4.101. Evaluate $\oint_C \frac{dz}{\sqrt{z^2+2z+2}}$ around the unit circle $|z|=1$ starting with $z=1$, assuming the integrand positive for this value.

4.102. Let n be a positive integer. Show that

$$\int_0^{2\pi} e^{\sin n\theta}\cos(\theta-\cos n\theta)\,d\theta = \int_0^{2\pi} e^{\sin n\theta}\sin(\theta-\cos n\theta)\,d\theta = 0$$

ANSWERS TO SUPPLEMENTARY PROBLEMS

4.32. (a) 88/3, (b) 32, (c) 40, (d) 24

4.33. (a) -48π, (b) 48π

4.34. (a) $\frac{511}{3}-\frac{49}{5}i$, (b) $\frac{518}{3}-57i$, (c) $\frac{518}{3}-8i$

4.35. $-1+i$

4.36. $-\frac{44}{3}-\frac{8}{3}i$ in all cases

4.38. (a) $-\frac{4}{3}+\frac{8}{3}i$, (b) $-\frac{1}{3}+\frac{79}{30}i$

4.39. (a) 0, (b) $4\pi i$

4.40. 0 in all cases

4.41. $(96\pi^5a^5+80\pi^3a^3+30\pi a)/15$

4.42. $\frac{248}{15}$

4.43. $2\pi i$ in all cases

4.44. $8\pi(1+i)$

4.45. Common value $=-8$

4.46. -18

4.48. πab

4.49. $\frac{3\pi a^2}{8}$

4.50. Common value $=120\pi$

4.51. (b) $-2\pi e^{\pi^2}$

4.52. (b) 24

4.54. (a) $18\pi i$, (b) $8i$, (c) $40\pi i$

4.55. $6\pi i a^2$

4.59. $\hat{z}=\frac{2ai}{\pi}$, $\hat{\bar{z}}=\frac{-2ai}{\pi}$

4.70. One possibility is $p=x^2-y^2+2y-x$,
$q=2x+y-2xy, f(z)=iz^2+(2-i)z$

4.72. $338-266i$

4.73. $\frac{1}{2}e^{-2}(1-e^{-2})$

4.74. (b) 0

4.79. (a) $-\frac{1}{2}e^{-2z}+c$, (b) $-\frac{1}{2}\cos z^2+c$,
(c) $\frac{1}{3}\ln(z^3+3z+2)+c$, (d) $\frac{1}{10}\sin^5 2z+c$,
(e) $\frac{1}{12}\ln\cosh(4z^3)+c$

4.80. (a) $\frac{1}{2}z\sin 2z+\frac{1}{4}\cos 2z+c$, (b) $-e^{-z}(z^2+2z+2)+c$,
(c) $\frac{1}{2}z^2\ln z-\frac{1}{4}+c$,
(d) $(z^3+6z)\cosh z-3(z^2+2)\sinh z+c$

4.81. (a) $\frac{2}{3}$, (b) $-\frac{2}{5}$, (c) $\frac{1}{4}\cosh 2-\frac{1}{2}\sinh 2+\frac{1}{2}\pi i\sinh 2$

4.85. $\frac{4}{5}\left(1+\sqrt{z+1}\right)^{5/2}-\frac{4}{3}\left(1+\sqrt{z+1}\right)^{3/2}+c$

4.92. $\frac{\pi}{2}$

4.94. $\frac{32}{3}$

CHAPTER 5

Cauchy's Integral Formulas and Related Theorems

5.1 Cauchy's Integral Formulas

Let $f(z)$ be analytic inside and on a simple closed curve C and let a be any point inside C [Fig. 5-1]. Then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \tag{5.1}$$

where C is traversed in the positive (counterclockwise) sense.

Also, the nth derivative of $f(z)$ at $z = a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \quad n = 1, 2, 3, \ldots \tag{5.2}$$

The result (5.1) can be considered a special case of (5.2) with $n = 0$ if we define $0! = 1$.

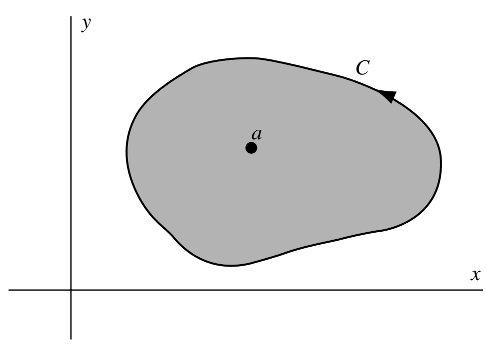

Fig. 5-1

The results (5.1) and (5.2) are called *Cauchy's integral formulas* and are quite remarkable because they show that if a function $f(z)$ is known *on* the simple closed curve C, then the values of the function and all its derivatives can be found at all points *inside* C. Thus, if a function of a complex variable has a first derivative, i.e., is analytic, in a simply-connected region $\mathcal{R}$, all its higher derivatives exist in $\mathcal{R}$. This is not necessarily true for functions of real variables.

5.2 Some Important Theorems

The following is a list of some important theorems that are consequences of Cauchy's integral formulas.

1. **Morera's theorem** (converse of Cauchy's theorem)
If $f(z)$ is continuous in a simply-connected region $\mathcal{R}$ and if $\oint_C f(z)\,dz = 0$ around every simple closed curve C in $\mathcal{R}$, then $f(z)$ is analytic in $\mathcal{R}$.

2. **Cauchy's inequality**
Suppose $f(z)$ is analytic inside and on a circle C of radius r and center at $z = a$. Then

$$|f^{(n)}(a)| \leq \frac{M \cdot n!}{r^n} \quad n = 0, 1, 2, \ldots \tag{5.3}$$

where M is a constant such that $|f(z)| < M$ on C, i.e., M is an upper bound of $|f(z)|$ on C.

3. **Liouville's theorem**
Suppose that for all z in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded, i.e., $|f(z)| < M$ for some constant M. Then $f(z)$ must be a constant.

4. **Fundamental theorem of algebra**
Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$ with degree $n \geq 1$ and $a_n \neq 0$ has at least one root.
From this it follows that $P(z) = 0$ has exactly n roots, due attention being paid to multiplicities of roots.

5. **Gauss' mean value theorem**
Suppose $f(z)$ is analytic inside and on a circle C with center at a and radius r. Then $f(a)$ is the mean of the values of $f(z)$ on C, i.e.,

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + re^{i\theta}\right) d\theta \tag{5.4}$$

6. **Maximum modulus theorem**
Suppose $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant. Then the maximum value of $|f(z)|$ occurs on C.

7. **Minimum modulus theorem**
Suppose $f(z)$ is analytic inside and on a simple closed curve C and $f(z) \neq 0$ inside C. Then $|f(z)|$ assumes its minimum value on C.

8. **The argument theorem**
Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C. Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N - P \tag{5.5}$$

where N and P are, respectively, the number of zeros and poles of $f(z)$ inside C.
For a generalization of this theorem, see Problem 5.90.

