Show, by means of the beta function, that

$$\int_t^z\frac{dx}{(z-x)^{1-\alpha}(x-t)^{\alpha}}=\frac{\pi}{\sin\pi\alpha},\qquad 0<\alpha<1.$$

8.4.7 Show that the Dirichlet integral

$$\iint x^p y^q\,dx\,dy=\frac{p!q!}{(p+q+2)!}=\frac{B(p+1,q+1)}{p+q+2},$$

where the range of integration is the triangle bounded by the positive x- and y-axes and the line $x+y=1$.

8.4.8 Show that

$$\int_0^{\infty}\int_0^{\infty}e^{-(x^2+y^2+2xy\cos\theta)}\,dx\,dy=\frac{\theta}{2\sin\theta}.$$

What are the limits on θ?
Hint. Consider oblique xy-coordinates.

ANS. $-\pi<\theta<\pi$.

8.4.9 Evaluate (using the beta function)

(a)

$$\int_0^{\pi/2}\cos^{1/2}\theta\,d\theta=\frac{(2\pi)^{3/2}}{16[(\frac{1}{4})!]^2},$$

(b)

$$\int_0^{\pi/2}\cos^n\theta\,d\theta=\int_0^{\pi/2}\sin^n\theta\,d\theta=\frac{\sqrt{\pi}[(n-1)/2]!}{2(n/2)!}$$

$$=\begin{cases}\dfrac{(n-1)!!}{n!!} & \text{for } n \text{ odd},\\ \dfrac{\pi}{2}\cdot\dfrac{(n-1)!!}{n!!} & \text{for } n \text{ even}.\end{cases}$$

8.4.10 Evaluate $\int_0^1 (1-x^4)^{-1/2}\,dx$ as a beta function.

$$\text{ANS. } \frac{[(\frac{1}{4})!]^2 \cdot 4}{(2\pi)^{1/2}} = 1.311028777.$$

8.4.11 Given

$$J_\nu(z) = \frac{2}{\pi^{1/2}(\nu - \frac{1}{2})!}\left(\frac{z}{2}\right)^\nu \int_0^{\pi/2} \sin^{2\nu}\theta \cos(z\cos\theta)\,d\theta, \qquad \Re(\nu) > -\tfrac{1}{2},$$

show, with the aid of beta functions, that this reduces to the Bessel series

$$J_\nu(z) = \sum_{s=0}^{\infty} (-1)^s \frac{1}{s!(s+\nu)!}\left(\frac{z}{2}\right)^{2s+\nu},$$

identifying the initial J_ν as an integral representation of the Bessel function, J_ν (Section 11.1).

8.4.12 Given the associated Legendre function

$$P_m^m(x) = (2m-1)!!\left(1-x^2\right)^{m/2},$$

Section 12.5, show that

(a) $$\int_{-1}^{1} \left[P_m^m(x)\right]^2 dx = \frac{2}{2m+1}(2m)!, \qquad m = 0, 1, 2, \ldots,$$

(b) $$\int_{-1}^{1} \left[P_m^m(x)\right]^2 \frac{dx}{1-x^2} = 2 \cdot (2m-1)!, \qquad m = 1, 2, 3, \ldots.$$

8.4.13 Show that

(a) $$\int_0^1 x^{2s+1}\left(1-x^2\right)^{-1/2} dx = \frac{(2s)!!}{(2s+1)!!},$$

(b) $$\int_0^1 x^{2p}\left(1-x^2\right)^q dx = \frac{1}{2}\frac{(p-\frac{1}{2})!q!}{(p+q+\frac{1}{2})!}.$$

8.4.14 A particle of mass m moving in a symmetric potential that is well described by $V(x) = A|x|^n$ has a total energy $\frac{1}{2}m(dx/dt)^2 + V(x) = E$. Solving for dx/dt and integrating we find that the period of motion is

$$\tau = 2\sqrt{2m}\int_0^{x_{\max}} \frac{dx}{(E - Ax^n)^{1/2}},$$

where $x_{\max}$ is a classical turning point given by $Ax_{\max}^n = E$. Show that

$$\tau = \frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{1/n}\frac{\Gamma(1/n)}{\Gamma(1/n + \frac{1}{2})}.$$

