ANSWERS TO SUPPLEMENTARY PROBLEMS

5.30. (a) e^2, (b) 0

5.31. $2\pi i$

5.32. (a) $-2\pi i$, (b) 0

5.33. (a) 0, (b) $-\frac{1}{2}$

5.35. $-\pi i$

5.38. (a) $\pi i/32$, (b) $21\pi i/16$

5.39. $\frac{1}{2}(\sin t - t\cos t)$

5.50. (b) $1, 1, -1 \pm i$

5.51. (a) i, $\frac{1}{2}\left(-i \pm \sqrt{15}\right)$,
(b) $\frac{1}{2}\left(-1 \pm \sqrt{3}i\right)$, $\frac{1}{2}\left(1 \pm \sqrt{3}i\right)$

5.52. 1/4

5.54. 17

5.61. $10\pi i$

5.62. -2

5.63. (a) $14\pi i$, (b) $12\pi i$, (c) $2\pi i$

5.64. $4\pi i$

5.74. $1 - (2/\pi)\tan^{-1}(y/x)$

5.75. $1 - \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{y}{x+1}\right) - \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{y}{x-1}\right)$

5.79. i

5.80. $-2\pi i t^2$

5.91. $-a_1/a_0$, (b) $(a_1^2 - 2a_0a_2)/a_0^2$

CHAPTER 6

Infinite Series
Taylor's and Laurent's Series

6.1 Sequences of Functions

The ideas of Chapter 2, pages 48 and 49, for sequences and series of constants are easily extended to sequences and series of functions.

Let $u_1(z), u_2(z), \ldots, u_n(z), \ldots$, denoted briefly by $\{u_n(z)\}$, be a sequence of functions of z defined and single-valued in some region of the z plane. We call $U(z)$ the *limit* of $u_n(z)$ as $n \to \infty$, and write $\lim_{n\to\infty} u_n(z) = U(z)$, if given any positive number ϵ, we can find a number N [depending in general on both ϵ and z] such that

$$|u_n(z) - U(z)| < \epsilon \quad \text{for all } n > N$$

In such a case, we say that the sequence *converges* or is *convergent* to $U(z)$.

If a sequence converges for all values of z (points) in a region $\mathcal{R}$, we call $\mathcal{R}$ the *region of convergence* of the sequence. A sequence that is not convergent at some value (point) z is called *divergent* at z.

The theorems on limits given on page 49 can be extended to sequences of functions.

6.2 Series of Functions

From the sequence of functions $\{u_n(z)\}$, let us form a new sequence $\{S_n(z)\}$ defined by

$$\begin{aligned} S_1(z) &= u_1(z) \\ S_2(z) &= u_1(z) + u_2(z) \\ &\vdots \\ S_n(z) &= u_1(z) + u_2(z) + \cdots + u_n(z) \end{aligned}$$

where $S_n(z)$, called the nth *partial sum*, is the sum of the first n terms of the sequence $\{u_n(z)\}$.

The sequence $S_1(z), S_2(z), \ldots$ or $\{S_n(z)\}$ is symbolized by

$$u_1(z) + u_2(z) + \cdots = \sum_{n=1}^{\infty} u_n(z) \tag{6.1}$$

called an *infinite series*. If $\lim_{n\to\infty} S_n(z) = S(z)$, the series is called *convergent* and $S(z)$ is its *sum*; otherwise, the series is called *divergent*. We sometimes write $\sum_{n=1}^{\infty} u_n(z)$ as $\sum u_n(z)$ or $\sum u_n$ for brevity.

As we have already seen, a necessary condition that the series (1) converges is $\lim_{n\to\infty} u_n(z) = 0$, but this is not sufficient. See, for example, Problem 2.150, and also Problems 6.67(c), 6.67(d), and 6.111(a).

If a series converges for all values of z (points) in a region $\mathcal{R}$, we call $\mathcal{R}$ the *region of convergence* of the series.

6.3 Absolute Convergence

A series $\sum_{n=1}^{\infty} u_n(z)$ is called *absolutely convergent* if the series of absolute values, i.e., $\sum_{n=1}^{\infty} |u_n(z)|$, converges.

If $\sum_{n=1}^{\infty} u_n(z)$ converges but $\sum_{n=1}^{\infty} |u_n(z)|$ does not converge, we call $\sum_{n=1}^{\infty} u_n(z)$ *conditionally convergent*.

6.4 Uniform Convergence of Sequences and Series

In the definition of limit of a sequence of functions, it was pointed out that the number N depends in general on ϵ and the particular value of z. It may happen, however, that we can find a number N such that $|u_n(z) - U(z)| < \epsilon$ for all $n > N$, where the same number N holds for all z in a region $\mathcal{R}$ [i.e., N depends only on ϵ and not on the particular value of z (point) in the region]. In such a case, we say that $u_n(z)$ *converges uniformly*, or is *uniformly convergent*, to $U(z)$ for all z in $\mathcal{R}$.

Similarly, if the sequence of partial sums $\{S_n(z)\}$ converges uniformly to $S(z)$ in a region, we say that the infinite series (6.1) *converges uniformly*, or is *uniformly convergent*, to $S(z)$ in the region.

We call $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \cdots = S(z) - S_n(z)$ the *remainder* of the infinite series (6.1) after n terms. Then, we can equivalently say that the series is uniformly convergent to $S(z)$ in $\mathcal{R}$ if, given any $\epsilon > 0$, we can find a number N such that for all z in $\mathcal{R}$,

$$|R_n(z)| = |S(z) - S_n(z)| < \epsilon \quad \text{for all } n > N$$

6.5 Power Series

A series having the form

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots = \sum_{n=0}^{\infty} a_n(z-a)^n \tag{6.2}$$

is called a *power series* in $z - a$. We shall sometimes shorten (6.2) to $\sum a_n(z-a)^n$.

Clearly the power series (6.2) converges for $z = a$, and this may indeed be the only point for which it converges [see Problem 6.13(b)]. In general, however, the series converges for other points as well. In such cases, we can show that there exists a positive number R such that (6.2) converges for $|z - a| < R$ and diverges for $|z - a| > R$, while for $|z - a| = R$, it may or may not converge.

Geometrically, if Γ is a circle of radius R with center at $z = a$, then the series (6.2) converges at all points inside Γ and diverges at all points outside Γ, while it may or may not converge on the circle Γ. We can consider the special cases $R = 0$ and $R = \infty$, respectively, to be the cases where (6.2) converges only at $z = a$ or converges for all (finite) values of z. Because of this geometrical interpretation, R is often called the *radius of convergence* of (6.2) and the corresponding circle is called the *circle of convergence*.

6.6 Some Important Theorems

For reference purposes, we list here some important theorems involving sequences and series. Many of these will be familiar from their analogs for real variables.

A. General Theorems

THEOREM 6.1. If a sequence has a limit, the limit is unique [i.e., it is the only one].

THEOREM 6.2. Let $u_n = a_n + ib_n$, $n = 1, 2, 3, \ldots$, where a_n and b_n are real. Then, a necessary and sufficient condition that $\{u_n\}$ converge is that $\{a_n\}$ and $\{b_n\}$ converge.

THEOREM 6.3. Let $\{a_n\}$ be a real sequence with the property that

(i) $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$
(ii) $|a_n| < M$ (a constant)

Then $\{a_n\}$ converges.

If the first condition in Property (i) holds, the sequence is called *monotonic increasing*; if the second condition holds, it is called *monotonic decreasing*. If Property (ii) holds, the sequence is said to be *bounded*. Thus, the theorem states that every bounded monotonic (increasing or decreasing) sequence has a limit.

THEOREM 6.4. A necessary and sufficient condition that $\{u_n\}$ converges is that given any $\epsilon > 0$, we can find a number N such that $|u_p - u_q| < \epsilon$ for all $p > N$, $q > N$.

This result, which has the advantage that the limit itself is not present, is called *Cauchy's convergence criterion*.

THEOREM 6.5. A necessary condition that $\sum u_n$ converge is that $\lim_{n\to\infty} u_n = 0$. However, the condition is not sufficient.

THEOREM 6.6. Multiplication of each term of a series by a constant different from zero does not affect the convergence or divergence. Removal (or addition) of a finite number of terms from (or to) a series does not affect the convergence or divergence.

THEOREM 6.7. A necessary and sufficient condition that $\sum_{n=1}^{\infty} (a_n + ib_n)$ converges, where a_n and b_n are real, is that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

B. Theorems on Absolute Convergence

THEOREM 6.8. If $\sum_{n=1}^{\infty} |u_n|$ converges, then $\sum_{n=1}^{\infty} u_n$ converges. In words, an absolutely convergent series is convergent.

THEOREM 6.9. The terms of an absolutely convergent series can be rearranged in any order and all such rearranged series converge to the same sum. Also, the sum, difference, and product of absolutely convergent series is absolutely convergent.

These are not so for conditionally convergent series (see Problem 6.127).

C. Special Tests for Convergence

THEOREM 6.10. (*Comparison tests*)

(a) If $\sum |v_n|$ converges and $|u_n| \leq |v_n|$, then $\sum u_n$ converges absolutely.

(b) If $\sum |v_n|$ diverges and $|u_n| \geq |v_n|$, then $\sum |u_n|$ diverges but $\sum u_n$ may or may not converge.

THEOREM 6.11. (*Ratio test*) Let $\lim_{n\to\infty}|u_{n+1}/u_n| = L$. Then $\sum u_n$ converges (absolutely) if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test fails.

THEOREM 6.12. (*nth Root test*) Let $\lim_{n\to\infty}\sqrt[n]{|u_n|} = L$. Then $\sum u_n$ converges (absolutely) if $L < 1$ and diverges if $L > 1$. If $L = 1$, the test fails.

THEOREM 6.13. (*Integral test*) If $f(x) \geq 0$ for $x \geq a$, then $\sum f(n)$ converges or diverges according as $\lim_{M\to\infty}\int_a^M f(x)\,dx$ converges or diverges.

THEOREM 6.14. (*Raabe's test*) Let $\lim_{n\to\infty} n\big(1 - |u_{n+1}/u_n|\big) = L$. Then $\sum u_n$ converges (absolutely) if $L > 1$ and diverges or converges conditionally if $L < 1$. If $L = 1$, the test fails.

THEOREM 6.15. (*Gauss' test*) Suppose $|u_{n+1}/u_n| = 1 - (L/n) + (c_n/n^2)$ where $|c_n| < M$ for all $n > N$. Then $\sum u_n$ converges (absolutely) if $L > 1$ and diverges or converges conditionally if $L \leq 1$.

THEOREM 6.16. (*Alternating series test*) If $a_n \geq 0$, $a_{n+1} \leq a_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} a_n = 0$, then $a_1 - a_2 + a_3 - \cdots = \sum(-1)^{n-1}a_n$ converges.

D. Theorems on Uniform Convergence

THEOREM 6.17. (*Weierstrass M test*) $|u_n(z)| \leq M_n$, where M_n is independent of z in a region $\mathcal{R}$ and $\sum M_n$ converges, then $\sum u_n(z)$ is uniformly convergent in $\mathcal{R}$.

THEOREM 6.18. The sum of a uniformly convergent series of continuous functions is continuous, i.e., if $u_n(z)$ is continuous in $\mathcal{R}$ and $S(z) = \sum u_n(z)$ is uniformly convergent in $\mathcal{R}$, then $S(z)$ is continuous in $\mathcal{R}$.

THEOREM 6.19. Suppose $\{u_n(z)\}$ are continuous in $\mathcal{R}$, $S(z) = \sum u_n(z)$ is uniformly convergent in $\mathcal{R}$ and C is a curve in $\mathcal{R}$. Then

$$\int_C S(z)\,dz = \int_C u_1(z)\,dz + \int_C u_2(z)\,dz + \cdots$$

or

$$\int_C \left\{\sum u_n(z)\right\} dz = \sum \int_C u_n(z)\,dz$$

In words, a uniformly convergent series of continuous functions can be integrated term by term.

THEOREM 6.20. Suppose $u_n'(z) = (d/dz)u_n(z)$ exists in $\mathcal{R}$, $\sum u_n'(z)$ converges uniformly in $\mathcal{R}$ and $\sum u_n(z)$ converges in $\mathcal{R}$. Then $(d/dz)\sum u_n(z) = \sum u_n'(z)$.

THEOREM 6.21. Suppose $\{u_n(z)\}$ are analytic and $\sum u_n(z)$ is uniformly convergent in $\mathcal{R}$. Then $S(z) = \sum u_n(z)$ is analytic in $\mathcal{R}$.

E. Theorems on Power Series

THEOREM 6.22. A power series converges uniformly and absolutely in any region that lies entirely inside its circle of convergence.

THEOREM 6.23. (a) A power series can be differentiated term by term in any region that lies entirely inside its circle of convergence.
(b) A power series can be integrated term by term along any curve C that lies entirely inside its circle of convergence.
(c) The sum of a power series is continuous in any region that lies entirely inside its circle of convergence.

These follow from Theorems 6.17–6.19 and 6.21.

THEOREM 6.24. (*Abel's theorem*) Let $\sum a_n z^n$ have radius of convergence R and suppose that z_0 is a point on the circle of convergence such that $\sum a_n z_0^n$ converges. Then, $\lim_{z\to z_0} \sum a_n z^n = \sum a_n z_0^n$ where $z \to z_0$ from within the circle of convergence. Extensions to other power series are easily made.

THEOREM 6.25. Suppose $\sum a_n z^n$ converges to zero for all z such that $|z| < R$ where $R > 0$. Then $a_n = 0$. Equivalently, if $\sum a_n z^n = \sum b_n z^n$ for all z such that $|z| < R$, then $a_n = b_n$.

6.7 Taylor's Theorem

Let $f(z)$ be analytic inside and on a simple closed curve C. Let a and $a + h$ be two points inside C. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots \tag{6.3}$$

or writing $z = a + h$, $h = z - a$,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots \tag{6.4}$$

This is called *Taylor's theorem* and the series (6.3) or (6.4) is called a *Taylor series* or *expansion* for $f(a+h)$ or $f(z)$.