9. **Rouché's theorem**
Suppose $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and suppose $|g(z)| < |f(z)|$ on C. Then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

10. **Poisson's integral formulas for a circle**
Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$. Then, if $z = re^{i\theta}$ is any point inside C, we have

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi \tag{5.6}$$

If $u(r, \theta)$ and $v(r, \theta)$ are the real and imaginary parts of $f(re^{i\theta})$ while $u(R, \phi)$ and $v(R, \phi)$ are the real and imaginary parts of $f(Re^{i\phi})$, then

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)u(R, \phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi \tag{5.7}$$

$$v(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)v(R, \phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi \tag{5.8}$$

These results are called *Poisson's integral formulas for a circle.* They express the values of a harmonic function inside a circle in terms of its values on the boundary.

11. Poisson's integral formulas for a half plane

Let $f(z)$ be analytic in the upper half $y \geq 0$ of the z plane and let $\zeta = \xi + i\eta$ be any point in this upper half plane. Then

$$f(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta f(x)}{(x - \xi)^2 + \eta^2}\, dx \tag{5.9}$$

In terms of the real and imaginary parts of $f(\zeta)$, this can be written

$$u(\xi, \eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta u(x, 0)}{(x - \xi)^2 + \eta^2}\, dx \tag{5.10}$$

$$v(\xi, \eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta v(x, 0)}{(x - \xi)^2 + \eta^2}\, dx \tag{5.11}$$

These are called *Poisson's integral formulas for a half plane.* They express the values of a harmonic function in the upper half plane in terms of the values on the x axis [the boundary] of the half plane.

SOLVED PROBLEMS

Cauchy's Integral Formulas

5.1. Let $f(z)$ be analytic inside and on the boundary C of a simply-connected region $\mathcal{R}$. Prove *Cauchy's integral formula*

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - a}\, dz$$

Solution

Method 1. The function $f(z)/(z - a)$ is analytic inside and on C except at the point $z = a$ (see Fig. 5-2). By Theorem 4.4, page 117, we have

$$\oint_C \frac{f(z)}{z - a}\, dz = \oint_\Gamma \frac{f(z)}{z - a}\, dz \tag{1}$$

where we can choose Γ as a circle of radius ϵ with center at a. Then an equation for Γ is $|z - a| = \epsilon$ or $z - a = \epsilon e^{i\theta}$ where $0 \leq \theta < 2\pi$. Substituting $z = a + \epsilon e^{i\theta}$, $dz = i\epsilon e^{i\theta}$, the integral on the right of (1) becomes

$$\oint_\Gamma \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\epsilon e^{i\theta})i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\,d\theta \;= i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta$$

Thus we have from (1),

$$\oint_C \frac{f(z)}{z-a}\,dz = i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta \tag{2}$$

Taking the limit of both sides of (2) and making use of the continuity of $f(z)$, we have

$$\begin{aligned}\oint_C \frac{f(z)}{z-a}\,dz &= \lim_{\epsilon\to 0} i\int_0^{2\pi} f(a+\epsilon e^{i\theta})\,d\theta \\ &= i\int_0^{2\pi} \lim_{\epsilon\to 0} f(a+\epsilon e^{i\theta})\,d\theta = i\int_0^{2\pi} f(a)\,d\theta = 2\pi i\,f(a)\end{aligned} \tag{3}$$

so that we have, as required,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

Method 2. The right side of equation (1) of Method 1 can be written as

$$\begin{aligned}\oint_\Gamma \frac{f(z)}{z-a}\,dz &= \oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + \oint_\Gamma \frac{f(a)}{z-a}\,dz \\ &= \oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz + 2\pi i\,f(a)\end{aligned}$$

using Problem 4.21. The required result will follow if we can show that

$$\oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz = 0$$

But by Problem 3.21,

$$\oint_\Gamma \frac{f(z)-f(a)}{z-a}\,dz = \oint_\Gamma f'(a)\,dz + \oint_\Gamma \eta\,dz = \oint_\Gamma \eta\,dz$$

Then choosing Γ so small that for all points on Γ we have $|\eta| < \delta/2\pi$, we find

$$\left|\oint_\Gamma \eta\,dz\right| < \left(\frac{\delta}{2\pi}\right)(2\pi\epsilon) = \epsilon$$

Thus $\oint_\Gamma \eta\,dz = 0$ and the proof is complete.

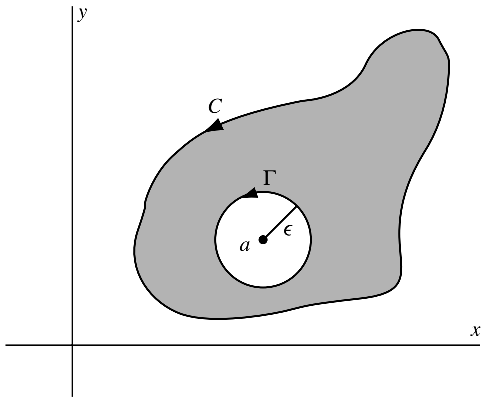

Fig. 5-2

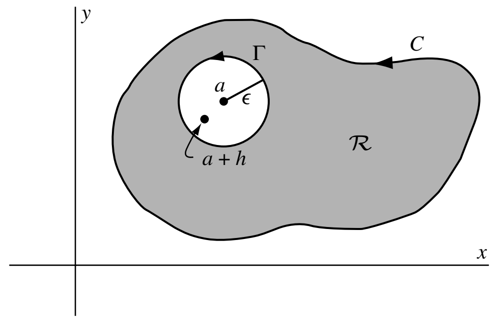

Fig. 5-3

5.2. Let $f(z)$ be analytic inside and on the boundary C of a simply-connected region $\mathcal{R}$. Prove that

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz$$

Solution

From Problem 5.1, if a and $a+h$ lie in $\mathcal{R}$, we have

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{1}{2\pi i} \oint_C \frac{1}{h}\left\{\frac{1}{z-(a+h)} - \frac{1}{z-a}\right\} f(z)\,dz = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)} \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-a)^2} + \frac{h}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2} \end{aligned}$$

The result follows on taking the limit as $h \to 0$ if we can show that the last term approaches zero.

To show this we use the fact that if Γ is a circle of radius ϵ and center a which lies entirely in $\mathcal{R}$ (see Fig. 5-3), then

$$\frac{h}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2} = \frac{h}{2\pi i} \oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

Choosing h so small in absolute value that $a+h$ lies in Γ and $|h| < \epsilon/2$, we have by Problem 1.7(c), and the fact that Γ has equation $|z-a| = \epsilon$,

$$|z-a-h| \geq |z-a| - |h| > \epsilon - \epsilon/2 = \epsilon/2$$

Also since $f(z)$ is analytic in $\mathcal{R}$, we can find a positive number M such that $|f(z)| < M$.

Then, since the length of Γ is $2\pi\epsilon$, we have

$$\left| \frac{h}{2\pi i} \oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2} \right| \leq \frac{|h|}{2\pi} \frac{M(2\pi\epsilon)}{(\epsilon/2)(\epsilon^2)} = \frac{2|h|M}{\epsilon^2}$$

and it follows that the left side approaches zero as $h \to 0$, thus completing the proof.

It is of interest to observe that the result is equivalent to

$$\frac{d}{da} f(a) = \frac{d}{da}\left\{ \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \right\} = \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial a}\left\{\frac{f(z)}{z-a}\right\} dz$$

which is an extension to contour integrals of *Leibnitz's rule* for differentiating under the integral sign.

