(a) $u(x_0) \sim -\dfrac{2}{\pi x_0^2} \displaystyle\int_0^\infty x v(x)\, dx,$

(b) $v(x_0) \sim \dfrac{2}{\pi x_0} \displaystyle\int_0^\infty u(x)\, dx.$

In quantum mechanics relations of this form are often called **sum rules**.

7.2.5 (a) Given the integral equation

$$\frac{1}{1+x_0^2} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)}{x-x_0} dx,$$

use Hilbert transforms to determine $u(x_0)$.

(b) Verify that the integral equation of part (a) is satisfied.

(c) From $f(z)|_{y=0} = u(x) + iv(x)$, replace x by z and determine $f(z)$. Verify that the conditions for the Hilbert transforms are satisfied.

(d) Are the crossing conditions satisfied?

ANS. (a) $u(x_0) = \dfrac{x_0}{1+x_0^2}$, (c) $f(z) = (z+i)^{-1}$.

7.2.6 (a) If the real part of the complex index of refraction (squared) is constant (no optical dispersion), show that the imaginary part is zero (no absorption).

(b) Conversely, if there is absorption, show that there must be dispersion. In other words, if the imaginary part of $n^2 - 1$ is not zero, show that the real part of $n^2 - 1$ is not constant.

7.2.7 Given $u(x) = x/(x^2+1)$ and $v(x) = -1/(x^2+1)$, show by direct evaluation of each integral that

$$\int_{-\infty}^{\infty} |u(x)|^2\, dx = \int_{-\infty}^{\infty} |v(x)|^2\, dx.$$

ANS. $\displaystyle\int_{-\infty}^{\infty} |u(x)|^2\, dx = \int_{-\infty}^{\infty} |v(x)|^2\, dx = \frac{\pi}{2}.$

7.2.8 Take $u(x) = \delta(x)$, a delta function, and **assume** that the Hilbert transform equations hold.

(a) Show that

$$\delta(w) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dy}{y(y-w)}.$$

(b) With changes of variables $w = s - t$ and $x = s - y$, transform the δ representation of part (a) into

$$\delta(s-t) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dx}{(x-s)(s-t)}.$$

Note. The δ function is discussed in Section 1.15.

7.2.9 Show that

$$\delta(x) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{dt}{t(t-x)}$$

is a valid representation of the delta function in the sense that

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0).$$

Assume that $f(x)$ satisfies the condition for the existence of a Hilbert transform.
Hint. Apply Eq. (7.84) twice.

7.3 METHOD OF STEEPEST DESCENTS

Analytic Landscape

In analyzing problems in mathematical physics, one often finds it desirable to know the behavior of a function for large values of the variable or some parameter s, that is, the asymptotic behavior of the function. Specific examples are furnished by the gamma function (Chapter 8) and various Bessel functions (Chapter 11). All these analytic functions are defined by integrals

$$I(s) = \int_C F(z,s)\,dz, \tag{7.100}$$

where F is analytic in z and depends on a real parameter s. We write $F(z)$ whenever possible.

So far we have evaluated such definite integrals of analytic functions along the real axis by deforming the path C to C' in the complex plane, so $|F|$ becomes small for all z on C'. This method succeeds as long as only isolated poles occur in the area between C and C'. The poles are taken into account by applying the residue theorem of Section 7.1. The residues give a measure of the simple poles, where $|F| \to \infty$, which usually dominate and determine the value of the integral.

The behavior of the integral in Eq. (7.100) clearly depends on the absolute value $|F|$ of the integrand. Moreover, the contours of $|F|$ often become more pronounced as s becomes large. Let us focus on a plot of $|F(x+iy)|^2 = U^2(x,y) + V^2(x,y)$, rather than the real part $\Re F = U$ and the imaginary part $\Im F = V$ separately. Such a plot of $|F|^2$ over the complex plane is called the **analytic landscape**, after Jensen, who, in 1912, proved that it has **only saddle points and troughs but no peaks**. Moreover, the troughs reach down all the way to the complex plane. In the absence of (simple) poles, **saddle points** are next in line to **dominate the integral** in Eq. (7.100). Hence the name **saddle point method**. At a saddle point the real (or imaginary) part U of F has a local maximum, which implies that

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0,$$

and therefore by the use of the Cauchy–Riemann conditions of Section 6.2,

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0,$$

so V has a minimum, or vice versa, and $F'(z) = 0$. Jensen's theorem prevents U and V from having either a maximum or a minimum. See Fig. 7.18 for a typical shape (and Exercises 6.2.3 and 6.2.4). **Our strategy will be to choose the path C so that it runs over the saddle point, which gives the dominant contribution, and in the valleys elsewhere.** If there are several saddle points, we treat each alike, and their contributions will add to $I(s \to \infty)$.