The region of convergence of the series (6.4) is given by $|z - a| < R$, where the radius of convergence R is the distance from a to the nearest singularity of the function $f(z)$. On $|z - a| = R$, the series may or may not converge. For $|z - a| > R$, the series diverges.

If the nearest singularity of $f(z)$ is at infinity, the radius of convergence is infinite, i.e., the series converges for all z.

If $a = 0$ in (6.3) or (6.4), the resulting series is often called a *Maclaurin series*.

6.8 Some Special Series

The following list shows some special series together with their regions of convergence. In the case of multiple-valued functions, the principal branch is used.

1. $e^z \quad = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^2}{3!} + \cdots + \dfrac{z^n}{n!} + \cdots \qquad |z| < \infty$

2. $\sin z \quad = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots (-1)^{n-1}\dfrac{z^{2n-1}}{(2n-1)!} + \cdots \qquad |z| < \infty$

3. $\cos z \quad = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots (-1)^{n-1}\dfrac{z^{2n-2}}{(2n-2)!} + \cdots \qquad |z| < \infty$

4. $\ln(1+z) \quad = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \cdots (-1)^{n-1}\dfrac{z^n}{n} + \cdots \qquad |z| < 1$

5. $\tan^{-1} z \quad = z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \cdots (-1)^{n-1}\dfrac{z^{2n-1}}{2n-1} + \cdots \qquad |z| < 1$

6. $(1+z)^p \quad = 1 + pz + \dfrac{p(p-1)}{2!}z^2 + \cdots + \dfrac{p(p-1)\cdots(p-n+1)}{n!}z^n + \cdots \qquad |z| < 1$

From the list above, note that the last is the *binomial theorem* or *formula*. If $(1+z)^p$ is multiple-valued, the result is valid for that branch of the function which has the value 1 when $z = 0$.

6.9 Laurent's Theorem

Let C_1 and C_2 be concentric circles of radii R_1 and R_2, respectively, and center at a [Fig. 6-1]. Suppose that $f(z)$ is single-valued and analytic on C_1 and C_2 and, in the ring-shaped region $\mathcal{R}$ [also called the *annulus* or *annular region*] between C_1 and C_2, is shown shaded in Fig. 6-1. Let $a + h$ be any point in $\mathcal{R}$. Then we have

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \cdots \tag{6.5}$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, 1, 2, \ldots \\ a_{-n} &= \frac{1}{2\pi i}\oint_{C_1} (z-a)^{n-1} f(z)\,dz \quad n = 1, 2, 3, \ldots \end{aligned} \tag{6.6}$$

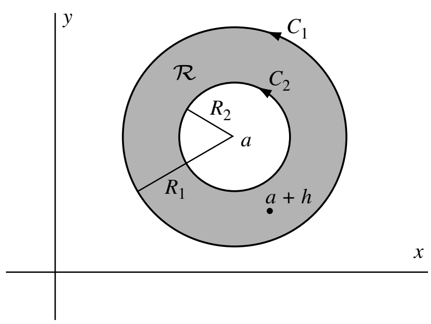

Fig. 6-1

C_1 and C_2 being traversed in the positive direction with respect to their interiors.

In the above integrations, we can replace C_1 and C_2 by any concentric circle C between C_1 and C_2 [see Problem 6.100]. Then, the coefficients (6.6) can be written in a single formula,

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.7}$$

With an appropriate change of notation, we can write the above as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \tag{6.8}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.9}$$

This is called *Laurent's theorem* and (6.5) or (6.8) with coefficients (6.6), (6.7), or (6.9) is called a *Laurent series* or *expansion*.

The part $a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$ is called the *analytic part* of the Laurent series, while the remainder of the series, which consists of inverse powers of $z - a$, is called the *principal part*. If the principal part is zero, the Laurent series reduces to a Taylor series.

6.10 Classification of Singularities

It is possible to classify the singularities of a function $f(z)$ by examination of its Laurent series. For this purpose, we assume that in Fig. 6-1, $R_2 = 0$, so that $f(z)$ is analytic inside and on C_1 except at $z = a$, which is an isolated singularity [see page 81]. In the following, all singularities are assumed isolated unless otherwise indicated.

1. **Poles.** If $f(z)$ has the form (6.8) in which the principal part has only a finite number of terms given by

$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n}$$

where $a_{-n} \neq 0$, then $z = a$ is called a *pole of order n*. If $n = 1$, it is called a *simple pole*.
If $f(z)$ has a pole at $z = a$, then $\lim_{z\to a} f(z) = \infty$ [see Problem 6.32].

2. **Removable singularities.** If a single-valued function $f(z)$ is not defined at $z = a$ but $\lim_{z\to a} f(z)$ exists, then $z = a$ is called a *removable singularity*. In a such case, we define $f(z)$ at $z = a$ as equal to $\lim_{z\to a} f(z)$, and $f(z)$ will then be analytic at a.

EXAMPLE 6.1: If $f(z) = \sin z/z$, then $z = 0$ is a removable singularity since $f(0)$ is not defined but $\lim_{z\to 0} \sin z/z = 1$. We define $f(0) = \lim_{z\to 0} \sin z/z = 1$. Note that in this case

$$\frac{\sin z}{z} = \frac{1}{z}\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right\} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

3. **Essential singularities.** If $f(z)$ is single-valued, then any singularity that is not a pole or removable singularity is called an *essential singularity*. If $z = a$ is an essential singularity of $f(z)$, the principal part of the Laurent expansion has infinitely many terms.

EXAMPLE 6.2: Since $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$, $z = 0$ is an essential singularity.

The following two related theorems are of interest (see Problems 6.153–6.155):

Casorati–Weierstrass theorem. In any neighborhood of an isolated essential singularity a, an otherwise analytic function $f(z)$ comes arbitrarily close to any complex number an infinite number of times. In symbols, given any positive numbers δ and ϵ and any complex number A, there exists a value of z inside the circle $|z - a| = \delta$ for which $|f(z) - A| < \epsilon$.
Picard's theorem. In the neighborhood of an isolated essential singularity a, an otherwise analytic function $f(z)$ takes on every complex value with perhaps one exception.

4. **Branch points.** A point $z = z_0$ is called a *branch point* of a multiple-valued function $f(z)$ if the branches of $f(z)$ are interchanged when z describes a closed path about z_0 [see page 45]. A branch point is a non-isolated singularity. Since each of the branches of a multiple-valued function is analytic, all of the theorems for analytic functions, in particular Taylor's theorem, apply.

EXAMPLE 6.3: The branch of $f(z) = z^{1/2}$, which has the value 1 for $z = 1$, has a Taylor series of the form $a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots$ with radius of convergence $R = 1$ [the distance from $z = 1$ to the nearest singularity, namely the branch point $z = 0$].

5. **Singularities at infinity.** By letting $z = 1/w$ in $f(z)$, we obtain the function $f(1/w) = F(w)$. Then the nature of the singularity for $f(z)$ at $z = \infty$ [the point at infinity] is defined to be the same as that of $F(w)$ at $w = 0$.

EXAMPLE 6.4: $f(z) = z^3$ has a pole of order 3 at $z = \infty$, since $F(w) = f(1/w) = 1/w^3$ has a pole of order 3 at $w = 0$. Similarly, $f(z) = e^z$ has an essential singularity at $z = \infty$, since $F(w) = f(1/w) = e^{1/w}$ has an essential singularity at $w = 0$.

6.11 Entire Functions

A function that is analytic everywhere in the finite plane [i.e., everywhere except at ∞] is called an *entire function* or *integral function*. The functions e^z, $\sin z$, $\cos z$ are entire functions.

An entire function can be represented by a Taylor series that has an infinite radius of convergence. Conversely, if a power series has an infinite radius of convergence, it represents an entire function.

Note that by Liouville's theorem [Chapter 5, page 145], a function which is analytic *everywhere including* ∞ must be a constant.

6.12 Meromorphic Functions

A function that is analytic everywhere in the finite plane except at a finite number of poles is called a *meromorphic function*.

EXAMPLE 6.5: $z/(z-1)(z+3)^2$, which is analytic everywhere in the finite plane except at the poles $z = 1$ (simple pole) and $z = -3$ (pole of order 2), is a meromorphic function.

6.13 Lagrange's Expansion

Let z be that root of $z = a + \zeta\phi(z)$ which has the value $z = a$ when $\zeta = 0$. Then, if $\phi(z)$ is analytic inside and on a circle C containing $z = a$, we have

$$z = a + \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{[\phi(a)]^n\} \tag{6.10}$$

More generally, if $F(z)$ is analytic inside and on C, then

$$F(z) = F(a) + \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{F'(a)[\phi(a)]^n\} \tag{6.11}$$

The expansion (6.11) and the special case (6.10) are often referred to as *Lagrange's expansions*.

6.14 Analytic Continuation

Suppose that we do not know the precise form of an analytic function $f(z)$ but only know that inside some circle of convergence C_1 with center at a [Fig. 6-2], $f(z)$ is represented by a Taylor series

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \tag{6.12}$$

Choosing a point b inside C_1, we can find the value of $f(z)$ and its derivatives at b from (6.13) and thus arrive at a new series

$$b_0 + b_1(z-b) + b_2(z-b)^2 + \cdots \tag{6.13}$$

having circle of convergence C_2. If C_2 extends beyond C_1, then the values of $f(z)$ and its derivatives can be obtained in this extended portion and so we have achieved more information concerning $f(z)$.

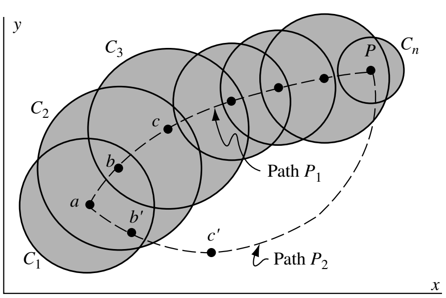

Fig. 6-2

We say, in this case, that $f(z)$ has been *extended analytically* beyond C_1 and call the process *analytic continuation* or *analytic extension*.

The process can, of course, be repeated indefinitely. Thus, choosing point c inside C_2, we arrive at a new series having circle of convergence C_3 which may extend beyond C_1 and C_2, etc.

The collection of all such power series representations, i.e., all possible analytic continuations, is defined as the analytic function $f(z)$ and each power series is sometimes called an *element* of $f(z)$.

In performing analytic continuations, we must avoid singularities. For example, there cannot be any singularity in Fig. 6-2 that is both inside C_2 and on the boundary of C_1, otherwise (6.13) would diverge at this point. In some cases, the singularities on a circle of convergence are so numerous that analytic continuation is impossible. In these cases the boundary of the circle is called a *natural boundary* or barrier [see Problem 6.30]. The function represented by a series having a natural boundary is called a *lacunary* function.

In going from circle C_1 to circle C_n [Fig. 6-2], we have chosen the path of centers $a, b, c, \ldots, p$, which we represent by *path* P_1. Many other paths are also possible, e.g., $a, b', c', \ldots, p$ represented briefly by *path* P_2. A question arises as to whether one obtains the same series representation valid inside C_n when one chooses different paths. The answer is *yes*, so long as the region bounded by paths P_1 and P_2 has no singularity.

For a further discussion of analytic continuation, see Chapter 10.

SOLVED PROBLEMS

Sequences and Series of Functions

6.1. Using the definition, prove that $\lim_{n\to\infty}\left(1+\dfrac{z}{n}\right)=1$ for all z.

Solution

Given any number $\epsilon > 0$, we must find N such that $|1 + z/n - 1| < \epsilon$ for $n > N$. Then $|z/n| < \epsilon$, i.e., $|z|/n < \epsilon$ if $n > |z|/\epsilon = N$.

6.2. (a) Prove that the series $z(1-z)+z^2(1-z)+z^3(1-z)+\cdots$ converges for $|z|<1$, and (b) find its sum.

Solution

The sum of the first n terms of the series is

$$\begin{aligned} S_n(z) &= z(1-z)+z^2(1-z)+\cdots+z^n(1-z) \\ &= z - z^2 + z^2 - z^3 + \cdots + z^n - z^{n+1} = z - z^{n+1} \end{aligned}$$

Now $|S_n(z) - z| = |-z^{n+1}| = |z|^{n+1} < \epsilon$ for $(n+1)\ln|z| < \ln\epsilon$, i.e., $n+1 > \ln\epsilon/\ln|z|$ or $n > (\ln\epsilon/\ln|z|) - 1$.

If $z = 0$, $S_n(0) = 0$ and $|S_n(0) - 0| < \epsilon$ for all n.

Hence $\lim_{n\to\infty} S_n(z) = z$, the required sum for all z such that $|z| < 1$.

Another Method. Since $S_n(z) = z - z^{n+1}$, we have [by Problem 2.41, in which we showed that $\lim_{n\to\infty} z^n = 0$ if $|2| < 1$]

$$\text{Required sum} = S(z) = \lim_{n\to\infty} S_n(z) = \lim_{n\to\infty}(z - z^{n+1}) = z.$$

Absolute and Uniform Convergence

6.3. (a) Prove that the series in Problem 6.2 converges uniformly to the sum z for $|z| \le \frac{1}{2}$.

(b) Does the series converge uniformly for $|z| \le 1$? Explain.

Solution

(a) In Problem 6.2, we have shown that $|S_n(z) - z| < \epsilon$ for all $n > (\ln\epsilon/\ln|z|) - 1$, i.e., the series converges to the sum z for $|z| < 1$ and thus for $|z| \le \frac{1}{2}$.

Now if $|z| \le \frac{1}{2}$, the largest value of $(\ln\epsilon/\ln|z|) - 1$ occurs where $|z| = \frac{1}{2}$ and is given by $(\ln\epsilon/\ln(1/2)) - 1 = N$. It follows that $|S_n(z) - z| < \epsilon$ for all $n > N$ where N depends only on ϵ and not on the particular z in $|z| \le \frac{1}{2}$. Thus, the series converges uniformly to z for $|z| \le \frac{1}{2}$.