5.3. Prove that under the conditions of Problem 5.2,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \quad n = 0, 1, 2, 3, \ldots$$

Solution

The cases where $n = 0$ and 1 follow from Problems 5.1 and 5.2, respectively, provided we define $f^{(0)}(a) = f(a)$ and $0! = 1$.

To establish the case where $n = 2$, we use Problem 5.2 where a and $a + h$ lie in $\mathcal{R}$ to obtain

$$\begin{aligned}\frac{f'(a+h) - f'(a)}{h} &= \frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2}\right\} f(z)\,dz \\ &= \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz + \frac{h}{2\pi i}\oint_C \frac{3(z-a)-2h}{(z-a-h)^2(z-a)^3}\,f(z)\,dz\end{aligned}$$

The result follows on taking the limit as $h \to 0$ if we can show that the last term approaches zero. The proof is similar to that of Problem 5.2, for using the fact that the integral around C equals the integral around Γ, we have

$$\left|\frac{h}{2\pi i}\oint_\Gamma \frac{3(z-a)-2h}{(z-a-h)^2(z-a)^3}\,f(z)\,dz\right| \le \frac{|h|}{2\pi}\frac{M(2\pi\epsilon)}{(\epsilon/2)^2(\epsilon^3)} = \frac{4|h|M}{\epsilon^4}$$

Since M exists such that $|\{3(z-a) - 2h\}\, f(z)| < M$.

In a similar manner, we can establish the result for $n = 3, 4, \ldots$ (see Problems 5.36 and 5.37).

The result is equivalent to (see last paragraph of Problem 5.2)

$$\frac{d^n}{da^n} f(a) = \frac{d^n}{da^n}\left\{\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)}\,dz\right\} = \frac{1}{2\pi i}\oint_C \frac{\partial^n}{\partial a^n}\left\{\frac{f(z)}{z-a}\right\}dz$$

5.4. Suppose $f(z)$ is analytic in a region $\mathcal{R}$. Prove that $f'(z), f''(z), \ldots$ are analytic in $\mathcal{R}$.

Solution

This follows from Problems 5.2 and 5.3.

5.5. Evaluate:

(a) $\displaystyle\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$, (b) $\displaystyle\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$ where C is the circle $|z| = 3$.

Solution

(a) Since $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz$$

By Cauchy's integral formula with $a = 2$ and $a = 1$, respectively, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz = 2\pi i\{\sin \pi(2)^2 + \cos \pi(2)^2\} = 2\pi i$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz = 2\pi i\{\sin \pi(1)^2 + \cos \pi(1)^2\} = -2\pi i$$

since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C. Then, the required integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

(b) Let $f(z) = e^{2z}$ and $a = -1$ in the Cauchy integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \tag{1}$$

If $n = 3$, then $f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2}$. Hence (1) becomes

$$8e^{-2} = \frac{3!}{2\pi i} \oint_C \frac{e^{2z}}{(z+1)^4} dz$$

from which we see that the required integral has the value $8\pi i e^{-2}/3$.

5.6. Prove Cauchy's integral formula for multiply-connected regions.

Solution

We present a proof for the multiply-connected region $\mathcal{R}$ bounded by the simple closed curves C_1 and C_2 as indicated in Fig. 5-4. Extensions to other multiply-connected regions are easily made (see Problem 5.40).

Construct a circle Γ having center at any point a in $\mathcal{R}$ so that Γ lies entirely in $\mathcal{R}$. Let $\mathcal{R}'$ consist of the set of points in $\mathcal{R}$ that are exterior to Γ. Then, the function $f(z)/(z-a)$ is analytic inside and on the boundary of $\mathcal{R}'$. Hence, by Cauchy's theorem for multiply-connected regions (Problem 4.16),

$$\frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-a} dz - \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-a} dz - \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z-a} dz = 0 \tag{1}$$

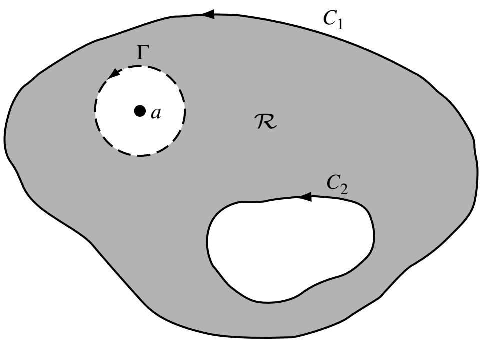

Fig. 5-4

But, by Cauchy's integral formula for simply-connected regions, we have

$$f(a) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z-a} dz \tag{2}$$

so that from (1),

$$f(a) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-a} dz - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z-a} dz \tag{3}$$

Then, if C represents the entire boundary of $\mathcal{R}$ (suitably traversed so that an observer moving around C always has $\mathcal{R}$ lying to his left), we can write (3) as

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz$$

In a similar manner, we can show that the other Cauchy integral formulas

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad n = 1, 2, 3, \ldots$$

hold for multiply-connected regions (see Problem 5.40).

Morera's Theorem

5.7. Prove *Morera's theorem* (the converse of Cauchy's theorem): Suppose $f(z)$ is continuous in a simply-connected region $\mathcal{R}$ and suppose

$$\oint_C f(z)\,dz = 0$$

around every simple closed curve C in $\mathcal{R}$. Then $f(z)$ is analytic in $\mathcal{R}$.

Solution

If $\oint_C f(z)\,dz = 0$ independent of C, it follows by Problem 4.17, that $F(z) = \int_a^z f(z)\,dz$ is independent of the path joining a and z, so long as this path is in $\mathcal{R}$.

Then, by reasoning identical with that used in Problem 4.18, it follows that $F(z)$ is analytic in $\mathcal{R}$ and $F'(z) = f(z)$. However, by Problem 5.2, it follows that $F'(z)$ is also analytic if $F(z)$ is. Hence, $f(z)$ is analytic in $\mathcal{R}$.

Cauchy's Inequality

5.8. Let $f(z)$ be analytic inside and on a circle C of radius r and center at $z = a$. Prove *Cauchy's inequality*

$$|f^{(n)}(a)| \le \frac{M \cdot n!}{r^n} \qquad n = 0, 1, 2, 3, \ldots$$

where M is a constant such that $|f(z)| < M$.

Solution

We have by Cauchy's integral formulas,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, 1, 2, 3, \ldots$$

Then, by Problem 4.3, since $|z - a| = r$ on C and the length of C is $2\pi r$,

$$|f^{(n)}(a)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \le \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\cdot 2\pi r = \frac{M \cdot n!}{r^n}$$

Liouville's Theorem

5.9. Prove *Liouville's theorem*: Suppose for all z in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded [i.e., we can find a constant M such that $|f(z)| < M$]. Then $f(z)$ must be a constant.

Solution

Let a and b be any two points in the z plane. Suppose that C is a circle of radius r having center at a and enclosing point b (see Fig. 5-5).