8.4.15 Referring to Exercise 8.4.14,

(a) Determine the limit as $n \to \infty$ of

$$\frac{2}{n}\sqrt{\frac{2\pi m}{E}}\left(\frac{E}{A}\right)^{1/n}\frac{\Gamma(1/n)}{\Gamma(1/n+\frac{1}{2})}.$$

(b) Find $\lim_{n\to\infty}\tau$ from the behavior of the integrand $(E - Ax^n)^{-1/2}$.

(c) Investigate the behavior of the physical system (potential well) as $n \to \infty$. Obtain the period from inspection of this limiting physical system.

8.4.16 Show that

$$\int_0^\infty \frac{\sinh^\alpha x}{\cosh^\beta x}dx = \frac{1}{2}B\left(\frac{\alpha+1}{2}, \frac{\beta-\alpha}{2}\right), \qquad -1 < \alpha < \beta.$$

Hint. Let $\sinh^2 x = u$.

8.4.17 The beta distribution of probability theory has a probability density

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1},$$

with x restricted to the interval (0, 1). Show that

(a) $\langle x\rangle(\text{mean}) = \dfrac{\alpha}{\alpha+\beta}$.

(b) $\sigma^2(\text{variance}) \equiv \langle x^2\rangle - \langle x\rangle^2 = \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

8.4.18 From

$$\lim_{n\to\infty}\frac{\int_0^{\pi/2}\sin^{2n}\theta\, d\theta}{\int_0^{\pi/2}\sin^{2n+1}\theta\, d\theta} = 1$$

derive the Wallis formula for π:

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdots.$$

8.4.19 Tabulate the beta function $B(p, q)$ for p and $q = 1.0(0.1)2.0$ independently.

Check value. $B(1.3, 1.7) = 0.40774$.

8.4.20 (a) Write a subroutine that will calculate the incomplete beta function $B_x(p, q)$. For $0.5 < x \le 1$ you will find it convenient to use the relation

$$B_x(p, q) = B(p, q) - B_{1-x}(q, p).$$

(b) Tabulate $B_x(\frac{3}{2}, \frac{3}{2})$. Spot check your results by using the Gauss–Legendre quadrature.

8.5 THE INCOMPLETE GAMMA FUNCTIONS AND RELATED FUNCTIONS

Generalizing the Euler definition of the gamma function (Eq. (8.5)), we define the incomplete gamma functions by the variable limit integrals

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1}\,dt, \qquad \Re(a) > 0$$

and

$$\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1}\,dt. \tag{8.67}$$

Clearly, the two functions are related, for

$$\gamma(a,x) + \Gamma(a,x) = \Gamma(a). \tag{8.68}$$

The choice of employing $\gamma(a,x)$ or $\Gamma(a,x)$ is purely a matter of convenience. If the parameter a is a positive integer, Eq. (8.67) may be integrated completely to yield

$$\begin{aligned} \gamma(n,x) &= (n-1)!\left(1 - e^{-x}\sum_{s=0}^{n-1}\frac{x^s}{s!}\right) \\ \Gamma(n,x) &= (n-1)!e^{-x}\sum_{s=0}^{n-1}\frac{x^s}{s!}, \qquad n = 1, 2, \ldots. \end{aligned} \tag{8.69}$$

For nonintegral a, a power-series expansion of $\gamma(a,x)$ for small x and an asymptotic expansion of $\Gamma(a,x)$ (denoted as $I(x,p)$) are developed in Exercise 5.7.7 and Section 5.10:

$$\begin{aligned} \gamma(a,x) &= x^a \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!(a+n)}, \qquad |x| \sim 1 \text{ (small } x), \\ \Gamma(a,x) &= x^{a-1}e^{-x}\sum_{n=0}^{\infty}\frac{(a-1)!}{(a-1-n)!}\cdot\frac{1}{x^n} \\ &= x^{a-1}e^{-x}\sum_{n=0}^{\infty}(-1)^n\frac{(n-a)!}{(-a)!}\cdot\frac{1}{x^n}, \qquad x \gg 1 \text{ (large } x). \end{aligned} \tag{8.70}$$

These incomplete gamma functions may also be expressed quite elegantly in terms of confluent hypergeometric functions (compare Section 13.5).