(b) The same argument given in part (a) serves to show that the series converges uniformly to sum z for $|z| \leq .9$ or $|z| \leq .99$ by using $N = (\ln \epsilon/\ln(.9)) - 1$ and $N = (\ln \epsilon/\ln(.99)) - 1$, respectively.

However, it is clear that we cannot extend the argument to $|z| \leq 1$ since this would require $N = (\ln \epsilon/\ln 1) - 1$, which is infinite, i.e., there is no finite value of N that can be used in this case. Thus, the series does not converge uniformly for $|z| \leq 1$.

6.4. (a) Prove that the sequence $\{1/1 + nz\}$ is uniformly convergent to zero for all z such that $|z| \geq 2$.

(b) Can the region of uniform convergence in (a) be extended? Explain.

Solution

(a) We have $|(1/1 + nz) - 0| < \epsilon$ when $1/|1 + nz| < \epsilon$ or $|1 + nz| > 1/\epsilon$. Now, $|1 + nz| \leq |1| + |nz| = 1 + n|z|$ and $1 + n|z| \geq |1 + nz| > 1/\epsilon$ for $n > (1/\epsilon - 1/|z|)$. Thus, the sequence converges to zero for $|z| > 2$.

To determine whether it converges uniformly to zero, note that the largest value of $(1/\epsilon - 1/|z|)$ in $|z| \geq 2$ occurs for $|z| = 2$ and is given by $\frac{1}{2}\{(1/\epsilon) - 1\} = N$. It follows that $|(1/1 + nz) - 0| < \epsilon$ for all $n > N$ where N depends only on ϵ and not on the particular z in $|z| \geq 2$. Thus, the sequence is uniformly convergent to zero in this region.

(b) If δ is any positive number, the largest value of $((1/\epsilon) - 1)/|z|$ in $|z| \geq \delta$ occurs for $|z| = \delta$ and is given by $((1/\epsilon) - 1)/\delta$. As in part (a), it follows that the sequence converges uniformly to zero for all z such that $|z| \geq \delta$, i.e., in any region that excludes all points in a neighborhood of $z = 0$.

Since δ can be chosen arbitrarily close to zero, it follows that the region of (a) can be extended considerably.

6.5. Show that (a) the sum function in Problem 6.2 is discontinuous at $z = 1$, (b) the limit in Problem 6.4 is discontinuous at $z = 0$.

Solution

(a) From Problem 6.2, $S_n(z) = z - z^{n+1}$, $S(z) = \lim_{n\to\infty} S_n(z)$. If $|z| < 1$, $S(z) = \lim_{n\to\infty} S_n(z) = z$. If $z = 1$, $S_n(z) = S_n(1) = 0$ and $\lim_{n\to\infty} S_n(1) = 0$. Hence, $S(z)$ is discontinuous at $z = 1$.

(b) From Problem 6.4, if we write $u_n(z) = 1/1 + nz$ and $U(z) = \lim_{n\to\infty} u_n(z)$, we have $U(z) = 0$ if $z \neq 0$ and 1 if $z = 0$. Thus, $U(z)$ is discontinuous at $z = 0$.

These are consequences of the fact [see Problem 6.16] that if a series of continuous functions is uniformly convergent in a region $\mathcal{R}$, then the sum function must be continuous in $\mathcal{R}$. Hence, if the sum function is not continuous, the series cannot be uniformly convergent. A similar result holds for sequences.

6.6. Prove that the series of Problem 6.2 is absolutely convergent for $|z| < 1$.

Solution

Let $T_n(z) = |z(1 - z)| + |z^2(1 - z)| + \cdots + |z^n(1 - z)| = |1 - z|\{|z| + |z|^2 + |z|^3 + \cdots + |z|^n\}$

$$= |1 - z||z|\left\{\frac{1 - |z|^n}{1 - |z|}\right\}$$

If $|z| < 1$, then $\lim_{n\to\infty} |z|^n = 0$ and $\lim_{n\to\infty} T_n(z)$ exists so that the series converges absolutely.

Note that the series of absolute values converges in this case to $|1 - z||z|/1 - |z|$.

Special Convergence Tests

6.7. Suppose $\sum |v_n|$ converges and $|u_n| \leq |v_n|$, $n = 1, 2, 3, \ldots$. Prove that $\sum |u_n|$ also converges (i.e., establish the comparison test for convergence).

Solution

Let $S_n = |u_1| + |u_2| + \cdots + |u_n|$, $T_n = |v_1| + |v_2| + \cdots + |v_n|$.

Since $\sum |v_n|$ converges, $\lim_{n\to\infty} T_n$ exists and equals T, say. Also since $|v_n| \geq 0$, $T_n \leq T$.

Then $S_n = |u_1| + |u_2| + \cdots + |u_n| \leq |v_1| + |v_2| + \cdots + |v_n| \leq T$ or $0 \leq S_n \leq T$.

Thus, S_n is a bounded monotonic increasing sequence and must have a limit [Theorem 6.3, page 171], i.e., $\sum |u_n|$ converges.

6.8. Prove that $\dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for any constant $p > 1$.

Solution

We have

$$\frac{1}{1^p} = \frac{1}{1^{p-1}}$$

$$\frac{1}{2^p} + \frac{1}{3^p} \leq \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}}$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} \leq \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{1}{4^{p-1}}$$

etc., where we consider 1, 2, 4, 8, ... terms of the series. It follows that the sum of any finite number of terms of the given series is less than the geometric series

$$\frac{1}{1^{p-1}} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \frac{1}{8^{p-1}} + \cdots = \frac{1}{1 - 1/2^{p-1}}$$

which converges for $p > 1$. Thus the given series, sometimes called the *p series*, converges.

By using a method analogous to that used here together with the comparison test for divergence [Theorem 6.10(b), page 171], we can show that $\sum_{n=1}^{\infty} 1/n^p$ diverges for $p \leq 1$.

6.9. Prove that an absolutely convergent series is convergent.

Solution

Given that $\sum |u_n|$ converges, we must show that $\sum u_n$ converges. Let

$$S_M = u_1 + u_2 + \cdots + u_M \quad \text{and} \quad T_M = |u_1| + |u_2| + \cdots + |u_M|$$

Then

$$\begin{aligned} S_M + T_M &= (u_1 + |u_1|) + (u_2 + |u_2|) + \cdots + (u_M + |u_M|) \\ &\leq 2|u_1| + 2|u_2| + \cdots + 2|u_M| \end{aligned}$$

Since $\sum |u_n|$ converges and $u_n + |u_n| \geq 0$ for $n = 1, 2, 3, \ldots$, it follows that $S_M + T_M$ is a bounded monotonic increasing sequence and so $\lim_{M\to\infty} (S_M + T_M)$ exists.

Also since $\lim_{M\to\infty} T_M$ exists [because, by hypothesis, the series is absolutely convergent],

$$\lim_{M\to\infty} S_M = \lim_{M\to\infty} (S_M + T_M - T_M) = \lim_{M\to\infty} (S_M + T_M) - \lim_{M\to\infty} T_M$$

must also exist and the result is proved.

6.10. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges (absolutely) for $|z| \leq 1$.

Solution

If $|z| \leq 1$, then $\left|\dfrac{z^n}{n(n+1)}\right| = \dfrac{|z|^n}{n(n+1)} \leq \dfrac{1}{n(n+1)} \leq \dfrac{1}{n^2}$.

Taking $u_n = z^n/n(n+1)$, $v_n = 1/n^2$ in the comparison test and recognizing that $\sum 1/n^2$ converges by Problem 6.8 with $p = 2$, we see that $\sum |u_n|$ converges, i.e., $\sum u_n$ converges absolutely.

6.11. Establish the ratio test for convergence.

Solution

We must show that if $\lim_{n\to\infty}|u_{n+1}/u_n| = L < 1$, then $\sum |u_n|$ converges or, by Problem 6.9, $\sum u_n$ is (absolutely) convergent.

By hypothesis, we can choose an integer N so large that for all $n \geq N$, $|u_{n+1}/u_n| \leq r$ where r is some constant such that $L < r < 1$. Then

$$|u_{N+1}| \leq r|u_N|$$
$$|u_{N+2}| \leq r|u_{N+1}| < r^2|u_N|$$
$$|u_{N+3}| \leq r|u_{N+2}| < r^3|u_N|$$

etc. By addition,

$$|u_{N+1}| + |u_{N+2}| + \cdots \leq |u_N|(r + r^2 + r^3 + \cdots)$$

and so $\sum |u_n|$ converges by the comparison test since $0 < r < 1$.

6.12. Find the region of convergence of the series $\sum_{n=1}^{\infty} \dfrac{(z+2)^{n-1}}{(n+1)^3 4^n}$.

Solution

If $u_n = \dfrac{(z+2)^{n-1}}{(n+1)^3 4^n}$, then $u_{n+1} = \dfrac{(z+2)^n}{(n+2)^3 4^{n+1}}$. Hence, excluding $z = -2$ for which the given series converges, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(z+2)}{4}\frac{(n+1)^3}{(n+2)^3}\right| = \frac{|z+2|}{4}$$

Then the series converges (absolutely) for $|z+2|/4 < 1$, i.e., $|z+2| < 4$. The point $z = -2$ is included in $|z+2| < 4$.

If $|z+2|/4 = 1$, i.e., $|z+2| = 4$, the ratio test fails. However, it is seen that in this case

$$\left|\frac{(z+2)^{n-1}}{(n+1)^3 4^n}\right| = \frac{1}{4(n+1)^3} \leq \frac{1}{n^3}$$

and since $\sum 1/n^3$ converges [p series with $p = 3$], the given series converges (absolutely).

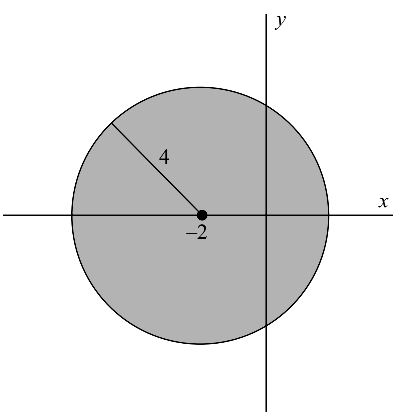

Fig. 6-3

It follows that the given series converges (absolutely) for $|z+2| \leq 4$. Geometrically, this is the set of all points inside and on the circle of radius 4 with center at $z = -2$, called the *circle of convergence* [shown shaded in Fig. 6-3]. The *radius of convergence* is equal to 4.

6.13. Find the region of convergence of the series (a) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$, (b) $\sum_{n=1}^{\infty} n! z^n$.

Solution

(a) If $u_n = (-1)^{n-1} z^{2n-1}/(2n-1)!$, then $u_{n+1} = (-1)^n z^{2n+1}/(2n+1)$. Hence, excluding $z = 0$ for which the given series converges, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|-\frac{z^2(2n-1)!}{(2n+1)!}\right| = \lim_{n\to\infty}\frac{(2n-1)!|z|^2}{(2n+1)(2n)(2n-1)!}$$
$$= \lim_{n\to\infty}\frac{|z|^2}{(2n+1)(2n)} = 0$$

for all finite z. Thus the series converges (absolutely) for all z, and we say that the series converges for $|z| < \infty$. We can equivalently say that the circle of convergence is infinite or that the radius of convergence is infinite.

(b) If $u_n = n!z^n$, $u_{n+1} = (n+1)!z^{n+1}$. Then excluding $z = 0$ for which the given series converges, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!z^{n+1}}{n!z^n}\right| = \lim_{n\to\infty}(n+1)|z| = \infty$$

Thus, the series converges only for $z = 0$.

Theorems on Uniform Convergence

6.14. Prove the Weierstrass M test, i.e., if in a region $\mathcal{R}$, $|u_n(z)| \le M_n$, $n = 1, 2, 3, \ldots,$ where M_n are positive constants such that $\sum M_n$ converges, then $\sum u_n(z)$ is uniformly (and absolutely) convergent in $\mathcal{R}$.

Solution

The remainder of the series $\sum u_n(z)$ after n terms is $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \cdots$. Now

$$\begin{aligned}|R_n(z)| = |u_{n+1}(z) + u_{n+2}(z) + \cdots| &\le |u_{n+1}(z)| + |u_{n+2}(z)| + \cdots \\ &\le M_{n+1} + M_{n+2} + \cdots\end{aligned}$$

But $M_{n+1} + M_{n+2} + \cdots$ can be made less than ϵ by choosing $n > N$, since $\sum M_n$ converges. Since N is clearly independent of z, we have $|R_n(z)| < \epsilon$ for $n > N$, and the series is uniformly convergent. The absolute convergence follows at once from the comparison test.

6.15. Test for uniform convergence in the indicated region:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{z^n}{n\sqrt{n+1}}$, $|z| \le 1$; (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+z^2}$, $1 < |z| < 2$; (c) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos nz}{n^3}$, $|z| \le 1$.

Solution

(a) If $u_n(z) = \dfrac{z^n}{n\sqrt{n+1}}$, then $|u_n(z)| = \dfrac{|z|^n}{n\sqrt{n+1}} \le \dfrac{1}{n^{3/2}}$ if $|z| \le 1$. Calling $M_n = 1/n^{3/2}$, we see that $\sum M_n$ converges (p series with $p = 3/2$). Hence, by the Weierstrass M test, the given series converges uniformly (and absolutely) for $|z| \le 1$.

(b) The given series is $\dfrac{1}{1^2+z^2} + \dfrac{1}{2^2+z^2} + \dfrac{1}{3^2+z^2} + \cdots$. The first two terms can be omitted without affecting the uniform convergence of the series. For $n \ge 3$ and $1 < |z| < 2$, we have

$$|n^2 + z^2| \ge |n^2| - |z^2| \ge n^2 - 4 \ge \frac{1}{2}n^2 \quad \text{or} \quad \left|\frac{1}{n^2+z^2}\right| \le \frac{2}{n^2}$$

Since $\sum_{n=3}^{\infty} 2/n^2$ converges, it follows from the Weierstrass M test (with $M_n = 2/n^2$) that the given series converges uniformly (and absolutely) for $1 < |z| < 2$.