From Cauchy's integral formula, we have

$$f(b) - f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-b}\,dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$
$$= \frac{b-a}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-b)(z-a)}$$

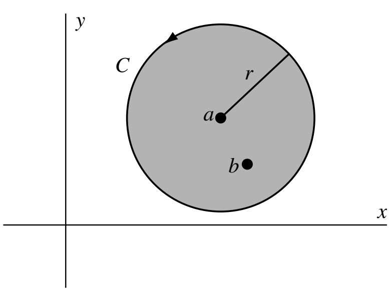

Fig. 5-5

Now we have

$$|z-a| = r, \qquad |z-b| = |z-a+a-b| \ge |z-a| - |a-b| = r - |a-b| \ge r/2$$

if we choose r so large that $|a-b| < r/2$. Then, since $|f(z)| < M$ and the length of C is $2\pi r$, we have by Problem 4.3,

$$|f(b)-f(a)| = \frac{|b-a|}{2\pi}\left|\oint_C \frac{f(z)\,dz}{(z-b)(z-a)}\right| \leq \frac{|b-a|M(2\pi r)}{2\pi(r/2)r} = \frac{2|b-a|M}{r}$$

Letting $r \to \infty$, we see that $|f(b)-f(a)| = 0$ or $f(b) = f(a)$, which shows that $f(z)$ must be a constant.

Another Method. Letting $n = 1$ in Problem 5.8 and replacing a by z we have,

$$|f'(z)| \leq M/r$$

Letting $r \to \infty$, we deduce that $|f'(z)| = 0$ and so $f'(z) = 0$. Hence, $f(z) =$ constant, as required.

Fundamental Theorem of Algebra

5.10. Prove the *fundamental theorem of algebra*: Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$, where the degree $n \geq 1$ and $a_n \neq 0$, has at least one root.

Solution

If $P(z) = 0$ has no root, then $f(z) = 1/P(z)$ is analytic for all z. Also, $|f(z)| = 1/|P(z)|$ is bounded (and in fact approaches zero) as $|z| \to \infty$.

Then by Liouville's theorem (Problem 5.9), it follows that $f(z)$ and thus $P(z)$ must be a constant. Thus, we are led to a contradiction and conclude that $P(z) = 0$ must have at least one root or, as is sometimes said, $P(z)$ has at least one *zero*.

5.11. Prove that every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$, where the degree $n \geq 1$ and $a_n \neq 0$, has exactly n roots.

Solution

By the fundamental theorem of algebra (Problem 5.10), $P(z)$ has at least one root. Denote this root by α. Then $P(\alpha) = 0$. Hence

$$\begin{aligned} P(z) - P(\alpha) &= a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n - (a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n) \\ &= a_1(z-\alpha) + a_2(z^2-\alpha^2) + \cdots + a_n(z^n - \alpha^n) \\ &= (z-\alpha)Q(z) \end{aligned}$$

where $Q(z)$ is a polynomial of degree $(n-1)$.

Applying the fundamental theorem of algebra again, we see that $Q(z)$ has at least one zero, which we can denote by β [which may equal α], and so $P(z) = (z-\alpha)(z-\beta)R(z)$. Continuing in this manner, we see that $P(z)$ has exactly n zeros.

Gauss' Mean Value Theorem

5.12. Let $f(z)$ be analytic inside and on a circle C with center at a. Prove *Gauss' mean value theorem* that the mean of the values of $f(z)$ on C is $f(a)$.

Solution

By Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \tag{1}$$

If C has radius r, the equation of C is $|z-a|=r$ or $z=a+re^{i\theta}$. Thus, (1) becomes

$$f(a)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(a+re^{i\theta})ire^{i\theta}}{re^{i\theta}}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}f(a+re^{i\theta})\,d\theta$$

which is the required result.

Maximum Modulus Theorem

5.13. Prove the *maximum modulus theorem*: Suppose $f(z)$ is analytic inside and on a simple closed curve C. Then the maximum value of $|f(z)|$ occurs on C, unless $f(z)$ is a constant.

Solution

Method 1

Since $f(z)$ is analytic and hence continuous inside and on C, it follows that $|f(z)|$ does have a maximum value M for at least one value of z inside or on C. Suppose this maximum value is not attained on the boundary of C but is attained at an interior point a, i.e., $|f(a)|=M$. Let C_1 be a circle inside C with center at a (see Fig. 5-6). If we exclude $f(z)$ from being a constant inside C_1, then there must be a point inside C_1, say b, such that $|f(b)|<M$ or, what is the same thing, $|f(b)|=M-\epsilon$ where $\epsilon>0$.

Now, by the continuity of $|f(z)|$ at b, we see that for any $\epsilon>0$ we can find $\delta>0$ such that

$$||f(z)|-|f(b)||<\frac{1}{2}\epsilon \quad \text{whenever } |z-b|<\delta \qquad (1)$$

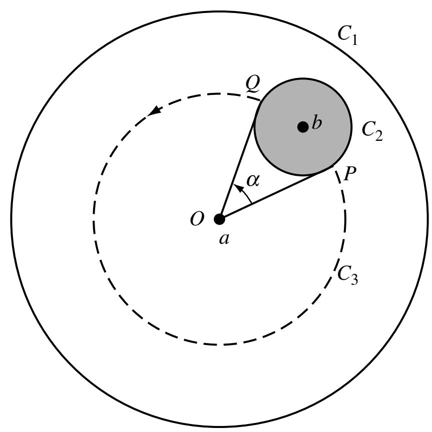

Fig. 5-6

i.e.,

$$|f(z)|<|f(b)|+\frac{1}{2}\epsilon=M-\epsilon+\frac{1}{2}\epsilon=M-\frac{1}{2}\epsilon \qquad (2)$$

for all points interior to a circle C_2 with center at b and radius δ, as shown shaded in the figure.

Construct a circle C_3 with a center at a that passes through b (dashed in Fig. 5-6). On part of this circle [namely that part PQ included in C_2], we have from (2), $|f(z)|<M-\frac{1}{2}\epsilon$. On the remaining part of the circle, we have $|f(z)|\le M$.

If we measure θ counterclockwise from OP and let $\angle POQ=\alpha$, it follows from Problem 5.12 that if $r=|b-a|$,

$$f(a)=\frac{1}{2\pi}\int_0^{\alpha}f(\alpha+re^{i\theta})\,d\theta+\frac{1}{2\pi}\int_{\alpha}^{2\pi}f(a+re^{i\theta})\,d\theta$$

Then

$$\begin{aligned}|f(a)|&\le\frac{1}{2\pi}\int_0^{\alpha}|f(a+re^{i\theta})|\,d\theta+\frac{1}{2\pi}\int_{\alpha}^{2\pi}|f(a+re^{i\theta})|\,d\theta\\&\le\frac{1}{2\pi}\int_0^{\alpha}\left(M-\frac{1}{2}\epsilon\right)d\theta+\frac{1}{2\pi}\int_{\alpha}^{2\pi}M\,d\theta\\&=\frac{\alpha}{2\pi}\left(M-\frac{1}{2}\epsilon\right)+\frac{M}{2\pi}(2\pi-\alpha)\\&\ne M-\frac{\alpha\epsilon}{4\pi}\end{aligned}$$

i.e., $|f(a)|=M\le M-(\alpha\epsilon/4\pi)$, an impossible situation. By virtue of this contradiction, we conclude that $|f(z)|$ cannot attain its maximum at any interior point of C and so must attain its maximum on C.