Exponential Integral

Although the incomplete gamma function $\Gamma(a,x)$ in its general form (Eq. (8.67)) is only infrequently encountered in physical problems, a special case is quite common and very

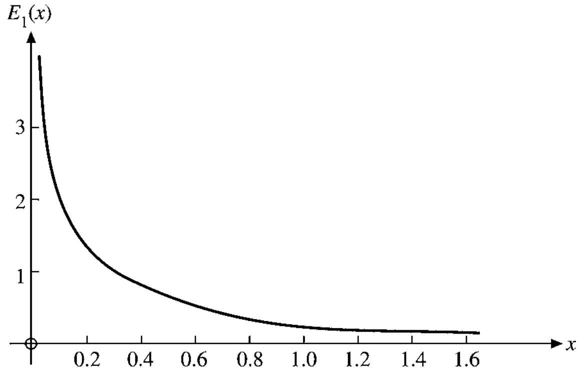

FIGURE 8.7 The exponential integral, $E_1(x) = -\mathrm{Ei}(-x)$.

useful. We define the exponential integral by[10]

$$-\mathrm{Ei}(-x) \equiv \int_x^\infty \frac{e^{-t}}{t} dt = E_1(x). \tag{8.71}$$

(See Fig. 8.7.) Caution is needed here, for the integral in Eq. (8.71) diverges logarithmically as $x \to 0$. To obtain a series expansion for small x, we start from

$$E_1(x) = \Gamma(0, x) = \lim_{a\to 0}\big[\Gamma(a) - \gamma(a, x)\big]. \tag{8.72}$$

We may split the divergent term in the series expansion for $\gamma(a, x)$,

$$E_1(x) = \lim_{a\to 0}\left[\frac{a\Gamma(a) - x^a}{a}\right] - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!}. \tag{8.73}$$

Using l'Hôpital's rule (Exercise 5.6.8) and

$$\frac{d}{da}\big\{a\Gamma(a)\big\} = \frac{d}{da}a! = \frac{d}{da}e^{\ln(a!)} = a!\psi(a+1), \tag{8.74}$$

and then Eq. (8.40),[11] we obtain the rapidly converging series

$$E_1(x) = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!}. \tag{8.75}$$

An asymptotic expansion $E_1(x) \approx e^{-x}[\frac{1}{x} - \frac{1!}{x^2} + \cdots]$ for $x \to \infty$ is developed in Section 5.10.

[10]The appearance of the two minus signs in $-\mathrm{Ei}(-x)$ is a historical monstrosity. AMS-55, Chapter 5, denotes this integral as $E_1(x)$. See Additional Readings for the reference.
[11]$dx^a/da = x^a \ln x$.

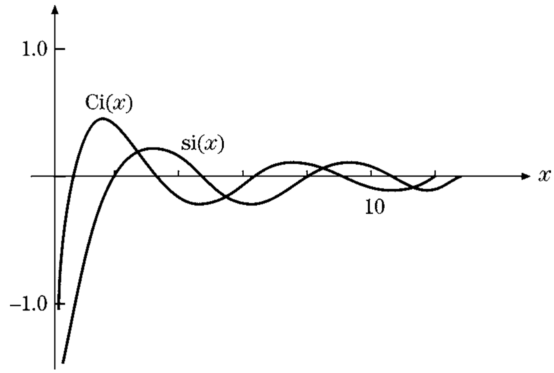

FIGURE 8.8 Sine and cosine integrals.