To prove that there are no peaks, assume there is one at z_0. That is, $|F(z_0)|^2 > |F(z)|^2$ for all z of a neighborhood $|z - z_0| \le r$. If

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is the Taylor expansion at z_0, the mean value $m(F)$ on the circle $z = z_0 + r\exp(i\varphi)$ becomes

$$\begin{aligned} m(F) &\equiv \frac{1}{2\pi}\int_0^{2\pi} \left|F\big(z_0 + re^{i\varphi}\big)\right|^2 d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{m,n=0}^{\infty} a_m^* a_n r^{m+n} e^{i(n-m)\varphi}\, d\varphi \\ &= \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \ge |a_0|^2 = \left|F(z_0)\right|^2, \end{aligned} \tag{7.101}$$

using orthogonality, $\frac{1}{2\pi}\int_0^{2\pi} \exp i(n-m)\varphi\, d\varphi = \delta_{nm}$. Since $m(F)$ is the mean value of $|F|^2$ on the circle of radius r, there must be a point z_1 on it so that $|F(z_1)|^2 \ge m(F) \ge |F(z_0)|^2$, which contradicts our assumption. Hence there can be no such peak.

Next, let us assume there is a minimum at z_0 so that $0 < |F(z_0)|^2 < |F(z)|^2$ for all z of a neighborhood of z_0. In other words, the dip in the valley does not go down to the complex plane. Then $|F(z)|^2 > 0$ and, since $1/F(z)$ is analytic there, it has a Taylor expansion and z_0 would be a peak of $1/|F(z)|^2$, which is impossible. This proves Jensen's theorem. We now turn our attention back to the integral in Eq. (7.100).

Saddle Point Method

Since each saddle point z_0 necessarily lies above the complex plane, that is, $|F(z_0)|^2 > 0$, we write F in exponential form, $e^{f(z,s)}$, in its vicinity without loss of generality. Note that having no zero in the complex plane is a characteristic property of the exponential function. Moreover, any saddle point with $F(z) = 0$ becomes a trough of $|F(z)|^2$ because $|F(z)|^2 \ge 0$. A case in point is the function z^2 at $z = 0$, where $d(z^2)/dz = 2z = 0$. Here $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$, and $2xy$ has a saddle point at $z = 0$, and so has $x^2 - y^2$, but $|z|^4$ has a trough there.

At z_0 the tangential plane is horizontal; that is, $\frac{\partial F}{\partial z}|_{z=z_0} = 0$, or equivalently $\frac{\partial f}{\partial z}|_{z=z_0} = 0$. **This condition locates the saddle point.** Our next goal is to determine the **direction of steepest descent.** At z_0, f has a power series

$$f(z) = f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + \cdots, \tag{7.102}$$

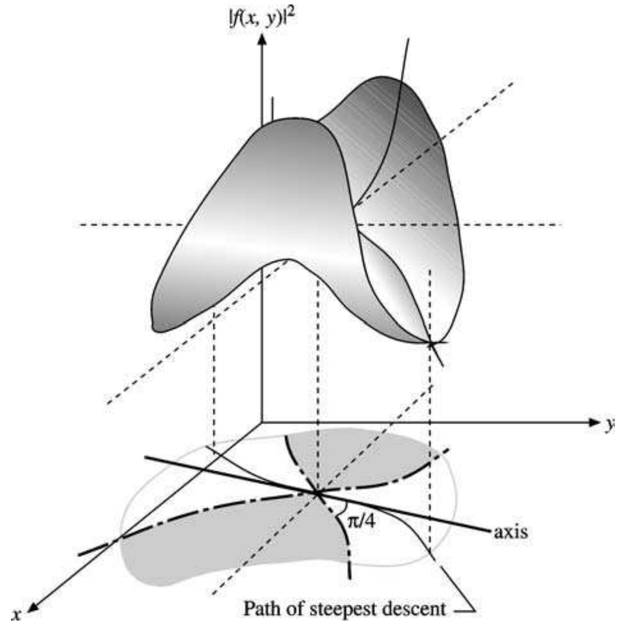

FIGURE 7.18 A saddle point.