Method 2

From Problem 5.12, we have

$$|f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a+re^{i\theta})|\,d\theta \tag{3}$$

Let us suppose that $|f(a)|$ is a maximum so that $|f(a+re^{i\theta})| \leq |f(a)|$. If $|f(a+re^{i\theta})| < |f(a)|$ for one value of θ then, by continuity of f, it would hold for a finite arc, say $\theta_1 < \theta < \theta_2$. But, in such case, the mean value of $|f(a+re^{i\theta})|$ is less than $|f(a)|$, which would contradict (3). It follows, therefore, that in any δ neighborhood of a, i.e., for $|z-a| < \delta$, $f(z)$ must be a constant. If $f(z)$ is not a constant, the maximum value of $|f(z)|$ must occur on C.

For another method, see Problem 5.57.

Minimum Modulus Theorem

5.14. Prove the *minimum modulus theorem*: Let $f(z)$ be analytic inside and on a simple closed curve C. Prove that if $f(z)\neq 0$ inside C, then $|f(z)|$ must assume its minimum value on C.

Solution

Since $f(z)$ is analytic inside and on C and since $f(z)\neq 0$ inside C, it follows that $1/f(z)$ is analytic inside C. By the maximum modulus theorem, it follows that $1/|f(z)|$ cannot assume its maximum value inside C and so $|f(z)|$ cannot assume its minimum value inside C. Then, since $|f(z)|$ has a minimum, this minimum must be attained on C.

5.15. Give an example to show that if $f(z)$ is analytic inside and on a simple closed curve C and $f(z) = 0$ at some point inside C, then $|f(z)|$ need not assume its minimum value on C.

Solution

Let $f(z) = z$ for $|z| \leq 1$, so that C is a circle with center at the origin and radius 1. We have $f(z) = 0$ at $z = 0$. If $z = re^{i\theta}$, then $|f(z)| = r$ and it is clear that the minimum value of $|f(z)|$ does not occur on C but occurs inside C where $r = 0$, i.e., at $z = 0$.

The Argument Theorem

5.16. Let $f(z)$ be analytic inside and on a simple closed curve C except for a pole $z = \alpha$ of order (multiplicity) p inside C. Suppose also that inside C, $f(z)$ has only one zero $z = \beta$ of order (multiplicity) n and no zeros on C. Prove that

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = n - p$$

Solution

Let C_1 and Γ_1 be non-overlapping circles lying inside C and enclosing $z = \alpha$ and $z = \beta$, respectively. [See Fig. 5-7.] Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}\,dz \tag{1}$$

Since $f(z)$ has a pole of order p at $z = \alpha$, we have

$$f(z) = \frac{F(z)}{(z-\alpha)^p} \tag{2}$$

where $F(z)$ is analytic and different from zero inside and on C_1. Then, taking logarithms in (2) and differentiating, we find

$$\frac{f'(z)}{f(z)} = \frac{F'(z)}{F(z)} - \frac{p}{z-\alpha} \tag{3}$$

so that

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_1} \frac{F'(z)}{F(z)}dz - \frac{p}{2\pi i}\oint_{C_1} \frac{dz}{z-\alpha} = 0 - p = -p \tag{4}$$

Since $f(z)$ has a zero of order n at $z = \beta$, we have

$$f(z) = (z-\beta)^n G(z) \tag{5}$$

where $G(z)$ is analytic and different from zero inside and on Γ_1.

Then, by logarithmic differentiation, we have

$$\frac{f'(z)}{f(z)} = \frac{n}{z-\beta} + \frac{G'(z)}{G(z)} \tag{6}$$

so that

$$\frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}dz = \frac{n}{2\pi i}\oint_{\Gamma_1} \frac{dz}{z-\beta} + \frac{1}{2\pi i}\oint \frac{G'(z)}{G(z)}dz = n \tag{7}$$

Hence, from (1), (4), and (7), we have the required result

$$\frac{1}{2\pi i}\oint_{C} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}dz = n - p$$

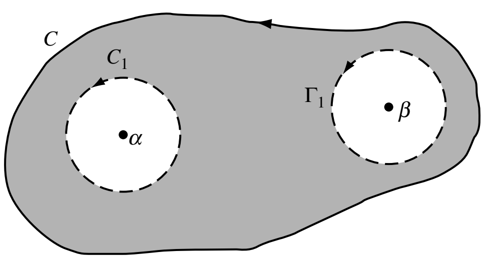

Fig. 5-7

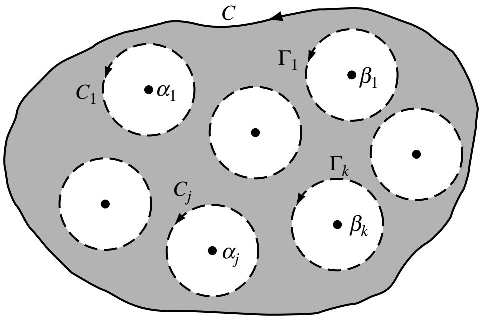

Fig. 5-8

5.17. Let $f(z)$ be analytic inside and on a simple closed curve C except for a finite number of poles inside C. Suppose that $f(z) \neq 0$ on C. If N and P are, respectively, the number of zeros and poles of $f(z)$ inside C, counting multiplicities, prove that

$$\frac{1}{2\pi i}\oint_{C} \frac{f'(z)}{f(z)}dz = N - P$$

Solution

Let $\alpha_1, \alpha_2, \ldots, \alpha_j$ and $\beta_1, \beta_2, \ldots, \beta_k$ be the respective poles and zeros of $f(z)$ lying inside C [Fig. 5-8] and suppose their multiplicities are $p_1, p_2, \ldots, p_j$ and $n_1, n_2, \ldots, n_k$.

Enclose each pole and zero by non-overlapping circles $C_1, C_2, \ldots, C_j$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$. This can always be done since the poles and zeros are isolated.

Then, we have, using the results of Problem 5.16,

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz &= \sum_{r=1}^{j}\frac{1}{2\pi i}\oint_{\Gamma_r}\frac{f'(z)}{f(z)}dz + \sum_{r=1}^{k}\frac{1}{2\pi i}\oint_{C_r}\frac{f'(z)}{f(z)}dz \\ &= \sum_{r=1}^{j} n_r - \sum_{r=1}^{k} p_r \\ &= N - P\end{aligned}$$

Rouché's Theorem

5.18. Prove *Rouché's theorem*: Suppose $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and suppose $|g(z)| < |f(z)|$ on C. Then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

Solution

Let $F(z) = g(z)/f(z)$ so that $g(z) = f(z)F(z)$ or briefly $g = fF$. Then, if N_1 and N_2 are the number of zeros inside C of $f + g$ and f, respectively, we have by Problem 5.17, using the fact that these functions have no poles inside C,

$$N_1 = \frac{1}{2\pi i}\oint_C \frac{f' + g'}{f + g}\,dz, \qquad N_2 = \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz$$

Then

$$\begin{aligned}N_1 - N_2 &= \frac{1}{2\pi i}\oint_C \frac{f' + f'F + fF'}{f + fF}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz = \frac{1}{2\pi i}\oint_C \frac{f'(1+F) + fF'}{f(1+F)}\,dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz \\ &= \frac{1}{2\pi i}\oint_C \left\{\frac{f'}{f} + \frac{F'}{1+F}\right\}dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}\,dz = \frac{1}{2\pi i}\oint_C \frac{F'}{1+F}\,dz \\ &= \frac{1}{2\pi i}\int_C F'(1 - F + F^2 - F^3 + \cdots)\,dz = 0\end{aligned}$$

using the given fact that $|F| < 1$ on C so that the series is uniformly convergent on C and term by term integration yields the value zero. Thus, $N_1 = N_2$ as required.