Note that the convergence, and thus uniform convergence, breaks down if $|z| = 1$ or $|z| = 2$ [namely at $z = \pm i$ and $z = \pm 2i$]. Hence, the series cannot converge uniformly for $1 \le |z| \le 2$.

(c) If $z = x + iy$, we have

$$\begin{aligned}\frac{\cos nz}{n^3} &= \frac{e^{inz} + e^{-inz}}{2n^3} = \frac{e^{inx-ny} + e^{-inx+ny}}{2n^3} \\ &= \frac{e^{-ny}(\cos nx + i\sin nx)}{2n^3} + \frac{e^{ny}(\cos nx - i\sin nx)}{2n^3}\end{aligned}$$

The series

$$\sum_{n=1}^{\infty}\frac{e^{ny}(\cos nx - i\sin nx)}{2n^3} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{e^{-ny}(\cos nx + i\sin nx)}{2n^3}$$

cannot converge for $y > 0$ and $y < 0$, respectively [since, in these cases, the nth term does not approach zero]. Hence, the series does not converge for all z such that $|z| \le 1$, and so cannot possibly be uniformly convergent in this region.

The series does converge for $y = 0$, i.e., if z is real. In this case, $z = x$ and the series becomes $\sum_{n=1}^{\infty} \cos nx/n^3$. Then, since $|\cos nx/n^3| \le 1/n^3$ and $\sum_{n=1}^{\infty} 1/n^3$ converges, it follows from the Weierstrass M test (with $M_n = 1/n^3$) that the given series converges uniformly in any interval on the real axis.

6.16. Prove Theorem 6.18, page 172, i.e., if $u_n(z)$, $n = 1, 2, 3, \ldots$, are continuous in $\mathcal{R}$ and $\sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent to $S(z)$ in $\mathcal{R}$, then $S(z)$ is continuous in $\mathcal{R}$.

Solution

If $S_n(z) = u_1(z) + u_2(z) + \cdots + u_n(z)$, and $R_n(z) = u_{n+1}(z) + u_{n+2}(z) + \cdots$ is the remainder after n terms, it is clear that

$$S(z) = S_n(z) + R_n(z) \quad \text{and} \quad S(z+h) = S_n(z+h) + R_n(z+h)$$

and so

$$S(z+h) - S(z) = S_n(z+h) - S_n(z) + R_n(z+h) - R_n(z) \tag{1}$$

where z and $z + h$ are in $\mathcal{R}$.

Since $S_n(z)$ is the sum of a finite number of continuous functions, it must also be continuous. Then, given $\epsilon > 0$, we can find δ so that

$$|S_n(z+h) - S_n(z)| < \epsilon/3 \quad \text{whenever } |h| < \delta \tag{2}$$

Since the series, by hypothesis, is uniformly convergent, we can choose N so that for all z in $\mathcal{R}$,

$$|R_n(z)| < \epsilon/3 \quad \text{and} \quad |R_n(z+h)| < \epsilon/3 \quad \text{for } n > N \tag{3}$$

Then, from (1), (2), and (3),

$$|S(z+h) - S(z)| \le |S_n(z+h) - S_n(z)| + |R_n(z+h)| + |R_n(z)| < \epsilon$$

for $|h| < \delta$ and all z in $\mathcal{R}$, and so the continuity is established.

6.17. Prove Theorem 6.19, page 172, i.e., suppose $\{u_n(z)\}$, $n = 1, 2, 3, \ldots$, are continuous in $\mathcal{R}$, $S(z) = \sum_{n=1}^{\infty} u_n(z)$ is uniformly convergent in $\mathcal{R}$ and C is a curve in $\mathcal{R}$. Then

$$\int_C S(z)\,dz = \int_C \left(\sum_{n=1}^{\infty} u_n(z) \right) dz = \sum_{n=1}^{\infty} \int_C u_n(z)\,dz$$

Solution

As in Problem 6.16, we have $S(z) = S_n(z) + R_n(z)$ and, since these are continuous in $\mathcal{R}$ [by Problem 6.16], their integrals exist, i.e.,

$$\int_C S(z)\,dz = \int_C S_n(z)\,dz + \int_C R_n(z)\,dz = \int_C u_1(z)\,dz + \int_C u_2(z)\,dz + \cdots + \int_C u_n(z)\,dz + \int_C R_n(z)\,dz$$

By hypothesis, the series is uniformly convergent, so that, given any $\epsilon > 0$, we can find a number N independent of z in $\mathcal{R}$ such that $|R_n(z)| < \epsilon$ when $n > N$. Denoting by L the length of C, we have [using Property (e), page 112]

$$\left| \int_C R_n(z)\,dz \right| < \epsilon L$$

Then $\left| \int_C S(z)\,dz - \int_C S_n(z)\,dz \right|$ can be made as small as we like by choosing n large enough, and the result is proved.

Theorems on Power Series

6.18. Suppose a power series $\sum a_n z^n$ converges for $z = z_0 \neq 0$. Prove that it converges: (a) absolutely for $|z| < |z_0|$, (b) uniformly for $|z| \leq |z_1|$ where $|z_1| < |z_0|$.

Solution

(a) Since $\sum a_n z_0^n$ converges, $\lim_{n\to\infty} a_n z_0^n = 0$ and so we can make $|a_n z_0^n| < 1$ by choosing n large enough, i.e., $|a_n| < 1/|z_0|^n$ for $n > N$. Then

$$\sum_{N+1}^{\infty} |a_n z^n| = \sum_{N+1}^{\infty} |a_n||z|^n \leq \sum_{N+1}^{\infty} \frac{|z|^n}{|z_0|^n} \tag{1}$$

But the last series in (1) converges for $|z| < |z_0|$ and so, by the comparison test, the first series converges, i.e., the given series is absolutely convergent.

(b) Let $M_n = |z_1|^n/|z_0|^n$. Then $\sum M_n$ converges, since $|z_1| < |z_0|$. As in part (a), $|a_n z^n| < M_n$ for $|z| \leq |z_1|$ so that, by the Weierstrass M test, $\sum a_n z^n$ is uniformly convergent.

It follows that a power series is uniformly convergent in any region that lies entirely inside its circle of convergence.

6.19. Prove that both the power series $\sum_{n=0}^{\infty} a_n z^n$ and the corresponding series of derivatives $\sum_{n=0}^{\infty} n a_n z^{n-1}$ have the same radius of convergence.

Solution

Let $R > 0$ be the radius of convergence of $\sum a_n z^n$. Let $0 < |z_0| < R$. Then, as in Problem 6.18, we can choose N so that $|a_n| < 1/|z_0|^n$ for $n > N$.

Thus the terms of the series $\sum |n a_n z^{n-1}| = \sum n|a_n||z|^{n-1}$ can for $n > N$ be made less than corresponding terms of the series $\sum n(|z|^{n-1}/|z_0|^n)$, which converges, by the ratio test, for $|z| < |z_0| < R$.

Hence, $\sum n a_n z^{n-1}$ converges absolutely for all points such that $|z| < |z_0|$ (no matter how close $|z_0|$ is to R), i.e., for $|z| < R$.

If, however, $|z| > R$, $\lim_{n\to\infty} a_n z^n \neq 0$ and thus $\lim_{n\to\infty} n a_n z^{n-1} \neq 0$, so that $\sum n a_n z^{n-1}$ does not converge.

Thus, R is the radius of convergence of $\sum n a_n z^{n-1}$. This is also true if $R = 0$.

Note that the series of derivatives may or may not converge for values of z such that $|z| = R$.

6.20. Prove that in any region, which lies entirely within its circle of convergence, a power series (a) represents a continuous function, say $f(z)$, (b) can be integrated term by term to yield the integral of $f(z)$, (c) can be differentiated term by term to yield the derivative of $f(z)$.

Solution

We consider the power series $\sum a_n z^n$, although analogous results hold for $\sum a_n(z-a)^n$.

(a) This follows from Problem 6.16 and the fact that each term $a_n z^n$ of the series is continuous.

(b) This follows from Problem 6.17 and the fact that each term $a_n z^n$ of the series is continuous and thus integrable.

(c) From Problem 6.19, the derivative of a power series converges within the circle of convergence of the original power series and therefore is uniformly convergent in any region entirely within the circle of convergence. Thus, the required result follows from Theorem 6.20, page 172.

6.21. Prove that the series $\sum_{n=1}^{\infty} z^n/n^2$ has a finite value at all points inside and on its circle of convergence but that is not true for the series of derivatives.

Solution

By the ratio test, the series converges for $|z| < 1$ and diverges for $|z| > 1$. If $|z| = 1$, then $|z^n/n^2| = 1/n^2$ and the series is convergent (absolutely). Thus, the series converges for $|z| \leq 1$ and so has a finite value inside and on its circle of convergence.

The series of derivatives is $\sum_{n=1}^{\infty} z^{n-1}/n$. By the ratio test, the series converges for $|z| < 1$. However, the series does not converge for all z such that $|z| = 1$, for example, if $z = 1$, the series diverges.

Taylor's Theorem

6.22. Prove Taylor's theorem: If $f(z)$ is analytic inside a circle C with center at a, then for all z inside C,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \cdots$$

Solution

Let z be any point inside C. Construct a circle C_1 with center at a and enclosing z (see Fig. 6-4). Then, by Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw \tag{1}$$

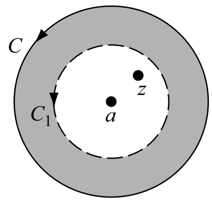

Fig. 6-4

We have

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left\{\frac{1}{1-(z-a)/(w-a)}\right\}\\ &= \frac{1}{w-a}\left\{1 + \left(\frac{z-a}{w-a}\right) + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^{n-1}\right.\\ &\quad \left. + \left(\frac{z-a}{w-a}\right)^n \frac{1}{1-(z-a)/(w-a)}\right\}\end{aligned}$$

or

$$\frac{1}{w-z} = \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n \frac{1}{w-z} \tag{2}$$

Multiplying both sides of (2) by $f(w)$ and using (1), we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^2}\,dw + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^n}\,dw + U_n \tag{3}$$

where

$$U_n = \frac{1}{2\pi i}\oint_{C_1} \left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\,dw$$

Using Cauchy's integral formulas

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw \quad n = 0, 1, 2, 3, \ldots$$

(3) becomes

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + U_n$$

If we can now show that $\lim_{n\to\infty} U_n = 0$, we will have proved the required result. To do this, we note that since w is on C_1,

$$\left|\frac{z-a}{w-a}\right| = \gamma < 1$$

where γ is a constant. Also, we have $|f(w)| < M$ where M is a constant, and

$$|w-z| = |(w-a)-(z-a)| \geq r_1 - |z-a|$$

where r_1 is the radius of C_1. Hence, from Property (e), Page 112, we have

$$|U_n| = \frac{1}{2\pi}\left|\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n \frac{j(w)}{w-z}\,dw\right|$$
$$\leq \frac{1}{2\pi}\frac{\gamma^n M}{r_1 - |z-a|}\cdot 2\pi r_1 = \frac{\gamma^n M r_1}{r_1 - |z-a|}$$

and we see that $\lim_{n\to\infty} U_n = 0$, completing the proof.

6.23. Let $f(z) = \ln(1+z)$, where we consider the branch that has the zero value when $z = 0$. (a) Expand $f(z)$ in a Taylor series about $z = 0$. (b) Determine the region of convergence for the series in (a). (c) Expand $\ln(1 + z/1 - z)$ in a Taylor series about $z = 0$.

Solution

(a)

$$\begin{aligned}
f(z) &= \ln(1+z), & f(0) &= 0\\
f'(z) &= \frac{1}{1+z} = (1+z)^{-1}, & f'(0) &= 1\\
f''(z) &= -(1+z)^{-2}, & f''(0) &= -1\\
f'''(z) &= (-1)(-2)(1+z)^{-3}, & f'''(0) &= 2!\\
&\vdots & &\vdots\\
f^{(n+1)}(z) &= (-1)^n n!(1+z)^{-(n+1)}, & f^{(n+1)}(0) &= (-1)^n n!
\end{aligned}$$

Then

$$f(z) = \ln(1+z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots$$
$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

Another Method. If $|z| < 1$,

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

Then integrating from 0 to z yields

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

(b) The nth term is $u_n = (-1)^{n-1}z^n/n$. Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| = |z|$$

and the series converges for $|z| < 1$. The series can be shown to converge for $|z| = 1$ except for $z = -1$.

This result also follows from the fact that the series converges in a circle that extends to the nearest singularity (i.e., $z = -1$) of $f(z)$.

(c) From the result in (a) we have, on replacing z by $-z$,

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$
$$\ln(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots$$

both series convergent for $|z| < 1$. By subtraction, we have

$$\ln\left(\frac{1+z}{1-z}\right) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right) = \sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$$

which converges for $|z| < 1$. We can also show that this series converges for $|z| = 1$ except for $z = \pm 1$.

6.24. (a) Expand $f(z) = \sin z$ in a Taylor series about $z = \pi/4$
(b) Determine the region of convergence of this series.