or

$$f(z) = f(z_0) + \frac{1}{2}\big(f''(z_0) + \varepsilon\big)(z - z_0)^2, \tag{7.103}$$

upon collecting all higher powers in the (small) ε. Let us take $f''(z_0) \neq 0$ for simplicity. Then

$$f''(z_0)(z - z_0)^2 = -t^2, \qquad t \text{ real}, \tag{7.104}$$

defines a line through z_0 (saddle point **axis** in Fig. 7.18). At z_0, $t = 0$. Along the axis $\Im f''(z_0)(z - z_0)^2$ is zero and $v = \Im f(z) \approx \Im f(z_0)$ is constant if ε in Eq. (7.103) is neglected. Equation (7.104) can also be expressed in terms of angles,

$$\arg(z - z_0) = \frac{\pi}{2} - \frac{1}{2}\arg f''(z_0) = \text{constant}. \tag{7.105}$$

Since $|F(z)|^2 = \exp(2\Re f)$ varies monotonically with $\Re f$, $|F(z)|^2 \approx \exp(-t^2)$ falls off exponentially from its maximum at $t = 0$ along this axis. Hence the name **steepest descent**. The line through z_0 defined by

$$f''(z_0)(z - z_0)^2 = +t^2 \tag{7.106}$$

is orthogonal to this axis (**dashed** in Fig. 7.18), which is evident from its angle,

$$\arg(z - z_0) = -\frac{1}{2}\arg f''(z_0) = \text{constant}, \tag{7.107}$$

when compared with Eq. (7.105). Here $|F(z)|^2$ grows exponentially.

The curves $\Re f(z) = \Re f(z_0)$ go through z_0, so $\Re[(f''(z_0) + \varepsilon)(z - z_0)^2] = 0$, or $(f''(z_0) + \varepsilon)(z - z_0)^2 = it$ for real t. Expressing this in angles as

$$\arg(z - z_0) = \frac{\pi}{4} - \frac{1}{2}\arg\bigl(f''(z_0) + \varepsilon\bigr), \qquad t > 0, \tag{7.108a}$$

$$\arg(z - z_0) = -\frac{\pi}{4} - \frac{1}{2}\arg\bigl(f''(z_0) + \varepsilon\bigr), \qquad t < 0, \tag{7.108b}$$

and comparing with Eqs. (7.105) and (7.107) we note that these curves (**dot-dashed** in Fig. 7.18) divide the saddle point region into four sectors, two with $\Re f(z) > \Re f(z_0)$ (hence $|F(z)| > |F(z_0)|$), shown shaded in Fig. 7.18, and two with $\Re f(z) < \Re f(z_0)$ (hence $|F(z)| < |F(z_0)|$). They are at $\pm\frac{\pi}{4}$ angles from the axis. Thus, the integration path has to avoid the shaded areas, where $|F|$ rises. If a path is chosen to run up the slopes above the saddle point, the large imaginary part of $f(z)$ leads to rapid oscillations of $F(z) = e^{f(z)}$ and cancelling contributions to the integral.

So far, our **treatment has been general**, except for $f''(z_0) \neq 0$, which can be relaxed. Now we are ready to **specialize the integrand** F further in order to tie up the path selection with the asymptotic behavior as $s \to \infty$.

We assume that s appears linearly in the exponent, that is, we replace $\exp f(z, s) \to \exp(sf(z))$. This dependence on s ensures that the saddle point contribution at z_0 grows with $s \to \infty$ providing steep slopes, as is the case in most applications in physics. In order to account for the region far away from the saddle point that is not influenced by s, we include another analytic function, $g(z)$, which varies slowly near the saddle point and is independent of s.

Altogether, then, **our integral has the more appropriate and specific form**

$$I(s) = \int_C g(z)e^{sf(z)}\,dz. \tag{7.109}$$

The path of steepest descent is the saddle point axis when we neglect the higher-order terms, ε, in Eq. (7.103). With ε, the path of steepest descent is the curve close to the axis within the unshaded sectors, where $v = \Im f(z)$ is strictly constant, while $\Im f(z)$ is only approximately constant on the axis. We approximate $I(s)$ by the integral along the piece of the axis inside the patch in Fig. 7.18, where (compare with Eq. (7.104))

$$z = z_0 + xe^{i\alpha}, \qquad \alpha = \frac{\pi}{2} - \frac{1}{2}\arg f''(z_0), \qquad a \le x \le b. \tag{7.110}$$