5.19. Use Rouché's theorem (Problem 5.18) to prove that every polynomial of degree n has exactly n zeros (fundamental theorem of algebra).

Solution

Suppose the polynomial to be $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0$. Choose $f(z) = a_n z^n$ and $g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$.

If C is a circle having center at the origin and radius $r > 1$, then on C we have

$$\begin{aligned}\left|\frac{g(z)}{f(z)}\right| &= \frac{|a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}|}{|a_n z^n|} \le \frac{|a_0| + |a_1| r + |a_2| r^2 + \cdots + |a_{n-1}| r^{n-1}}{|a_n| r^n} \\ &\le \frac{|a_0| r^{n-1} + |a_1| r^{n-1} + |a_2| r^{n-1} + \cdots + |a_{n-1}| r^{n-1}}{|a_n| r^n} = \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n| r}\end{aligned}$$

Then, by choosing r large enough, we can make $|g(z)/f(z)| < 1$, i.e., $|g(z)| < |f(z)|$. Hence, by Rouché's theorem, the given polynomial $f(z) + g(z)$ has the same number of zeros as $f(z) = a_n z^n$. But, since this last function has n zeros all located at $z = 0$, $f(z) + g(z)$ also has n zeros and the proof is complete.

5.20. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Solution

Consider the circle C_1: $|z| = 1$. Let $f(z) = 12$, $g(z) = z^7 - 5z^3$. On C_1 we have

$$|g(z)| = |z^7 - 5z^3| \le |z^7| + |5z^3| \le 6 < 12 = |f(z)|$$

Hence, by Rouché's theorem, $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 1$ as $f(z) = 12$, i.e., there are no zeros inside C_1.

Consider the circle C_2: $|z| = 2$. Let $f(z) = z^7$, $g(z) = 12 - 5z^3$. On C_2 we have

$$|g(z)| = |12 - 5z^3| \le |12| + |5z^3| \le 60 < 2^7 = |f(z)|$$

Hence, by Rouché's theorem, $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 2$ as $f(z) = z^7$, i.e., all the zeros are inside C_2.

Hence, all the roots lie inside $|z| = 2$ but outside $|z| = 1$, as required.

Poisson's Integral Formulas for a Circle

5.21. (a) Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$, and let $z = re^{i\theta}$ be any point inside C (see Fig. 5-9). Prove that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f(Re^{i\phi})\, d\phi$$

(b) Let $u(r, \theta)$ and $v(r, \theta)$ be the real and imaginary parts of $f(re^{i\theta})$. Prove that

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, u(R, \phi)\, d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

$$v(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, v(R, \phi)\, d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

The results are called *Poisson's integral formulas for the circle.*

Solution

(a) Since $z = re^{i\theta}$ is any point inside C, we have by Cauchy's integral formula

$$f(z) = f(re^{i\theta}) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - z}\, dw \qquad (1)$$

The *inverse of the point* z with respect to C lies outside C and is given by $R^2/\bar{z}$. Hence, by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w - R^2/\bar{z}}\, dw \qquad (2)$$

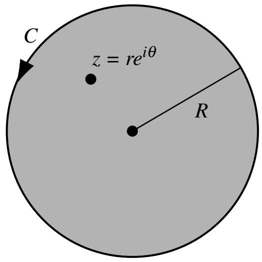

Fig. 5-9

If we subtract (2) from (1), we find

$$f(z) = \frac{1}{2\pi i} \oint_C \left\{ \frac{1}{w - z} - \frac{1}{w - R^2/\bar{z}} \right\} f(w)\, dw$$

$$= \frac{1}{2\pi i} \oint_C \frac{z - R^2/\bar{z}}{(w - z)(w - R^2/\bar{z})} f(w)\, dw \tag{3}$$

Now, let $z = re^{i\theta}$ and $w = Re^{i\phi}$. Then, since $\bar{z} = re^{-i\theta}$, (3) yields

$$f(re^{i\theta}) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\{re^{i\theta} - (R^2/r)e^{i\theta}\} f(Re^{i\phi}) iRe^{i\phi}\, d\phi}{\{Re^{i\phi} - re^{i\theta}\}\{Re^{i\phi} - (R^2/r)e^{i\theta}\}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r^2 - R^2)e^{i(\theta+\phi)} f(Re^{i\phi})\, d\phi}{(Re^{i\phi} - re^{i\theta})(re^{i\phi} - Re^{i\theta})}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})\, d\phi}{(Re^{i\phi} - re^{i\theta})(Re^{-i\phi} - re^{-i\theta})} = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})\, d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

(b) Since $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ and $f(Re^{i\phi}) = u(R, \phi) + iv(R, \phi)$, we have from part (a),

$$u(r, \theta) + iv(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2)\{u(R, \phi) + iv(R, \phi)\}\, d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) u(R, \phi)\, d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2} + \frac{i}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) v(R, \phi)\, d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

Then the required result follows on equating real and imaginary parts.

Poisson's Integral Formulas for a Half Plane

5.22. Derive Poisson's formulas for the half plane [see page 146].

Solution

Let C be the boundary of a semicircle of radius R [see Fig. 5-10] containing ζ as an interior point. Since C encloses ζ but does not enclose $\bar{\zeta}$, we have by Cauchy's integral formula,

$$f(\zeta) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \zeta} dz, \qquad 0 = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \bar{\zeta}} dz$$

Then, by subtraction,

$$f(\zeta) = \frac{1}{2\pi i} \oint_C f(z) \left\{ \frac{1}{z - \zeta} - \frac{1}{z - \bar{\zeta}} \right\} dz = \frac{1}{2\pi i} \oint_C \frac{(\zeta - \bar{\zeta}) f(z)\, dz}{(z - \zeta)(z - \bar{\zeta})}$$

Letting $\zeta = \xi + i\eta$, $\bar{\zeta} = \xi - i\eta$, this can be written

$$f(\zeta) = \frac{1}{\pi} \int_{-R}^{R} \frac{\eta f(x)\, dx}{(x - \xi)^2 + \eta^2} + \frac{1}{\pi} \int_\Gamma \frac{\eta f(z)\, dz}{(z - \zeta)(z - \bar{\zeta})}$$

where Γ is the semicircular arc of C. As $R \to \infty$, this last integral approaches zero [see Problem 5.76] and we have

$$f(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta f(x)\, dx}{(x-\xi)^2 + \eta^2}$$

Writing $f(\zeta) = f(\xi + i\eta) = u(\xi, \eta) + iv(\xi, \eta)$, $f(x) = u(x, 0) + iv(x, 0)$, we obtain as required,

$$u(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta u(x, 0)\, dx}{(x-\xi)^2 + \eta^2}, \quad v(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta v(x, 0)\, dx}{(x-\xi)^2 + \eta^2}$$