Solution

(a) $f(z) = \sin z,\ f'(z) = \cos z,\ f''(z) = -\sin z,\ f'''(z) = -\cos z,\ f^{\mathrm{IV}}(z) = \sin z, \ldots$

$f(\pi/4) = \sqrt{2}/2,\ f'(\pi/4) = \sqrt{2}/2,\ f''(\pi/4) = -\sqrt{2}/2,\ f'''(\pi/4) = -\sqrt{2}/2,\ f^{\mathrm{IV}}(\pi/4) = \sqrt{2}/2, \ldots$

Then, since $a = \pi/4$,

$$\begin{aligned} f(z) &= f(a) + f'(a)(z-a) + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)(z-a)^3}{3!} + \cdots \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(z-\pi/4) - \frac{\sqrt{2}}{2\cdot 2!}(z-\pi/4)^2 - \frac{\sqrt{2}}{2\cdot 3!}(z-\pi/4)^3 + \cdots \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\} \end{aligned}$$

Another Method. Let $u = z - \pi/4$ or $z = u + \pi/4$. Then, we have,

$$\begin{aligned} \sin z &= \sin(u + \pi/4) = \sin u \cos(\pi/4) + \cos u \sin(\pi/4) \\ &= \frac{\sqrt{2}}{2}(\sin u + \cos u) \\ &= \frac{\sqrt{2}}{2}\left\{\left(u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots\right) + \left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots\right)\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + u - \frac{u^2}{2!} - \frac{u^3}{3!} + \frac{u^4}{4!} + \cdots\right\} \\ &= \frac{\sqrt{2}}{2}\left\{1 + (z-\pi/4) - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right\} \end{aligned}$$

(b) Since the singularity of $\sin z$ nearest to $\pi/4$ is at infinity, the series converges for all finite values of z, i.e., $|z| < \infty$. This can also be established by the ratio test.

Laurent's Theorem

6.25. Prove *Laurent's theorem*: Suppose $f(z)$ is analytic inside and on the boundary of the ring-shaped region $\mathcal{R}$ bounded by two concentric circles C_1 and C_2 with center at a and respective radii r_1 and r_2 $(r_1 > r_2)$ (see Fig. 6-5). Then for all z in $\mathcal{R}$,

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{a-n}{(z-a)^n}$$

where

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw \qquad n = 0, 1, 2, \ldots$$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\,dw \qquad n = 1, 2, 3, \ldots$$

Solution

By Cauchy's integral formula [see Problem 5.23, page 159], we have

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\,dw \tag{1}$$

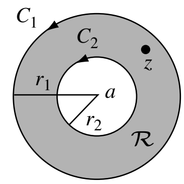

Fig. 6-5

Consider the first integral in (1). As in Problem 6.22, equation (2), we have

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{(w-a)\{1-(z-a)/(w-a)\}}\\ &= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n \frac{1}{w-z}\end{aligned} \tag{2}$$

so that

$$\begin{aligned}\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw\\ &\quad + \cdots + \frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw + U_n\\ &= a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1} + U_n\end{aligned} \tag{3}$$

where

$$a_0 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw, \quad a_1 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw, \quad \ldots, \quad a_{n-1} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw$$

and

$$U_n = \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\,dw$$

Let us now consider the second integral in (1). We have on interchanging w and z in (2),

$$\begin{aligned}-\frac{1}{w-z} &= \frac{1}{(z-a)\{1-(w-a)/(z-a)\}}\\ &= \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \cdots + \frac{(w-a)^{n-1}}{(z-a)^n} + \left(\frac{w-a}{z-a}\right)^n \frac{1}{z-w}\end{aligned}$$

so that

$$\begin{aligned}-\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{z-a}\,dw + \frac{1}{2\pi i}\oint_{C_2}\frac{w-a}{(z-a)^2}f(w)\,dw\\ &\quad + \cdots + \frac{1}{2\pi i}\oint_{C_3}\frac{(w-a)^{n-1}}{(z-a)^n}f(w)\,dw + V_n\\ &= \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + V_n\end{aligned} \tag{4}$$

where

$$a_{-1} = \frac{1}{2\pi i}\oint_{C_2} f(w)\,dw, \quad a_{-2} = \frac{1}{2\pi i}\oint_{C_2}(w-a)f(w)\,dw, \ldots, a_{-n} = \frac{1}{2\pi i}\oint_{C_2}(w-a)^{n-1}f(w)\,dw$$

and

$$V_n = \frac{1}{2\pi i}\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw$$

From (1), (3), and (4), we have

$$f(z) = \{a_0 + a_1(z-a) + \cdots + a_{n-1}(z-a)^{n-1}\} + \left\{\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n}\right\} + U_n + V_n \tag{5}$$

The required result follows if we can show that (a) $\lim_{n\to\infty} U_n = 0$ and (b) $\lim_{n\to\infty} V_n = 0$. The proof of (a) follows from Problem 6.22. To prove (b), we first note that since w is on C_2,

$$\left|\frac{w-a}{z-a}\right| = \kappa < 1$$

where κ is a constant. Also, we have $|f(w)| < M$ where M is a constant and

$$|z-w| = |(z-a)-(w-a)| \geq |z-a| - r_2$$

Hence, from Property (e), page 112, we have

$$|V_n| = \frac{1}{2\pi}\left|\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}\,dw\right| \leq \frac{1}{2\pi}\frac{\kappa^n M}{|z-a|-r_2}2\pi r_2 = \frac{\kappa^n M r_2}{|z-a|-r_2}$$

Then, $\lim_{n\to\infty} V_n = 0$ and the proof is complete.

6.26. Find Laurent series about the indicated singularity for each of the following functions:

(a) $\dfrac{e^{2z}}{(z-1)^3}$; $z=1$. (c) $\dfrac{z-\sin z}{z^3}$; $z=0$. (e) $\dfrac{1}{z^2(z-3)^2}$; $z=3$.

(b) $(z-3)\sin\dfrac{1}{z+2}$; $z=-2$. (d) $\dfrac{z}{(z+1)(z+2)}$; $z=-2$.

Name the singularity in each case and give the region of convergence of each series.

Solution

(a) Let $z-1=u$. Then $z=1+u$ and

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3}\cdot e^{2u} = \frac{e^2}{u^3}\left\{1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \frac{(2u)^4}{4!} + \cdots\right\}$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \cdots$$

$z=1$ is a *pole of order* 3, or *triple pole*.

The series converges for all values of $z \neq 1$.

(b) Let $z+2=u$ or $z=u-2$. Then

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u} = (u-5)\left\{\frac{1}{u} - \frac{1}{3!\,u^3} + \frac{1}{5!\,u^5} - \cdots\right\}$$

$$= 1 - \frac{5}{u} - \frac{1}{3!\,u^2} + \frac{5}{3!\,u^3} + \frac{1}{5!\,u^4} - \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots$$

$z=-2$ is an *essential singularity*.

The series converges for all values of $z \neq -2$.

(c)

$$\frac{z-\sin z}{z^3} = \frac{1}{z^3}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right\}$$

$$= \frac{1}{z^3}\left\{\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right\} = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots$$

$z = 0$ is a *removable singularity*.

The series converges for all values of z.

(d) Let $z + 2 = u$. Then

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)u} = \frac{2-u}{u} \cdot \frac{1}{1-u} = \frac{2-u}{u}(1 + u + u^2 + u^3 + \cdots)$$
$$= \frac{2}{u} + 1 + u + u^2 + \cdots = \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots$$

$z = -2$ is a *pole of order* 1, or *simple pole*.

The series converges for all values of z such that $0 < |z+2| < 1$.

(e) Let $z - 3 = u$. Then, by the binomial theorem,

$$\frac{1}{z^2(z-3)^2} = \frac{1}{u^2(3+u)^2} = \frac{1}{9u^2(1+u/3)^2}$$
$$= \frac{1}{9u^2}\left\{1 + (-2)\left(\frac{u}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3 + \cdots\right\}$$
$$= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots$$
$$= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \cdots$$

$z = 3$ is a *pole of order* 2 or *double pole*.

The series converges for all values of z such that $0 < |z-3| < 3$.

6.27. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for:

(a) $1 < |z| < 3$, (b) $|z| > 3$, (c) $0 < |z+1| < 2$, (d) $|z| < 1$.

Solution

(a) Resolving into partial fractions,

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

If $|z| > 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| < 3$,

$$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then, the required Laurent expansion valid for both $|z| > 1$ and $|z| < 3$, i.e., $1 < |z| < 3$, is

$$\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

(b) If $|z| > 1$, we have as in part (a),

$$\frac{1}{2(z+1)} = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| > 3$,

$$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \frac{27}{2z^4} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$, i.e., $|z| > 3$, is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(c) Let $z + 1 = u$. Then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$
$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

valid for $|u| < 2$, $u \neq 0$ or $0 < |z+1| < 2$.

(d) If $|z| < 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) = \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + \cdots$$

If $|z| < 3$, we have by part (a),

$$\frac{1}{2(z+3)} = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion, valid for both $|z| < 1$ and $|z| < 3$, i.e., $|z| < 1$, is by subtraction

$$\frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

This is a *Taylor series*.

Lagrange's Expansion

6.28. Prove Lagrange's expansion (6.11) on page 176.

Solution

Let us assume that C is taken so that there is only one simple zero of $z = \alpha + \zeta\phi(z)$ inside C. Then, from Problem 5.90, page 167, with $g(z) = z$ and $f(z) = z - a - \zeta\phi(z)$, we have

$$z = \frac{1}{2\pi i}\oint_C w\left\{\frac{1 - \zeta\phi'(w)}{w - a - \zeta\phi(w)}\right\} dw$$
$$= \frac{1}{2\pi i}\oint_C \frac{w}{w-a}\{1 - \zeta\phi'(w)\}\left\{\frac{1}{1 - \zeta\phi(w)/(w-a)}\right\} dw$$
$$= \frac{1}{2\pi i}\oint_C \frac{w}{w-a}\{1 - \zeta\phi'(w)\}\left\{\sum_{n=0}^{\infty}\zeta^n\phi^n(w)/(w-a)^n\right\} dw$$
$$= \frac{1}{2\pi i}\oint_C \frac{w}{w-a}\,dw + \sum_{n=1}^{\infty}\frac{\zeta^n}{2\pi i}\oint_C\left\{\frac{w\phi^n(w)}{(w-a)^{n+1}} - \frac{w\phi^{n-1}(w)\phi'(w)}{(w-a)^n}\right\} dw$$
$$= a - \sum_{n=1}^{\infty}\frac{\zeta^n}{2\pi i}\oint_C \frac{w}{n}\frac{d}{dw}\left\{\frac{\phi^n(w)}{(w-a)^n}\right\} dw = a + \sum_{n=1}^{\infty}\frac{\zeta^n}{2\pi i n}\oint_C \frac{\phi^n(w)}{(w-a)^n}\,dw$$
$$= a + \sum_{n=1}^{\infty}\frac{\zeta^n}{n!}\frac{d^{n-1}}{da^{n-1}}[\phi^n(a)]$$

Analytic Continuation

6.29. Show that the series (a) $\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$ and (b) $\sum_{n=0}^{\infty}\frac{(z-i)^n}{(2-i)^{n+1}}$ are analytic continuations of each other.

Solution

(a) By the ratio test, the series converges for $|z| < 2$ (shaded in Fig. 6-6). In this circle, the series, which is a geometric series with first term $\frac{1}{2}$ and ratio $z/2$ can be summed and represents the function

$$\frac{1/2}{1-z/2} = \frac{1}{2-z}$$

(b) By the ratio test, the series converges for $\left|(z-i)/(2-i)\right| < 1$, i.e., $|z-i| < \sqrt{5}$ (see Fig. 6-6). In this circle, the series, which is a geometric series with first term $1/(2-i)$ and ratio $(z-i)/(2-i)$, can be summed and represents the function

$$\frac{1/(2-i)}{1-(z-i)/(2-i)} = \frac{1}{2-z}$$

Since the power series represent, the same function in the regions common to the interiors of the circles $|z| = 2$ and $|z-i| = \sqrt{5}$, it follows that they are analytic continuations of each other.

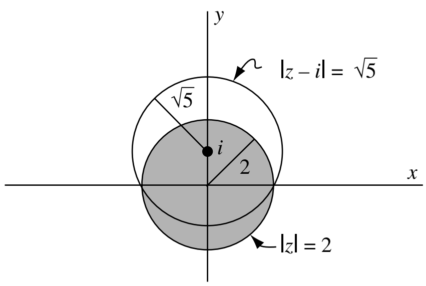

Fig. 6-6

6.30. Prove that the series $1 + z + z^2 + z^4 + z^8 + \cdots = 1 + \sum_{n=0}^{\infty} z^{2^n}$ cannot be continued analytically beyond $|z| = 1$.

Solution

Let $F(z) = 1 + z + z^2 + z^4 + z^8 + \cdots$. Then,

$$F(z) = z + F(z^2), \quad F(z) = z + z^2 + F(z^4), \quad F(z) = z + z^2 + z^4 + F(z^8) + \cdots.$$

From these, it is clear that the values of z given by $z = 1$, $z^2 = 1$, $z^4 = 1$, $z^8 = 1, \ldots$ are all singularities of $F(z)$. These singularities all lie on the circle $|z| = 1$. Given any small arc of this circle, there will be infinitely many such singularities. These represent an impassable barrier and analytic continuation beyond $|z| = 1$ is therefore impossible. The circle $|z| = 1$ constitutes a *natural boundary*.

Miscellaneous Problems

6.31. Let $\{f_k(z)\}$, $k = 1, 2, 3, \ldots$ be a sequence of functions analytic in a region $\mathcal{R}$. Suppose that

$$F(z) = \sum_{k=1}^{\infty} f_k(z)$$

is uniformly convergent in $\mathcal{R}$. Prove that $F(z)$ is analytic in $\mathcal{R}$.