We find

$$I(s) \approx e^{i\alpha}\int_a^b g\bigl(z_0 + xe^{i\alpha}\bigr)\exp\bigl[sf\bigl(z_0 + xe^{i\alpha}\bigr)\bigr]\,dx, \tag{7.111a}$$

and the omitted part is small and can be estimated because $\Re(f(z) - f(z_0))$ has an upper negative bound, $-R$ say, that depends on the size of the saddle point patch in Fig. 7.18 (that is, the values of a, b in Eq. (7.110)) that we choose. In Eq. (7.111) we use the power expansions

$$\begin{aligned} f\bigl(z_0 + xe^{i\alpha}\bigr) &= f(z_0) + \frac{1}{2}f''(z_0)e^{2i\alpha}x^2 + \cdots, \\ g\bigl(z_0 + xe^{i\alpha}\bigr) &= g(z_0) + g'(z_0)e^{i\alpha}x + \cdots, \end{aligned} \tag{7.111b}$$

and recall from Eq. (7.110) that

$$\frac{1}{2}f''(z_0)e^{2i\alpha} = -\frac{1}{2}\left|f''(z_0)\right| < 0.$$

We find for the leading term for $s \to \infty$:

$$I(s) = g(z_0)e^{sf(z_0)+i\alpha}\int_a^b e^{-\frac{1}{2}s|f''(z_0)|x^2}\,dx. \tag{7.112}$$

Since the integrand in Eq. (7.112) is essentially zero when x departs appreciably from the origin, we let $b \to \infty$ and $a \to -\infty$. The small error involved is straightforward to estimate. Noting that the remaining integral is just a Gauss error integral,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}a^2x^2}\,dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \frac{\sqrt{2\pi}}{a},$$

we finally obtain

$$\boxed{I(s) = \frac{\sqrt{2\pi}\,g(z_0)e^{sf(z_0)}e^{i\alpha}}{|sf''(z_0)|^{1/2}},} \tag{7.113}$$

where the phase α was introduced in Eqs. (7.110) and (7.105).

A note of warning: We assumed that the only significant contribution to the integral came from the immediate vicinity of the saddle point(s) $z = z_0$. This condition must be checked for each new problem (Exercise 7.3.5).

Example 7.3.1 Asymptotic Form of the Hankel Function $H_\nu^{(1)}(s)$

In Section 11.4 it is shown that the Hankel functions, which satisfy Bessel's equation, may be defined by

$$H_\nu^{(1)}(s) = \frac{1}{\pi i}\int_{C_1,0}^{\infty e^{i\pi}} e^{(s/2)(z-1/z)}\frac{dz}{z^{\nu+1}}, \tag{7.114}$$

$$H_\nu^{(2)}(s) = \frac{1}{\pi i}\int_{C_2,\infty e^{-i\pi}}^{0} e^{(s/2)(z-1/z)}\frac{dz}{z^{\nu+1}}. \tag{7.115}$$

The contour C_1 is the curve in the upper half-plane of Fig. 7.19. The contour C_2 is in the lower half-plane. We apply the method of steepest descents to the first Hankel function, $H_\nu^{(1)}(s)$, which is conveniently in the form specified by Eq. (7.109), with $f(z)$ given by

$$f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right). \tag{7.116}$$

By differentiating, we obtain

$$f'(z) = \frac{1}{2} + \frac{1}{2z^2}. \tag{7.117}$$

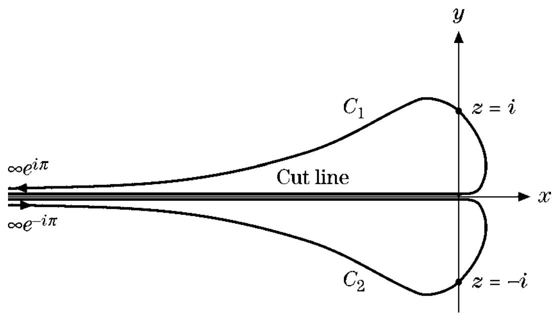

FIGURE 7.19 Hankel function contours.