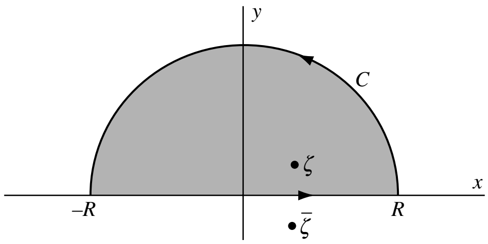

Fig. 5-10

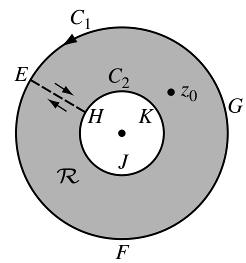

Fig. 5-11

Miscellaneous Problems

5.23. Let $f(z)$ be analytic in a region $\mathcal{R}$ bounded by two concentric circles C_1 and C_2 and on the boundary [Fig. 5-11]. Prove that, if z_0 is any point in $\mathcal{R}$, then

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z - z_0} dz$$

Solution

Method 1. Construct cross-cut EH connecting circles C_1 and C_2. Then $f(z)$ is analytic in the region bounded by $EFGEHKJHE$. Hence, by Cauchy's integral formula,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \oint_{EFGEHKJHE} \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \oint_{EFGE} \frac{f(z)}{z - z_0} dz + \frac{1}{2\pi i} \int_{EH} \frac{f(z)}{z - z_0} dz + \frac{1}{2\pi i} \oint_{HKJH} \frac{f(z)}{z - z_0} dz + \frac{1}{2\pi i} \int_{HE} \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z - z_0} dz \end{aligned}$$

since the integrals along EH and HE cancel.

Similar results can be established for the derivatives of $f(z)$.

Method 2. The result also follows from equation (3) of Problem 5.6 if we replace the simple closed curves C_1 and C_2 by the circles of Fig. 5-11.

5.24. Prove that, for $n = 1, 2, 3, \ldots,$

$$\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} 2\pi$$

Solution

Let $z = e^{i\theta}$. Then, $dz = ie^{i\theta}\, d\theta = iz\, d\theta$ or $d\theta = dz/iz$ and $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + 1/z)$. Hence, if C is the unit circle $|z| = 1$, we have

$$\begin{aligned}
\int_0^{2\pi} \cos^{2n}\theta\, d\theta &= \oint_C \left\{\frac{1}{2}\left(z + \frac{1}{z}\right)\right\}^{2n} \frac{dz}{iz} \\
&= \frac{1}{2^{2n} i} \oint_C \frac{1}{z}\left\{z^{2n} + \binom{2n}{1}(z^{2n-1})\left(\frac{1}{z}\right) + \cdots + \binom{2n}{k}(z^{2n-k})\left(\frac{1}{z}\right)^k + \cdots + \left(\frac{1}{z}\right)^{2n}\right\} dz \\
&= \frac{1}{2^{2n} i} \oint_C \left\{z^{2n-1} + \binom{2n}{1} z^{2n-3} + \cdots + \binom{2n}{k} z^{2n-2k-1} + \cdots + z^{-2n}\right\} dz \\
&= \frac{1}{2^{2n} i} \cdot 2\pi i \binom{2n}{n} = \frac{1}{2^{2n}} \binom{2n}{n} 2\pi \\
&= \frac{1}{2^{2n}} \frac{(2n)!}{n!n!} 2\pi = \frac{(2n)(2n-1)(2n-2)\cdots(n)(n-1)\cdots 1}{2^{2n} n! n!} 2\pi \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} 2\pi
\end{aligned}$$

5.25. Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a region $\mathcal{R}$. Prove that u and v are harmonic in $\mathcal{R}$.

Solution

In Problem 3.6, we proved that u and v are harmonic in $\mathcal{R}$, i.e., satisfy the equation $(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) = 0$, under the *assumption* of existence of the second partial derivatives of u and v, i.e., the existence of $f''(z)$.

This assumption is no longer necessary since we have in fact proved in Problem 5.4 that, if $f(z)$ is analytic in $\mathcal{R}$, then *all* the derivatives of $f(z)$ exist.

5.26. Prove *Schwarz's theorem*: Let $f(z)$ be analytic for $|z| \le R$, $f(0) = 0$, and $|f(z)| \le M$. Then

$$|f(z)| \le \frac{M|z|}{R}$$

Solution

The function $f(z)/z$ is analytic in $|z| \le R$. Hence, on $|z| = R$, we have by the maximum modulus theorem,

$$\left|\frac{f(z)}{z}\right| \le \frac{M}{R}$$

However, since this inequality must also hold for points inside $|z| = R$, we have for $|z| \le R$, $|f(z)| \le M|z|/R$ as required.

5.27. Let

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \ne 0 \\ 0 & x = 0 \end{cases}$$

where x is real. Show that the function $f(x)$ (a) has a first derivative at all values of x for which $0 \le x \le 1$ but (b) does not have a second derivative in $0 \le x \le 1$. (c) Reconcile these conclusions with the result of Problem 5.4.

Solution

(a) The only place where there is any question as to existence of the first derivative is at $x = 0$. But, at $x = 0$, the derivative is

$$\lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\Delta x)^2 \sin(1/\Delta x) - 0}{\Delta x} = \lim_{\Delta x \to 0} \Delta x \sin(1/\Delta x) = 0$$

and so exists.

At all other values of x in $0 \le x \le 1$, the derivative is given (using elementary differentiation rules) by

$$x^2 \cos(1/x)\{-1/x^2\} + (2x)\sin(1/x) = 2x \sin(1/x) - \cos(1/x)$$

(b) From part (a), we have

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x \ne 0 \\ 0 & x = 0 \end{cases}$$

The second derivative exists for all x such that $0 < x \le 1$. At $x = 0$, the second derivative is given by

$$\lim_{\Delta x \to 0} \frac{f'(0 + \Delta x) - f'(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x \sin(1/\Delta x) - \cos(1/\Delta x) - 0}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \{2 \sin(1/\Delta x) - (1/\Delta x)\cos(1/\Delta x)\}$$

which does not exist.

It follows that the second derivative of $f(x)$ does not exist in $0 \le x \le 1$.

(c) According to Problem 5.4, if $f(z)$ is analytic in a region $\mathcal{R}$, then all higher derivatives exist and are analytic in $\mathcal{R}$. The above results do not conflict with this, since the function $f(z) = z^2 \sin(1/z)$ is not analytic in any region which includes $z = 0$.

5.28. (a) Let $F(z)$ be analytic inside and on a simple closed curve C except for a pole of order m at $z = a$ inside C. Prove that

$$\frac{1}{2\pi i} \oint_C F(z)\, dz = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m F(z)\}$$

(b) How would you modify the result in (a) if more than one pole were inside C?