Solution

Let $S_n(z) = \sum_{k=1}^{n} f_k(z)$. By definition of uniform convergence, given any $\epsilon > 0$, we can find a positive integer N depending on ϵ and not on z such that for all z in $\mathcal{R}$,

$$|F(z) - S_n(z)| < \epsilon \qquad \text{for all } n > N \tag{1}$$

Now suppose that C is any simple closed curve lying entirely in $\mathcal{R}$ and denote its length by L. Then, by Problem 6.16, since $f_k(z)$, $k = 1, 2, 3, \ldots$ are continuous, $F(z)$ is also continuous so that $\oint_C F(z)\,dz$ exists. Also, using (1), we see that for $n > N$,

$$\left|\oint_C F(z)\,dz - \sum_{k=1}^{n} \oint_C f_k(z)\,dz\right| = \left|\oint_C \{F(z) - S_n(z)\}\,dz\right|$$

Because ϵ can be made as small as we please, we can see that

$$\oint_C F(z)\,dz = \sum_{k=1}^{\infty} \oint_C f_k(z)\,dz$$

But, by Cauchy's theorem, $\oint_C f_k(z)\,dz = 0$. Hence

$$\oint_C F(z)\,dz = 0$$

and so by Morera's theorem (page 145, Chapter 5), $F(z)$ must be analytic.

6.32. Prove that an analytic function cannot be bounded in the neighborhood of an isolated singularity.

Solution

Let $f(z)$ be analytic inside and on a circle C of radius r, except at the isolated singularity $z = a$ taken to be the center of C. Then, by Laurent's theorem, $f(z)$ has a Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k \tag{1}$$

where the coefficients a_k are given by equation (6.7), page 174. In particular,

$$a_{-n} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{-n+1}}\,dz \quad n = 1, 2, 3, \ldots \tag{2}$$

Now, if $|f(z)| < M$ for a constant M, i.e., if $f(z)$ is bounded, then from (2),

$$|a_{-n}| = \frac{1}{2\pi} \left| \oint_C (z-a)^{n-1} f(z)\,dz \right| \le \frac{1}{2\pi} r^{n-1} \cdot M \cdot 2\pi r = Mr^n$$

Hence, since r can be made arbitrarily small, we have $a_{-n} = 0$, $n = 1, 2, 3, \ldots$, i.e., $a_{-1} = a_{-2} = a_{-3} = \cdots = 0$, and the Laurent series reduces to a Taylor series about $z = a$. This shows that $f(z)$ is analytic at $z = a$ so that $z = a$ is not a singularity, contrary to hypothesis. This contradiction shows that $f(z)$ cannot be bounded in the neighborhood of an isolated singularity.

6.33. Prove that if $z \neq 0$, then

$$e^{1/2\alpha(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(\alpha) z^n$$

where

$$J_n(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - \alpha \sin\theta)\,d\theta \quad n = 0, 1, 2, \ldots$$

Solution

The point $z = 0$ is the only finite singularity of the function $e^{1/2\alpha(z-1/z)}$ and it follows that the function must have a Laurent series expansion of the form

$$e^{1/2\alpha(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(\alpha) z^n \tag{1}$$

which holds for $|z| > 0$. By equation (6.7), page 174, the coefficients $J_n(\alpha)$ are given by

$$J_n(\alpha) = \frac{1}{2\pi i} \oint_C \frac{e^{1/2\alpha(z-1/z)}}{z^{n+1}}\,dz \tag{2}$$

where C is any simple closed curve having $z = 0$ inside.

Let us, in particular, choose C to be a circle of radius 1 having center at the origin; that is, the equation of C is $|z| = 1$ or $z = e^{i\theta}$. Then (2) becomes

$$J_n(\alpha) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{1/2\alpha(e^{i\theta}-e^{-i\theta})}}{e^{i(n+1)\theta}} ie^{i\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i\alpha\sin\theta - in\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha\sin\theta - n\theta)\,d\theta + \frac{i}{2\pi}\int_0^{2\pi}\sin(\alpha\sin\theta - n\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta - \alpha\sin\theta)\,d\theta$$

using the fact that $I = \int_0^{2\pi}\sin(\alpha\sin\theta - n\theta)\,d\theta = 0$. This last result follows since, on letting $\theta = 2\pi - \phi$, we find

$$I = \int_0^{2\pi}\sin(-\alpha\sin\phi - 2\pi n + n\phi)\,d\phi = -\int_0^{2\pi}\sin(\alpha\sin\phi - n\phi)\,d\phi = -I$$

so that $I = -I$ and $I = 0$. The required result is thus established.

The function $J_n(\alpha)$ is called a *Bessel function* of the first kind of order n.

For further discussion of Bessel functions, see Chapter 10.

6.34. The *Legendre polynomials* $P_n(t)$, $n = 0, 1, 2, 3, \ldots$ are defined by *Rodrigues' formula*

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$$

(a) Prove that if C is any simple closed curve enclosing the point $z = t$, then

$$P_n(t) = \frac{1}{2\pi i}\cdot\frac{1}{2^n}\oint_C \frac{(z^2-1)^n}{(z-t)^{n+1}}\,dz$$

This is called *Schlaefli's representation* for $P_n(t)$, or *Schlaefli's formula*.

(b) Prove that

$$P_n(t) = \frac{1}{2\pi}\int_0^{2\pi}\left(t + \sqrt{t^2-1}\cos\theta\right)^n d\theta$$

Solution

(a) By Cauchy's integral formulas, if C encloses point t,

$$f^{(n)}(t) = \frac{d^n}{dt^n}f(t) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-t)^{n+1}}\,dz$$

Then, taking $f(t) = (t^2-1)^n$ so that $f(z) = (z^2-1)^n$, we have the required result

$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}(t^2-1)^n$$
$$= \frac{1}{2^n}\cdot\frac{1}{2\pi i}\oint_C\frac{(z^2-1)^n}{(z-t)^{n+1}}\,dz$$

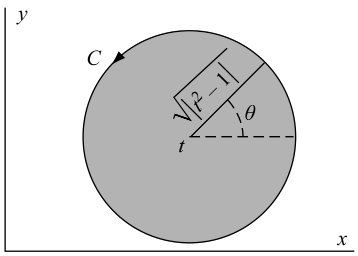

Fig. 6-7

(b) Choose C as a circle with center at t and radius $\sqrt{|t^2-1|}$ as shown in Fig. 6-7. Then, an equation for C is $|z-t| = \sqrt{|t^2-1|}$ or $z = t + \sqrt{t^2-1}\,e^{i\theta}$, $0 \le \theta < 2\pi$. Using this in part (a), we have

$$P_n(t) = \frac{1}{2^n} \cdot \frac{1}{2\pi i} \int_0^{2\pi} \frac{\{(t+\sqrt{t^2-1}\,e^{i\theta})^2 - 1\}^n \sqrt{t^2-1}\,ie^{i\theta}}{(\sqrt{t^2-1}\,e^{i\theta})^{n+1}}\, d\theta$$

$$= \frac{1}{2^n} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\{(t^2-1) + 2t\sqrt{t^2-1}\,e^{i\theta} + (t^2-1)e^{2i\theta}\}^n e^{-in\theta}}{(t^2-1)^{n/2}}\, d\theta$$

$$= \frac{1}{2^n} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\{(t^2-1)e^{-i\theta} + 2t\sqrt{t^2-1} + (t^2-1)e^{i\theta}\}^n}{(t^2-1)^{n/2}}\, d\theta$$

$$= \frac{1}{2^n} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\{2t\sqrt{t^2-1} + 2(t^2-1)\cos\theta\}^n}{(t^2-1)^{n/2}}\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (t + \sqrt{t^2-1}\cos\theta)^n\, d\theta$$

For further discussion of Legendre polynomials, see Chapter 10.

SUPPLEMENTARY PROBLEMS

Sequences and Series of Functions

6.35. Using the definition, prove: (a) $\lim_{n\to\infty} \frac{3n-2z}{n+z} = 3$, (b) $\lim_{n\to\infty} \frac{nz}{n^2+z^2} = 0$.

6.36. Let $\lim_{n\to\infty} u_n(z) = U(z)$ and $\lim_{n\to\infty} v_n(z) = V(z)$. Prove that (a) $\lim_{n\to\infty}\{u_n(z) \pm v_n(z)\} = U(z) \pm V(z)$, (b) $\lim_{n\to\infty}\{u_n(z)v_n(z)\} = U(z)V(z)$, (c) $\lim_{n\to\infty} u_n(z)/v_n(z) = U(z)/V(z)$ if $V(z) \ne 0$.

6.37. (a) Prove that the series $\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \cdots = \sum_{n=1}^{\infty} \frac{z^{n-1}}{2^n}$ converges for $|z| < 2$ and (b) find its sum.

6.38. (a) Determine the set of values of z for which the series $\sum_{n=0}^{\infty} (-1)^n (z^n + z^{n+1})$ converges and (b) find its sum.

6.39. (a) For what values of z does the series $\sum_{n=1}^{\infty} 1/(z^2+1)^n$ converge and (b) what is its sum?

6.40. Suppose $\lim_{n\to\infty} |u_n(z)| = 0$. Prove that $\lim_{n\to\infty} u_n(z) = 0$. Is the converse true? Justify your answer.

6.41. Prove that for all finite z, $\lim_{n\to\infty} z^n/n! = 0$.

6.42. Let $\{a_n\}$, $n = 1, 2, 3, \ldots$ be a sequence of positive numbers having zero as a limit. Suppose that $|u_n(z)| \le a_n$ for $n = 1, 2, 3, \ldots$. Prove that $\lim_{n\to\infty} u_n(z) = 0$.

6.43. Prove that the convergence or divergence of a series is not affected by adding (or removing) a finite number of terms.

6.44. Let $S_n = z + 2z^2 + 3z^3 + \cdots + nz^n$, $T_n = z + z^2 + z^3 + \cdots + z^n$. (a) Show that $S_n = (T_n - nz^{n+1})/(1-z)$. (b) Use (a) to find the sum of the series $\sum_{n=1}^{\infty} nz^n$ and determine the set of values for which the series converges.

6.45. Find the sum of the series $\sum_{n=0}^{\infty} (n+1)/2^n$.

Absolute and Uniform Convergence

6.46. (a) Prove that $u_n(z) = 3z + 4z^2/n$, $n = 1, 2, 3, \ldots$, converges uniformly to $3z$ for all z inside or on the circle $|z| = 1$. (b) Can the circle of part (a) be enlarged? Explain.

6.47. (a) Determine whether the sequence $u_n(z) = nz/(n^2 + z^2)$ [Problem 6.35(b)] converges uniformly to zero for all z inside $|z| = 3$. (b) Does the result of (a) hold for all finite values of z?

6.48. Prove that the series $1 + az + a^2z^2 + \cdots$ converges uniformly to $1/(1 - az)$ inside or on the circle $|z| = R$ where $R < 1/|a|$.

6.49. Investigate the (a) absolute and (b) uniform convergence of the series

$$\frac{z}{3} + \frac{z(3-z)}{3^2} + \frac{z(3-z)^2}{3^3} + \frac{z(3-z)^3}{3^4} + \cdots$$

6.50. Investigate the (a) absolute and (b) uniform convergence of the series in Problem 6.38.

6.51. Investigate the (a) absolute and (b) uniform convergence of the series in Problem 6.39.

6.52. Let $\{a_n\}$ be a sequence of positive constants having limit zero; and suppose that for all z in a region $\mathcal{R}$, $|u_n(z)| \le a_n$, $n = 1, 2, 3, \ldots$. Prove that $\lim_{n\to\infty} u_n(z) = 0$ uniformly in $\mathcal{R}$.

6.53. (a) Prove that the sequence $u_n(z) = nze^{-nz^2}$ converges to zero for all finite z such that $\text{Re}\{z^2\} > 0$, and represent this region geometrically. (b) Discuss the uniform convergence of the sequence in (a).

6.54. Suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely. Prove that $\sum_{n=0}^{\infty} c_n$, where $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$, converges absolutely.

6.55. Suppose each of two series is absolutely and uniformly convergent in $\mathcal{R}$. Prove that their product is absolutely and uniformly convergent in $\mathcal{R}$.

Special Convergence Tests

6.56. Test for convergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+1}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n-1}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{3n^2-n+2}$, (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+3}$, (e) $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{\sqrt{n^3+n+2}}$.

6.57. Investigate the convergence of:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+|z|}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+|z|}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+|z|}$, (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+z}$.

6.58. Investigate the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{ne^{n\pi i/4}}{e^n-1}$.

6.59. Find the region of convergence of:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(z+i)^n}{(n+1)(n+2)}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2\cdot 3^n}\left(\frac{z+1}{z-1}\right)^n$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!}$.

6.60. Investigate the region of absolute convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n(-1)^n(z-i)^n}{4^n(n^2+1)^{5/2}}$.

6.61. Find the region of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{e^{2\pi inz}}{(n+1)^{3/2}}$.

6.62. Prove that the series $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$ diverges although the nth term approaches zero.

6.63. Let N be a positive integer and suppose that for all $n > N$, $|u_n| > 1/(n \ln n)$. Prove that $\sum_{n=1}^{\infty} u_n$ diverges.

6.64. Establish the validity of the (a) nth root test [Theorem 6.12], (b) integral test [Theorem 6.13], on page 141.

6.65. Find the interval of convergence of $1 + 2z + z^2 + 2z^3 + z^4 + 2z^5 + \cdots$.

6.66. Prove Raabe's test (Theorem 6.14) on page 172.

6.67. Test for convergence: (a) $\dfrac{1}{2\ln^2 2} + \dfrac{1}{3\ln^2 3} + \dfrac{1}{4\ln^2 4} + \cdots$, (b) $\dfrac{1}{5} + \dfrac{1 \cdot 4}{5 \cdot 8} + \dfrac{1 \cdot 4 \cdot 7}{5 \cdot 8 \cdot 11} + \cdots$,

(c) $\dfrac{2}{5} + \dfrac{2 \cdot 7}{5 \cdot 10} + \dfrac{2 \cdot 7 \cdot 12}{5 \cdot 10 \cdot 15} + \cdots$, (d) $\dfrac{\ln 2}{2} + \dfrac{\ln 3}{3} + \dfrac{\ln 4}{4} + \cdots$.

Theorems on Uniform Convergence and Power Series

6.68. Determine the regions in which each of the following series is uniformly convergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{3^n + 1}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(z-i)^{2n}}{n^2}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)z^n}$, (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2 + |z|^2}$.