Further special forms related to the exponential integral are the sine integral, cosine integral (Fig. 8.8), and logarithmic integral, defined by[12]

$$\begin{aligned} \mathrm{si}(x) &= -\int_x^\infty \frac{\sin t}{t}dt \\ \mathrm{Ci}(x) &= -\int_x^\infty \frac{\cos t}{t}dt \\ \mathrm{li}(x) &= \int_0^x \frac{du}{\ln u} = \mathrm{Ei}(\ln x) \end{aligned} \tag{8.76}$$

for their principal branch, with the branch cut conventionally chosen to be along the negative real axis from the branch point at zero. By transforming from real to imaginary argument, we can show that

$$\mathrm{si}(x) = \frac{1}{2i}\big[\mathrm{Ei}(ix) - \mathrm{Ei}(-ix)\big] = \frac{1}{2i}\big[E_1(ix) - E_1(-ix)\big], \tag{8.77}$$

whereas

$$\mathrm{Ci}(x) = \frac{1}{2}\big[\mathrm{Ei}(ix) + \mathrm{Ei}(-ix)\big] = -\frac{1}{2}\big[E_1(ix) + E_1(-ix)\big], \qquad |\arg x| < \frac{\pi}{2}. \tag{8.78}$$

Adding these two relations, we obtain

$$\mathrm{Ei}(ix) = \mathrm{Ci}(x) + i\,\mathrm{si}(x), \tag{8.79}$$

to show that the relation among these integrals is exactly analogous to that among e^{ix}, $\cos x$, and $\sin x$. Reference to Eqs. (8.71) and (8.78) shows that $\mathrm{Ci}(x)$ agrees with the definitions of AMS-55 (see Additional Readings for the reference). In terms of E_1,

$$E_1(ix) = -\mathrm{Ci}(x) + i\,\mathrm{si}(x).$$

Asymptotic expansions of $\mathrm{Ci}(x)$ and $\mathrm{si}(x)$ are developed in Section 5.10. Power-series expansions about the origin for $\mathrm{Ci}(x)$, $\mathrm{si}(x)$, and $\mathrm{li}(x)$ may be obtained from those for

[12] Another sine integral is given by $\mathrm{Si}(x) = \mathrm{si}(x) + \pi/2$.

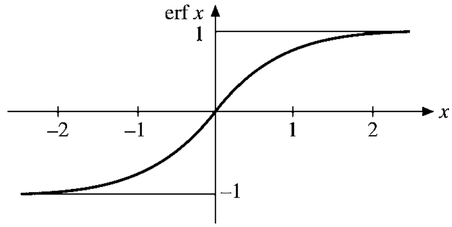

FIGURE 8.9 Error function, erf x.

the exponential integral, $E_1(x)$, or by direct integration, Exercise 8.5.10. The exponential, sine, and cosine integrals are tabulated in AMS-55, Chapter 5, (see Additional Readings for the reference) and can also be accessed by symbolic software such as Mathematica, Maple, Mathcad, and Reduce.

Error Integrals

The error integrals

$$\boxed{\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt, \qquad \operatorname{erfc} z = 1 - \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt} \tag{8.80a}$$

(normalized so that erf $\infty = 1$) are introduced in Exercise 5.10.4 (Fig. 8.9). Asymptotic forms are developed there. From the general form of the integrands and Eq. (8.6) we expect that erf z and erfc z may be written as incomplete gamma functions with $a = \frac{1}{2}$. The relations are

$$\operatorname{erf} z = \pi^{-1/2}\gamma\left(\tfrac{1}{2}, z^2\right), \qquad \operatorname{erfc} z = \pi^{-1/2}\Gamma\left(\tfrac{1}{2}, z^2\right). \tag{8.80b}$$

The power-series expansion of erf z follows directly from Eq. (8.70).

Exercises

8.5.1 Show that

$$\gamma(a, x) = e^{-x} \sum_{n=0}^{\infty} \frac{(a-1)!}{(a+n)!} x^{a+n}$$

(a) by repeatedly integrating by parts.
(b) Demonstrate this relation by transforming it into Eq. (8.70).