Setting $f'(z) = 0$, we obtain

$$z = i, -i. \tag{7.118}$$

Hence there are saddle points at $z = +i$ and $z = -i$. At $z = i$, $f''(i) = -i$, or $\arg f''(i) = -\pi/2$, so the saddle point direction is given by Eq. (7.110) as $\alpha = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3}{4}\pi$. For the integral for $H_\nu^{(1)}(s)$ we must choose the contour through the point $z = +i$ so that it starts at the origin, moves out tangentially to the positive real axis, and then moves around through the saddle point at $z = +i$ in the direction given by the angle $\alpha = 3\pi/4$ and then on out to minus infinity, asymptotic with the negative real axis. The path of steepest ascent, which we must avoid, has the phase $-\frac{1}{2}\arg f''(i) = \frac{\pi}{4}$, according to Eq. (7.107), and is orthogonal to the axis, our path of steepest descent.

Direct substitution into Eq. (7.113) with $\alpha = 3\pi/4$ now yields

$$\begin{aligned} H_\nu^{(1)}(s) &= \frac{1}{\pi i}\frac{\sqrt{2\pi i^{-\nu-1}}e^{(s/2)(i-1/i)}e^{3\pi i/4}}{|(s/2)(-2/i^3)|^{1/2}} \\ &= \sqrt{\frac{2}{\pi s}}e^{(i\pi/2)(-\nu-2)}e^{is}e^{i(3\pi/4)}. \end{aligned} \tag{7.119}$$

By combining terms, we obtain

$$H_\nu^{(1)}(s) \approx \sqrt{\frac{2}{\pi s}}e^{i(s-\nu(\pi/2)-\pi/4)} \tag{7.120}$$

as the leading term of the asymptotic expansion of the Hankel function $H_\nu^{(1)}(s)$. Additional terms, if desired, may be picked up from the power series of f and g in Eq. (7.111b). The other Hankel function can be treated similarly using the saddle point at $z = -i$. ■

Example 7.3.2 ASYMPTOTIC FORM OF THE FACTORIAL FUNCTION $\Gamma(1+s)$

In many physical problems, particularly in the field of statistical mechanics, it is desirable to have an accurate approximation of the gamma or factorial function of very large numbers. As developed in Section 8.1, the factorial function may be defined by the Euler integral

$$\Gamma(1+s) = \int_0^\infty \rho^s e^{-\rho}\, d\rho = s^{s+1} \int_0^\infty e^{s(\ln z - z)}\, dz. \tag{7.121}$$

Here we have made the substitution $\rho = zs$ in order to convert the integral to the form required by Eq. (7.109). As before, we assume that s is real and positive, from which it follows that the integrand vanishes at the limits 0 and ∞. By differentiating the z-dependence appearing in the exponent, we obtain

$$\frac{df(z)}{dz} = \frac{d}{dz}(\ln z - z) = \frac{1}{z} - 1, \qquad f''(z) = -\frac{1}{z^2}, \tag{7.122}$$

which shows that the point $z = 1$ is a saddle point and $\arg f''(1) = \arg(-1) = \pi$. According to Eq. (7.109) we let

$$z - 1 = xe^{i\alpha}, \qquad \alpha = \frac{\pi}{2} - \frac{1}{2}\arg f''(1) = \frac{\pi}{2} - \frac{\pi}{2} = 0, \tag{7.123}$$

with x small, to describe the contour in the vicinity of the saddle point. From this we see that the direction of steepest descent is along the real axis, a conclusion that we could have reached more or less intuitively.

Direct substitution into Eq. (7.113) with $\alpha = 0$ now gives

$$\Gamma(1+s) \approx \frac{\sqrt{2\pi}\, s^{s+1} e^{-s}}{|s(-1^{-2})|^{1/2}}. \tag{7.124}$$

Thus the first term in the asymptotic expansion of the factorial function is

$$\Gamma(1+s) \approx \sqrt{2\pi s}\, s^s e^{-s}. \tag{7.125}$$

This result is the first term in Stirling's expansion of the factorial function. The method of steepest descent is probably the easiest way of obtaining this first term. If more terms in the expansion are desired, then the method of Section 8.3 is preferable. ■

In the foregoing example the calculation was carried out by assuming s to be real. This assumption is not necessary. We may show (Exercise 7.3.6) that Eq. (7.125) also holds when s is replaced by the complex variable w, provided only that the real part of w be required to be large and positive.

Asymptotic limits of integral representations of functions are extremely important in many approximations and applications in physics:

$$\int_C g(z) e^{sf(z)}\, dz \sim \frac{\sqrt{2\pi}\, g(z_0) e^{sf(z_0)} e^{i\alpha}}{\sqrt{|sf''(z_0)|}}, \qquad f'(z_0) = 0.$$

The saddle point method is one method of choice for deriving them and belongs in the toolkit of every physicist and engineer.