6.69. Prove Theorem 6.20, page 172.

6.70. State and prove theorems for sequences analogous to Theorems 6.18, 6.19, and 6.20, page 172, for series.

6.71. (a) By differentiating both sides of the identity

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$$

find the sum of the series $\sum_{n=1}^{\infty} nz^n$ for $|z| < 1$. Justify all steps.

(b) Find the sum of the series $\sum_{n=1}^{\infty} n^2 z^n$ for $|z| < 1$.

6.72. Let z be real and such that $0 \le z \le 1$, and let $u_n(z) = nze^{-nz^3}$.

(a) Find $\displaystyle\lim_{n\to\infty} \int_0^1 u_n(z)\,dz$, (b) Find $\displaystyle\int_0^1 \left\{\lim_{n\to\infty} u_n(z)\right\} dz$

(c) Explain why the answers to (a) and (b) are not equal [see Problem 6.53].

6.73. Prove Abel's theorem [Theorem 6.24, page 173].

6.74. (a) Prove that $1/(1 + z^2) = 1 - z^2 + z^4 - z^6 + \cdots$ for $|z| < 1$.

(b) If we choose that branch of $f(z) = \tan^{-1} z$ such that $f(0) = 0$, use (a) to prove that

$$\tan^{-1} z = \int_0^z \frac{dz}{1+z^2} = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

(c) Prove that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$.

6.75. Prove Theorem 6.25, page 173.

6.76. (a) Determine $Y(z) = \sum_{n=0}^{\infty} a_n z^n$ such that for all z in $|z| \le 1$, $Y'(z) = Y(z)$, $Y(0) = 1$. State all theorems used and verify that the result obtained is a solution.

(b) Is the result obtained in (a) valid outside of $|z| \le 1$? Justify your answer.

(c) Show that $Y(z) = e^z$ satisfies the differential equation and conditions in (a).

(d) Can we identify the series in (a) with e^z? Explain.

6.77. (a) Use series methods on the differential equation $Y''(z) + Y(z) = 0$, $Y(0) = 0$, $Y'(0) = 1$ to obtain the series expansion

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

(b) How could you obtain a corresponding series for $\cos z$?

Taylor's Theorem

6.78. Expand each of the following functions in a Taylor series about the indicated point and determine the region of convergence in each case.

(a) e^{-z}; $z=0$ (c) $1/(1+z)$; $z=1$ (e) ze^{2z}; $z=-1$
(b) $\cos z$; $z=\pi/2$ (d) z^3-3z^2+4z-2; $z=2$

6.79. Suppose each of the following functions were expanded into a Taylor series about the indicated points. What would be the region of convergence? Do not perform the expansion.

(a) $\sin z/(z^2+4)$; $z=0$, (c) $(z+3)/(z-1)(z-4)$; $z=2$, (e) $e^z/z(z-1)$; $z=4i$, (g) $\sec \pi z$; $z=1$
(b) $z/(e^z+1)$; $z=0$, (d) $e^{-z^2}\sinh(z+2)$; $z=0$, (f) $z\coth 2z$; $z=0$,

6.80. Verify the expansions 1, 2, 3 for e^z, $\sin z$, and $\cos z$ on page 173.

6.81. Show that $\sin z^2 = z^2 - \dfrac{z^6}{3!} + \dfrac{z^{10}}{5!} - \dfrac{z^{14}}{7!} + \cdots, \quad |z| < \infty.$

6.82. Prove that $\tan^{-1} z = z - \dfrac{z^3}{3} + \dfrac{z^5}{5} - \dfrac{z^7}{7} + \cdots, \quad |z| < 1.$

6.83. Show that: (a) $\tan z = z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \cdots, \quad |z| < \pi/2,$

(b) $\sec z = 1 + \dfrac{z^2}{2} + \dfrac{5z^4}{24} + \cdots, \quad |z| < \pi/2,$ (c) $\csc z = \dfrac{1}{z} + \dfrac{z}{6} + \dfrac{7z^3}{360} + \cdots, \quad 0 < |z| < \pi$

6.84. By replacing z by iz in the expansion of Problem 6.82, obtain the result in Problem 6.23(c) on page 185.

6.85. How would you obtain series for (a) $\tanh z$, (b) $\operatorname{sech} z$, (c) $\operatorname{csch} z$ from the series in Problem 6.83?

6.86. Prove the uniqueness of the Taylor series expansion of $f(z)$ about $z=a$.

[*Hint.* Assume $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} d_n(z-a)^n$ and show that $c_n = d_n$, $n = 0, 1, 2, 3, \ldots$.]

6.87. Prove the binomial Theorem 6.6 on page 174.

6.88. Suppose we choose that branch of $\sqrt{1+z^3}$ having the value 1 for $z=0$. Show that

$$\frac{1}{\sqrt{1+z^3}} = 1 - \frac{1}{2}z^3 + \frac{1\cdot 3}{2\cdot 4}z^6 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^9 + \cdots \quad |z| < 1$$

6.89. (a) Choosing that branch of $\sin^{-1} z$ having the value zero for $z=0$, show that

$$\sin^{-1} z = z + \frac{1}{2}\frac{z^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{z^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{z^7}{7} + \cdots \quad |z| < 1$$

(b) Prove that the result in (a) is valid for $z=i$.

6.90. (a) Expand $f(z) = \ln(3-iz)$ in powers of $z-2i$, choosing that branch of the logarithm for which $f(0) = \ln 3$, and (b) determine the region of convergence.

Laurent's Theorem

6.91. Expand $f(z) = 1/(z-3)$ in a Laurent series valid for (a) $|z| < 3$, (b) $|z| > 3$.

6.92. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for:

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| > 1$, (e) $0 < |z-2| < 1$.

6.93. Expand $f(z) = 1/z(z-2)$ in a Laurent series valid for (a) $0 < |z| < 2$, (b) $|z| > 2$.

6.94. Find an expansion of $f(z) = z/(z^2+1)$ valid for $|z-3| > 2$.

6.95. Expand $f(z) = 1/(z-2)^2$ in a Laurent series valid for (a) $|z| < 2$, (b) $|z| > 2$.

6.96. Expand each of the following functions in a Laurent series about $z = 0$, naming the type of singularity in each case.

(a) $(1-\cos z)/z$, (b) e^z/z^3, (c) $z^{-1}\cosh z^{-1}$, (d) $z^2 e^{-z^4}$

6.97. Suppose $\tan z$ is expanded into a Laurent series about $z = \pi/2$. Show that: (a) the principal part is $-1/(z-\pi/2)$, (b) the series converges for $0 < |z-\pi/2| < \pi/2$, (c) $z = \pi/2$ is a simple pole.

6.98. Determine and classify all the singularities of the functions:

(a) $1/(2\sin z - 1)^2$, (b) $z/(e^{1/z}-1)$, (c) $\cos(z^2+z^{-2})$, (d) $\tan^{-1}(z^2+2z+2)$, (e) $z/(e^z-1)$.

6.99. (a) Expand $f(z) = e^{z/(z-2)}$ in a Laurent series about $z = 2$ and (b) determine the region of convergence of this series. (c) Classify the singularities of $f(z)$.

6.100. Establish the result (6.7), page 174, for the coefficients in a Laurent series.

6.101. Prove that the only singularities of a rational function are poles.

6.102. Prove the converse of Problem 6.101, i.e., if the only singularities of a function are poles, the function must be rational.

Lagrange's Expansion

6.103. Show that the root of the equation $z = 1 + \zeta z^p$, which is equal to 1 when $\zeta = 0$, is given by

$$z = 1 + \zeta + \frac{2p}{2!}\zeta^2 + \frac{(3p)(3p-1)}{3!}\zeta^3 + \frac{(4p)(4p-1)(4p-2)}{4!}\zeta^4 + \cdots$$

6.104. Calculate the root in Problem 6.103 if $p = 1/2$ and $\zeta = 1$, (a) by series and (b) exactly. Compare the two answers.

6.105. By considering the equation $z = \alpha + \frac{1}{2}\zeta(z^2-1)$, show that

$$\frac{1}{\sqrt{1-2a\zeta+\zeta^2}} = 1 + \sum_{n=1}^{\infty}\frac{\zeta^n}{2^n n!}\frac{d^n}{da^n}(a^2-1)^n$$

6.106. Show how Lagrange's expansion can be used to solve Kepler's problem of determining the root of $z = a + \zeta \sin z$ for which $z = a$ when $\zeta = 0$.

6.107. Prove the Lagrange expansion (6.11) on page 176.

Analytic Continuation

6.108. (a) Prove that

$$F_2(z) = \frac{1}{1+i}\sum_{n=0}^{\infty}\left(\frac{z+i}{1+i}\right)^n$$

is an analytic continuation of $F_1(z) = \sum_{n=0}^{\infty} z^n$, showing graphically the regions of convergence of the series.

(b) Determine the function represented by all analytic continuations of $F_1(z)$.

6.109. Let $F_1(z) = \sum_{n=0}^{\infty}\frac{z^{n+1}}{3^n}$.

(a) Find an analytic continuation of $F_1(z)$, which converges for $z = 3-4i$.

(b) Determine the value of the analytic continuation in (a) for $z = 3-4i$.

6.110. Prove that the series $z^{1!} + z^{2!} + z^{3!} + \cdots$ has the natural boundary $|z| = 1$.

Miscellaneous Problems

6.111. (a) Prove that $\sum_{n=1}^{\infty} 1/n^p$ diverges if the constant $p \le 1$.

(b) Prove that if p is complex, the series in (a) converges if $\text{Re}\{p\} > 1$.

(c) Investigate the convergence or divergence of the series in (a) if $\text{Re}\{p\} \le 1$.

6.112. Test for convergence or divergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+i}$ (c) $\displaystyle\sum_{n=1}^{\infty} n \sin^{-1}(1/n^3)$ (e) $\displaystyle\sum_{n=1}^{\infty} \coth^{-1} n$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n + \sin^2 n}{ie^n + (2-i)n}$ (d) $\displaystyle\sum_{n=2}^{\infty} \frac{(i)^n}{n \ln n}$ (f) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

6.113. Euler presented the following argument to show that $\sum_{-\infty}^{\infty} z^n = 0$:

$$\frac{z}{1-z} = z + z^2 + z^3 + \cdots = \sum_{1}^{\infty} z^n, \qquad \frac{z}{z-1} = \frac{1}{1-1/z} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \sum_{0}^{-\infty} z^n$$

Then adding, $\sum_{-\infty}^{\infty} z^n = 0$. Explain the fallacy.

6.114. Show that for $|z-1| < 1$, $\quad z \ln z = (z-1) + \dfrac{(z-1)^2}{1 \cdot 2} - \dfrac{(z-1)^3}{2 \cdot 3} + \dfrac{(z-1)^4}{3 \cdot 4} - \cdots$.

6.115. Expand $\sin^3 z$ in a Maclaurin series.

6.116. Given the series $z^2 + \dfrac{z^2}{1+z^2} + \dfrac{z^2}{(1+z^2)^2} + \dfrac{z^2}{(1+z^2)^3} + \cdots$.

(a) Show that the sum of the first n terms is $S_n(z) = 1 + z^2 - 1/(1+z^2)^{n-1}$.

(b) Show that the sum of the series is $1 + z^2$ for $z \ne 0$, and 0 for $z = 0$; and hence that $z = 0$ is a point of discontinuity.

(c) Show that the series is not uniformly convergent in the region $|z| \le \delta$ where $\delta > 0$.

6.117. If $F(z) = \dfrac{3z-3}{(2z-1)(z-2)}$, find a Laurent series of $F(z)$ about $z = 1$ convergent for $\frac{1}{2} < |z-1| < 1$.

6.118. Let $G(z) = (\tan^{-1} z)/z^4$. (a) Expand $G(z)$ in a Laurent series. (b) Determine the region of convergence of the series in (a). (c) Evaluate $\oint_C G(z)\, dz$ where C is a square with vertices at $2 \pm 2i$, $-2 \pm 2i$.

6.119. Consider each of the functions ze^{1/z^2}, $(\sin^2 z)/z$, $1/z(4-z)$ which have singularities at $z = 0$:
(a) give a Laurent expansion about $z = 0$ and determine the region of convergence;
(b) state in each case whether $z = 0$ is a removable singularity, essential singularity or a pole;
(c) evaluate the integral of the function about the circle $|z| = 2$.

6.120. (a) Investigate the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$. (b) Does your answer to (a) contradict Problem 6.8.

6.121. (a) Show that the following series, where $z = x + iy$, converges absolutely in the region bounded by $\sin^2 x + \sinh^2 y = 1$:

$$\frac{\sin z}{1^2+1} + \frac{\sin^2 z}{2^2+1} + \frac{\sin^3 z}{3^2+1} + \cdots$$

(b) Graph the region of (a).

6.122. If $|z| > 0$, prove that $\cosh(z + 1/z) = c_0 + c_1(z + 1/z) + c_2(z^2 + 1/z^2) + \cdots$ where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \cos n\phi \cosh(2\cos\phi)\, d\phi$$

6.123. If $f(z)$ has simple zeros at $1-i$ and $1+i$, double poles at $-1+i$ and $-1-i$, but no other finite singularities, prove that the function must be given by

$$f(z)=\kappa\frac{z^2-2z+2}{(z^2+2z+2)^2}$$

where κ is an arbitrary constant.

6.124. Prove that for all z, $e^z\sin z=\sum_{n=1}^{\infty}\frac{2^{n/2}\sin(n\pi/4)}{n!}z^n$.

6.125. Show that $\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$, justifying all steps. [*Hint.* Use Problem 6.23.]

6.126. Investigate the uniform convergence of the series $\sum_{n=1}^{\infty}\frac{z}{[1+(n-1)z][1+nz]}$.

$\Big[$*Hint.* Resolve the nth term into partial fractions and show that the nth partial sum is $S_n(z)=1-(1/1+nz)$.$\Big]$

6.127. Given $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges to S. Prove that the rearranged series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots=\frac{3}{2}S.$$

Explain.