8.5.2 Show that

(a) $\dfrac{d^m}{dx^m}\left[x^{-a}\gamma(a, x)\right] = (-1)^m x^{-a-m}\gamma(a+m, x),$

(b) $$\frac{d^m}{dx^m}\left[e^x \gamma(a,x)\right] = e^x \frac{\Gamma(a)}{\Gamma(a-m)} \gamma(a-m,x).$$

8.5.3 Show that $\gamma(a,x)$ and $\Gamma(a,x)$ satisfy the recurrence relations

(a) $\gamma(a+1,x) = a\gamma(a,x) - x^a e^{-x}$,
(b) $\Gamma(a+1,x) = a\Gamma(a,x) + x^a e^{-x}$.

8.5.4 The potential produced by a $1S$ hydrogen electron (Exercise 12.8.6) is given by

$$V(r) = \frac{q}{4\pi\varepsilon_0 a_0}\left\{\frac{1}{2r}\gamma(3,2r) + \Gamma(2,2r)\right\}.$$

(a) For $r \ll 1$, show that

$$V(r) = \frac{q}{4\pi\varepsilon_0 a_0}\left\{1 - \frac{2}{3}r^2 + \cdots\right\}.$$

(b) For $r \gg 1$, show that

$$V(r) = \frac{q}{4\pi\varepsilon_0 a_0} \cdot \frac{1}{r}.$$

Here r is expressed in units of a_0, the Bohr radius.
Note. For computation at intermediate values of r, Eqs. (8.69) are convenient.

8.5.5 The potential of a $2P$ hydrogen electron is found to be (Exercise 12.8.7)

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{24a_0}\left\{\frac{1}{r}\gamma(5,r) + \Gamma(4,r)\right\}$$
$$- \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{120a_0}\left\{\frac{1}{r^3}\gamma(7,r) + r^2\Gamma(2,r)\right\} P_2(\cos\theta).$$

Here r is expressed in units of a_0, the Bohr radius. $P_2(\cos\theta)$ is a Legendre polynomial (Section 12.1).

(a) For $r \ll 1$, show that

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{a_0}\left\{\frac{1}{4} - \frac{1}{120}r^2 P_2(\cos\theta) + \cdots\right\}.$$

(b) For $r \gg 1$, show that

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{a_0 r}\left\{1 - \frac{6}{r^2} P_2(\cos\theta) + \cdots\right\}.$$

8.5.6 Prove that the exponential integral has the expansion

$$\int_x^\infty \frac{e^{-t}}{t}dt = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!},$$

where γ is the Euler–Mascheroni constant.

8.5.7 Show that $E_1(z)$ may be written as

$$E_1(z) = e^{-z}\int_0^\infty \frac{e^{-zt}}{1+t}\,dt.$$

Show also that we must impose the condition $|\arg z| \le \pi/2$.

8.5.8 Related to the exponential integral (Eq. (8.71)) by a simple change of variable is the function

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}\,dt.$$

Show that $E_n(x)$ satisfies the recurrence relation

$$E_{n+1}(x) = \frac{1}{n}e^{-x} - \frac{x}{n}E_n(x), \qquad n = 1, 2, 3, \ldots.$$

8.5.9 With $E_n(x)$ as defined in Exercise 8.5.8, show that $E_n(0) = 1/(n-1)$, $n > 1$.

8.5.10 Develop the following power-series expansions:

(a) $\mathrm{si}(x) = -\dfrac{\pi}{2} + \displaystyle\sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$,

(b) $\mathrm{Ci}(x) = \gamma + \ln x + \displaystyle\sum_{n=1}^\infty \frac{(-1)^n x^{2n}}{2n(2n)!}$.

8.5.11 An analysis of a center-fed linear antenna leads to the expression

$$\int_0^x \frac{1-\cos t}{t}\,dt.$$

Show that this is equal to $\gamma + \ln x - \mathrm{Ci}(x)$.

8.5.12 Using the relation

$$\Gamma(a) = \gamma(a, x) + \Gamma(a, x),$$

show that if $\gamma(a, x)$ satisfies the relations of Exercise 8.5.2, then $\Gamma(a, x)$ must satisfy the same relations.