Exercises

7.3.1 Using the method of steepest descents, evaluate the second Hankel function, given by

$$H_\nu^{(2)}(s) = \frac{1}{\pi i}\int_{-\infty C_2}^{0} e^{(s/2)(z-1/z)}\frac{dz}{z^{\nu+1}},$$

with contour C_2 as shown in Fig. 7.19.

ANS. $H_\nu^{(2)}(s) \approx \sqrt{\dfrac{2}{\pi s}}\, e^{-i(s-\pi/4-\nu\pi/2)}$.

7.3.2 Find the steepest path and leading asymptotic expansion for the Fresnel integrals $\int_0^s \cos x^2\,dx$, $\int_0^s \sin x^2\,dx$.
Hint. Use $\int_0^1 e^{isz^2}\,dz$.

7.3.3 (a) In applying the method of steepest descent to the Hankel function $H_\nu^{(1)}(s)$, show that

$$\Re\big[f(z)\big] < \Re\big[f(z_0)\big] = 0$$

for z on the contour C_1 but away from the point $z = z_0 = i$.

(b) Show that

$$\Re\big[f(z)\big] > 0 \qquad \text{for} \quad 0 < r < 1, \qquad \begin{cases} \dfrac{\pi}{2} < \theta \le \pi \\ -\pi \le \theta < \dfrac{\pi}{2} \end{cases}$$

and

$$\Re\big[f(z)\big] < 0 \qquad \text{for} \quad r > 1, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

(Fig. 7.20). This is why C_1 may not be deformed to pass through the second saddle point, $z = -i$. Compare with and verify the dot-dashed lines in Fig. 7.18 for this case.

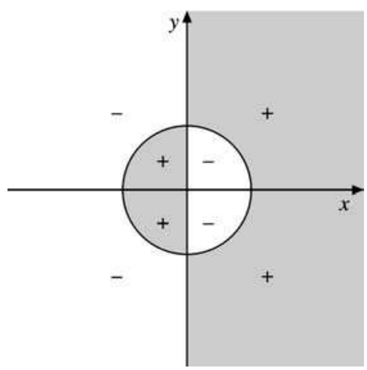

FIGURE 7.20

7.3.4 Determine the asymptotic dependence of the modified Bessel functions $I_\nu(x)$, given

$$I_\nu(x) = \frac{1}{2\pi i}\int_C e^{(x/2)(t+1/t)}\frac{dt}{t^{\nu+1}}.$$

The contour starts and ends at $t = -\infty$, encircling the origin in a positive sense. There are two saddle points. Only the one at $z = +1$ contributes significantly to the asymptotic form.

7.3.5 Determine the asymptotic dependence of the modified Bessel function of the second kind, $K_\nu(x)$, by using

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{(-x/2)(s+1/s)}\frac{ds}{s^{1-\nu}}.$$

7.3.6 Show that Stirling's formula,

$$\Gamma(1+s) \approx \sqrt{2\pi s}\, s^s e^{-s},$$

holds for complex values of s (with $\Re(s)$ large and positive).
Hint. This involves assigning a phase to s and then demanding that $\Im[sf(z)] = \text{constant}$ in the vicinity of the saddle point.

7.3.7 Assume $H_\nu^{(1)}(s)$ to have a negative power-series expansion of the form

$$H_\nu^{(1)}(s) = \sqrt{\frac{2}{\pi s}}\, e^{i(s-\nu(\pi/2)-\pi/4)}\sum_{n=0}^{\infty} a_{-n}s^{-n},$$

with the coefficient of the summation obtained by the method of steepest descent. Substitute into Bessel's equation and show that you reproduce the asymptotic series for $H_\nu^{(1)}(s)$ given in Section 11.6.

Additional Readings

Nussenzveig, H. M., *Causality and Dispersion Relations*, Mathematics in Science and Engineering Series, Vol. 95. New York: Academic Press (1972). This is an advanced text covering causality and dispersion relations in the first chapter and then moving on to develop the implications in a variety of areas of theoretical physics.

Wyld, H. W., *Mathematical Methods for Physics*. Reading, MA: Benjamin/Cummings (1976), Perseus Books (1999). This is a relatively advanced text that contains an extensive discussion of the dispersion relations.