[*Hint.* Take $\frac{1}{2}$ of the first series and write it as $0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+\cdots$; then add term by term to the first series. Note that $S=\ln 2$, as shown in Problem 6.125.]

6.128. Prove that the *hypergeometric series*

$$1+\frac{a\cdot b}{1\cdot c}z+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2+\frac{a(a+1)(a+2)b(b+1)(b+2)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}z^3+\cdots$$

(a) converges absolutely if $|z|<1$,
(b) diverges for $|z|>1$,
(c) converges absolutely for $z=1$ if $\text{Re}\{a+b-c\}<0$,
(d) satisfies the differential equation $z(1-z)Y''+\{c-(a+b+1)z\}Y'-abY=0$.

6.129. Prove that for $|z|<1$,

$$(\sin^{-1}z)^2=z^2+\frac{2}{3}\cdot\frac{z^4}{2}+\frac{2\cdot 4}{3\cdot 5}\cdot\frac{z^6}{3}+\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{z^8}{4}+\cdots$$

6.130. Prove that $\sum_{n=1}^{\infty}1/n^{1+i}$ diverges.

6.131. Show that $\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}-\frac{1}{4\cdot 5}+\cdots=2\ln 2-1$.

6.132. Locate and name all the singularities of $\frac{z^6+1}{(z-1)^3(3z+2)^2}\sin\left(\frac{z^2}{z-3}\right)$.

6.133. By using only properties of infinite series, prove that

(a) $\left\{1+a+\frac{a^2}{2!}+\frac{a^3}{3!}+\cdots\right\}\left\{1+b+\frac{b^2}{2!}+\frac{b^3}{3!}+\cdots\right\}=\left\{1+(a+b)+\frac{(a+b)^2}{2!}+\cdots\right\}$

(b) $\left\{1-\frac{a^2}{2!}+\frac{a^4}{4!}-\frac{a^6}{6!}+\cdots\right\}^2+\left\{a-\frac{a^3}{3!}+\frac{a^5}{5!}-\frac{a^7}{7!}+\cdots\right\}^2=1$

6.134. Suppose $f(z)=\sum_{n=0}^{\infty}a_nz^n$ converges for $|z|<R$ and $0\le r<R$. Prove that

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2\,d\theta=\sum_{n=0}^{\infty}|a_n|^2r^{2n}$$

6.135. Use Problem 6.134 to prove Cauchy's inequality (page 145), namely

$$|f^{(n)}(0)| \le \frac{M \cdot n!}{r^n} \qquad n = 0, 1, 2, \ldots$$

6.136. Suppose a function has six zeros of order 4, and four poles of orders 3, 4, 7, and 8, but no other singularities in the finite plane. Prove that it has a pole of order 2 at $z = \infty$.

6.137. State whether each of the following functions are entire, meromorphic or neither:

(a) z^2e^{-z}, (c) $(1 - \cos z)/z$, (e) $z\sin(1/z)$, (g) $\sin\sqrt{z}/\sqrt{z}$

(b) $\cot 2z$, (d) $\cosh z^2$, (f) $z + 1/z$, (h) $\sqrt{\sin z}$

6.138. Let $-\pi < \theta < \pi$. Prove that $\ln(2\cos\theta/2) = \cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \frac{1}{4}\cos 4\theta + \cdots$

6.139. (a) Expand $1/\ln(1+z)$ in a Laurent series about $z = 0$ and (b) determine the region of convergence.

6.140. Let $S(z) = a_0 + a_1z + a_2z^2 + \cdots$. Giving restrictions if any, prove that

$$\frac{S(z)}{1-z} = a_0 + (a_0 + a_1)z + (a_0 + a_1 + a_2)z^2 + \cdots$$

6.141. Show that the following series (a) is not absolutely convergent but (b) is uniformly convergent for all values of z.

$$\frac{1}{1+|z|} - \frac{1}{2+|z|} + \frac{1}{3+|z|} - \frac{1}{4+|z|} + \cdots$$

6.142. Prove that $\sum_{n=1}^{\infty} z^n/n$ converges at all points of $|z| \le 1$ except $z = 1$.

6.143. Prove that the solution of $z = a + \zeta e^z$, which has the value a when $\zeta = 0$, is given by

$$z = a + \sum_{n=1}^{\infty} \frac{n^{n-1}e^{na}\zeta^n}{n!}$$

if $|\zeta| < |e^{-(a+1)}|$.

6.144. Find the sum of the series $1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \cdots$.

6.145. Let $F(z)$ be analytic in the finite plane and suppose that $F(z)$ has period 2π, i.e., $F(z + 2\pi) = F(z)$. Prove that

$$F(z) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inz} \qquad \text{where } \alpha_n = \frac{1}{2\pi}\int_0^{2\pi} F(z)e^{-inz}\,dz$$

The series is called the *Fourier series* for $F(z)$.

6.146. Prove that the following series is equal to $\pi/4$ if $0 < \theta < \pi$, and to $-\pi/4$ if $-\pi < \theta < 0$:

$$\sin\theta + \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta + \cdots$$

6.147. Prove that $|z| = 1$ is a natural boundary for the series $\sum_{n=0}^{\infty} 2^{-n}z^{3^n}$.

6.148. Suppose $f(z)$ is analytic and not identically zero in the region $0 < |z - z_0| < R$, and suppose $\lim_{z\to z_0} f(z) = 0$. Prove that there exists a positive integer n such that $f(z) = (z - z_0)^n g(z)$ where $g(z)$ is analytic at z_0 and different from zero.

6.149. Suppose $f(z)$ is analytic in a deleted neighborhood of z_0 and $\lim_{z\to z_0} |f(z)| = \infty$. Prove that $z = z_0$ is a pole of $f(z)$.

6.150. Explain why Problem 6.149 does not hold for $f(x) = e^{1/x^2}$ where x is real.

6.151. (a) Show that the function $f(z) = e^{1/z}$ can assume any value except zero.
(b) Discuss the relationship of the result of (a) to the Casorati–Weierstrass theorem and Picard's theorem.

6.152. (a) Determine whether the function $g(z) = z^2 - 3z + 2$ can assume any complex value.
(b) Is there any relationship of the result in (a) to the theorems of Casorati–Weierstrass and Picard? Explain.

6.153. Prove the Casorati–Weierstrass theorem stated on page 175. [*Hint.* Use the fact that if $z = a$ is an essential singularity of $f(z)$, then it is also an essential singularity of $1/\{f(z) - A\}$.]

6.154. (a) Prove that along any ray through $z = 0$, $|z + e^z| \to \infty$.

(b) Does the result in (a) contradict the Casorati–Weierstrass theorem?

6.155. (a) Prove that an entire function $f(z)$ can assume any value whatsoever, with perhaps one exception.

(b) Illustrate the result of (a) by considering $f(z) = e^z$ and stating the exception in this case.

(c) What is the relationship of the result to the Casorati–Weierstrass and Picard theorems?

6.156. Prove that every entire function has a singularity at infinity. What type of singularity must this be? Justify your answer.

6.157. Prove that: (a) $\dfrac{\ln(1+z)}{1+z} = z - \left(1 + \dfrac{1}{2}\right)z^2 + \left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right)z^3 - \cdots, \quad |z| < 1$

(b) $\{\ln(1+z)\}^2 = z^2 - \left(1 + \dfrac{1}{2}\right)\dfrac{2z^3}{3} + \left(1 + \dfrac{1}{2} + \dfrac{1}{3}\right)\dfrac{2z^4}{4} - \cdots, \quad |z| < 1$

6.158. Find the sum of the following series if $|a| < 1$:

(a) $\sum_{n=1}^{\infty} na^n \sin n\theta$, (b) $\sum_{n=1}^{\infty} n^2 a^n \sin n\theta$

6.159. Show that $e^{\sin z} = 1 + z + \dfrac{z^2}{2} - \dfrac{z^4}{8} - \dfrac{z^5}{15} + \cdots, \quad |z| < \infty.$

6.160. (a) Show that $\sum_{n=1}^{\infty} z^n/n^2$ converges for $|z| \le 1$.

(b) Show that the function $F(z)$, defined as the collection of all possible analytic continuations of the series in (a), has a singular point at $z = 1$.

(c) Reconcile the results of (a) and (b).

6.161. Let $\sum_{n=1}^{\infty} a_n z^n$ converge inside a circle of convergence of radius R. There is a theorem which states that the function $F(z)$ defined by the collection of all possible continuations of this series, has at least one singular point on the circle of convergence. (a) Illustrate the theorem by several examples. (b) Can you prove the theorem?

6.162. Show that

$$u(r,\,\theta) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{U(\phi)\,d\phi}{R^2 - 2rR\cos(\theta - \phi) + r^2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \{a_n \cos n\theta + b_n \sin n\theta\}$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} U(\phi)\cos n\phi\,d\phi, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} U(\phi)\sin n\phi\,d\phi$$

6.163. Let

$$\frac{z}{e^z - 1} = 1 + B_1 z + \frac{B_2 z^2}{2!} + \frac{B_3 z^3}{3!} + \cdots$$

(a) Show that the numbers B_n, called the *Bernoulli numbers*, satisfy the recursion formula $(B+1)^n = B^n$ where B^k is formally replaced by B_k after expanding. (b) Using (a) or otherwise, determine $B_1, \ldots, B_6$.

6.164. (a) Prove that $\dfrac{z}{e^z - 1} = \dfrac{z}{2}\left(\coth\dfrac{z}{2} - 1\right)$.

(b) Use Problem 6.163 and part (a) to show that $B_{2k+1} = 0$ if $k = 1, 2, 3, \ldots$.

6.165. Derive the series expansions:

(a) $\coth z = \dfrac{1}{z} + \dfrac{z}{3} - \dfrac{z^3}{45} + \cdots + \dfrac{B_{2n}(2z)^{2n}}{(2n)!z} + \cdots, \qquad |z| < \pi$

(b) $\cot z = \dfrac{1}{z} - \dfrac{z}{3} - \dfrac{z^3}{45} + \cdots (-1)^n \dfrac{B_{2n}(2z)^{2n}}{(2n)!z} + \cdots, \qquad |z| < \pi$

(c) $\tan z = z + \dfrac{z^3}{3} + \dfrac{2z^5}{15} + \cdots (-1)^{n-1} \dfrac{2(2^{2n}-1)B_{2n}(2z)^{2n-1}}{(2n)!}, \qquad |z| < \pi/2$

(d) $\csc z = \dfrac{1}{z} + \dfrac{z}{6} + \dfrac{7z^3}{360} + \cdots (-1)^{n-1} \dfrac{2(2^{2n-1}-1)B_{2n}z^{2n-1}}{(2n)!} + \cdots, \qquad |z| < \pi$

[*Hint.* For (a), use Problem 6.164; for (b) replace z by iz in (a); for (c) use $\tan z = \cot z - 2\cot 2z$; for (d) use $\csc z = \cot z + \tan z/2$.]

ANSWERS TO SUPPLEMENTARY PROBLEMS

6.37. (a) $S_n(z) = \{1 - (z/2)^n\}/(2 - z)$ and $\lim_{n\to\infty} S_n(z)$ exists if $|z| < 2$, (b) $S(z) = 1/(2 - z)$

6.38. (a) $|z| < 1$, (b) 1

6.39. (a) All z such that $|z^2 + 1| > 1$, (b) $1/z^2$

6.44. (b) $z/(1 - z)^2$, $|z| < 1$

6.45. 4

6.49. (a) Converges absolutely if $|z - 3| < 3$ or $z = 0$. (b) Converges uniformly for $|z - 3| \le R$ where $0 < R < 3$; does not converge uniformly in any neighborhood that includes $z = 0$.

6.50. (a) Converges absolutely if $|z| < 1$. (b) Converges uniformly if $|z| \le R$ where $R < 1$.

6.51. (a) Converges absolutely if $|z^2 + 1| > 1$. (b) Converges uniformly if $|z^2 + 1| \ge R$ where $R > 1$.

6.53. (b) Not uniformly convergent in any region that includes $z = 0$.

6.56. (a) conv., (b) conv., (c) div., (d) conv., (e) div.

6.57. (a) Diverges for all finite z. (b) Converges for all z. (c) Converges for all z.

(d) Converges for all z except $z = -n^2$, $n = 1, 2, 3, \ldots$.

6.58. Conv.

6.59. (a) $|z + i| \le 1$, (b) $|(z + 1)/(z - 1)| \le 3$, (c) $|z| < \infty$

6.60. Conv. Abs. for $|z - i| \le 4$.

6.61. Converges if $\operatorname{Im} z \ge 0$.

6.65. $|z| < 1$.

6.67. (a) conv., (b) conv., (c) div., (d) div.

6.68. (a) $|z| \le R$ where $R < 3$, (b) $|z - i| \le 1$, (c) $|z| \ge R$ where $R > 1$, (d) all z.

6.71. (a) $z/(1 - z)^2$ [compare Problem 6.44], (b) $z(1 + z)/(1 - z)^3$

6.72. (a) 1/2, (b) 0

6.76. (a) $Y(z) = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \ldots$

6.79. (a) $|z| < 2$, (b) $|z| < \pi$, (c) $|z - 2| < 1$, (d) $|z| < \infty$, (e) $|z - 4i| < 4$, (f) $|z| < \pi/2$, (g) $|z - 1| < 1/2$

6.90. (a) $\ln 5 - \dfrac{i(z - 2i)}{5} + \dfrac{(z - 2i)^2}{2 \cdot 5^2} + \dfrac{i(z - 2i)^3}{3 \cdot 5^3} - \dfrac{(z - 2i)^4}{4 \cdot 5^4} - \cdots$ (b) $|z - 2i| < 5$

6.91. (a) $-\dfrac{1}{3} - \dfrac{1}{9}z - \dfrac{1}{27}z^2 - \dfrac{1}{81}z^3 - \cdots$ (b) $z^{-1} + 3z^{-2} + 9z^{-3} + 27z^{-4} + \cdots$