8.5.13 (a) Write a subroutine that will calculate the incomplete gamma functions $\gamma(n, x)$ and $\Gamma(n, x)$ for n a positive integer. Spot check $\Gamma(n, x)$ by Gauss–Laguerre quadratures.
(b) Tabulate $\gamma(n, x)$ and $\Gamma(n, x)$ for $x = 0.0(0.1)1.0$ and $n = 1, 2, 3$.

8.5.14 Calculate the potential produced by a $1S$ hydrogen electron (Exercise 8.5.4) (Fig. 8.10). Tabulate $V(r)/(q/4\pi\varepsilon_0 a_0)$ for $x = 0.0(0.1)4.0$. Check your calculations for $r \ll 1$ and for $r \gg 1$ by calculating the limiting forms given in Exercise 8.5.4.

8.5.15 Using Eqs. (5.182) and (8.75), calculate the exponential integral $E_1(x)$ for

(a) $x = 0.2(0.2)1.0$, (b) $x = 6.0(2.0)10.0$.

Program your own calculation but check each value, using a library subroutine if available. Also check your calculations at each point by a Gauss–Laguerre quadrature.

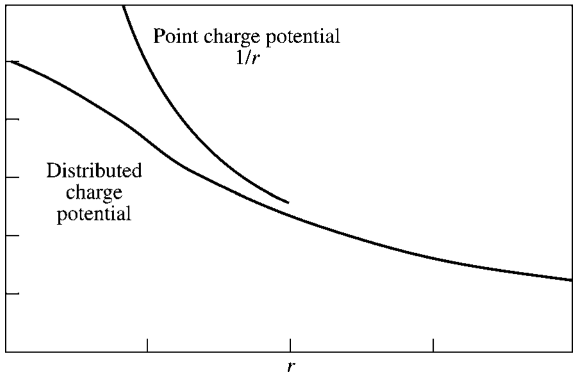

FIGURE 8.10 Distributed charge potential produced by a $1S$ hydrogen electron, Exercise 8.5.14.

You'll find that the power-series converges rapidly and yields high precision for small x. The asymptotic series, even for $x = 10$, yields relatively poor accuracy.

Check values. $E_1(1.0) = 0.219384$
$E_1(10.0) = 4.15697 \times 10^{-6}$.

8.5.16 The two expressions for $E_1(x)$, (1) Eq. (5.182), an asymptotic series and (2) Eq. (8.75), a convergent power series, provide a means of calculating the Euler–Mascheroni constant γ to high accuracy. Using double precision, calculate γ from Eq. (8.75), with $E_1(x)$ evaluated by Eq. (5.182).
Hint. As a convenient choice take x in the range 10 to 20. (Your choice of x will set a limit on the accuracy of your result.) To minimize errors in the alternating series of Eq. (8.75), accumulate the positive and negative terms separately.

ANS. For $x = 10$ and "double precision," $\gamma = 0.57721566$.

Additional Readings

Abramowitz, M., and I. A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (AMS-55). Washington, DC: National Bureau of Standards (1972), reprinted, Dover (1974). Contains a wealth of information about gamma functions, incomplete gamma functions, exponential integrals, error functions, and related functions — Chapters 4 to 6.

Artin, E., *The Gamma Function* (translated by M. Butler). New York: Holt, Rinehart and Winston (1964). Demonstrates that if a function $f(x)$ is smooth (log convex) and equal to $(n-1)!$ when $x = n =$ integer, it is the gamma function.

Davis, H. T., *Tables of the Higher Mathematical Functions*. Bloomington, IN: Principia Press (1933). Volume I contains extensive information on the gamma function and the polygamma functions.

Gradshteyn, I. S., and I. M. Ryzhik, *Table of Integrals, Series, and Products*. New York: Academic Press (1980).

Luke, Y. L., *The Special Functions and Their Approximations*, Vol. 1. New York: Academic Press (1969).

Luke, Y. L., *Mathematical Functions and Their Approximations*. New York: Academic Press (1975). This is an updated supplement to *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (AMS-55). Chapter 1 deals with the gamma function. Chapter 4 treats the incomplete gamma function and a host of related functions.