Solution

(a) If $F(z)$ has a pole of order m at $z = a$, then $F(z) = f(z)/(z-a)^m$ where $f(z)$ is analytic inside and on C, and $f(a) \ne 0$. Then, by Cauchy's integral formula,

$$\frac{1}{2\pi i} \oint_C F(z)\, dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^m}\, dz = \frac{f^{(m-1)}(a)}{(m-1)!} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m F(z)\}$$

(b) Suppose there are two poles at $z = a_1$ and $z = a_2$ inside C, of orders m_1 and m_2, respectively. Let Γ_1 and Γ_2 be circles inside C having radii ϵ_1 and ϵ_2 and centers at a_1 and a_2, respectively (see Fig. 5-12). Then

$$\frac{1}{2\pi i} \oint_C F(z)\, dz = \frac{1}{2\pi i} \oint_{\Gamma_1} F(z)\, dz + \frac{1}{2\pi i} \oint_{\Gamma_2} F(z)\, dz \qquad (1)$$

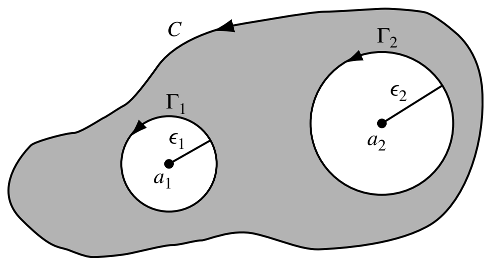

Fig. 5-12

If $F(z)$ has a pole of order m_1 at $z = a_1$, then

$$F(z) = \frac{f_1(z)}{(z-a_1)^{m_1}} \quad \text{where } f_1(z) \text{ is analytic and } f_1(a_1) \neq 0$$

If $F(z)$ has a pole of order m_2 at $z = a_2$, then

$$F(z) = \frac{f_2(z)}{(z-a_2)^{m_2}} \quad \text{where } f_2(z) \text{ is analytic and } f_2(a_2) \neq 0$$

Then, by (1) and part (a),

$$\begin{aligned}\frac{1}{2\pi i}\oint_C F(z)\,dz &= \frac{1}{2\pi i}\oint_{\Gamma_1}\frac{f_1(z)}{(z-a_1)^{m_1}}dz + \frac{1}{2\pi i}\oint_{\Gamma_2}\frac{f_2(z)}{(z-a_2)^{m_2}}dz \\ &= \lim_{z\to a_1}\frac{1}{(m_1-1)!}\frac{d^{m_1-1}}{dz^{m_1-1}}\{(z-a_1)^{m_1}F(z)\} \\ &\quad + \lim_{z\to a_2}\frac{1}{(m_2-1)!}\frac{d^{m_2-1}}{dz^{m_2-1}}\{(z-a_2)^{m_2}F(z)\}\end{aligned}$$

If the limits on the right are denoted by R_1 and R_2, we can write

$$\oint_C F(z)\,dz = 2\pi i(R_1 + R_2)$$

where R_1 and R_2 are called the *residues* of $F(z)$ at the poles $z = a_1$ and $z = a_2$.

In general, if $F(z)$ has a number of poles inside C with residues $R_1, R_2, \ldots$, then $\oint_C F(z)\,dz = 2\pi i$ times the sum of the residues. This result is called the *residue theorem*. Applications of this theorem, together with generalization to singularities other than poles, are treated in Chapter 7.

5.29. Evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz$ where C is the circle $|z| = 4$.

Solution

The poles of $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}$ are at $z = \pm\pi i$ inside C and are both of order two.

Residue at $z = \pi i$ is $\lim_{z\to\pi i}\frac{1}{1!}\frac{d}{dz}\left\{(z-\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi+i}{4\pi^3}$.

Residue at $z = -\pi i$ is $\lim_{z\to-\pi i}\frac{1}{1!}\frac{d}{dz}\left\{(z+\pi i)^2\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right\} = \frac{\pi-i}{4\pi^3}$.

Then $\oint_C \frac{e^z}{(z^2+\pi^2)^2}dz = 2\pi i$ (sum of residues) $= 2\pi i\left(\frac{\pi+i}{4\pi^3}+\frac{\pi-i}{4\pi^3}\right) = \frac{i}{\pi}$.

SUPPLEMENTARY PROBLEMS

Cauchy's Integral Formulas

5.30. Evaluate $\dfrac{1}{2\pi i}\oint_C \dfrac{e^z}{z-2}\,dz$ if C is: (a) the circle $|z| = 3$, (b) the circle $|z| = 1$.

5.31. Evaluate $\oint_C \dfrac{\sin\ 3z}{z+\pi/2}\,dz$ if C is the circle $|z| = 5$.

5.32. Evaluate $\oint_C \dfrac{e^{3z}}{z-\pi i}\,dz$ if C is: (a) the circle $|z-1| = 4$, (b) the ellipse $|z-2| + |z+2| = 6$.

5.33. Evaluate $\dfrac{1}{2\pi i}\oint_C \dfrac{\cos \pi 2}{z^2-1}\,dz$ around a rectangle with vertices at: (a) $2 \pm i,\ -2 \pm i$; (b) $-i,\ 2-i,\ 2+i,\ i$.

5.34. Show that $\dfrac{1}{2\pi i}\oint_C \dfrac{e^{zt}}{z^2+1}\,dz = \sin t$ if $t > 0$ and C is the circle $|z| = 3$.

5.35. Evaluate $\oint_C \dfrac{e^{iz}}{z^3}\,dz$ where C is the circle $|z| = 2$.

5.36. Suppose C is a simple closed curve enclosing $z = a$ and $f(z)$ is analytic inside and on C. Prove that $f'''(a) = \dfrac{3!}{2\pi i}\oint_C \dfrac{f(z)\,dz}{(z-a)^4}$.

5.37. Prove Cauchy's integral formulas for all positive integral values of n. [*Hint*: Use mathematical induction.]

5.38. Given C is the circle $|z| = 1$. Find the value of (a) $\oint_C \dfrac{\sin^6 z}{z-\pi/6}\,dz$, (b) $\oint_C \dfrac{\sin^6 z}{(z-\pi/6)^3}\,dz$.

5.39. Evaluate $\dfrac{1}{2\pi i}\oint_C \dfrac{e^{zt}}{(z^2+1)^2}\,dz$ when $t > 0$ and C is the circle $|z| = 3$.

5.40. Prove Cauchy's integral formulas for the multiply-connected region of Fig. 4-26, page 140.

Morera's Theorem

5.41. (a) Determine whether $G(z) = \int_1^z d\zeta/\zeta$ is independent of the path joining 1 and z.

(b) Discuss the relationship of your answer to part (a) with Morera's theorem.

5.42. Does Morera's theorem apply in a multiply-connected region? Justify your answer.

5.43. (a) Suppose $P(x, y)$ and $Q(x, y)$ are conjugate harmonic functions and C is any simple closed curve. Prove that $\oint_C P\,dx + Q\,dy = 0$.

(b) Suppose for all simple closed curves C in a region $\mathcal{R}$, $\oint_C P\,dx + Q\,dy = 0$. Is it true that P and Q are conjugate harmonic functions, i.e., is the converse of (a) true? Justify your conclusion.

Cauchy's Inequality

5.44. (a) Use Cauchy's inequality to obtain estimates for the derivatives of $\sin z$ at $z = 0$ and (b) determine how good these estimates are.

5.45. (a) Show that if $f(z) = 1/(1-z)$, then $f^{(n)}(z) = n!/(1-z)^{n+1}$.

(b) Use (a) to show that the Cauchy inequality is "best possible", i.e., the estimate of growth of the nth derivative cannot be improved for *all* functions.

5.46. Prove that the equality in Cauchy's inequality (5.3), page 145, holds in the case $n = m$ if and only if $f(z) = kM(z - a)^m / r^m$, where $|k| = 1$.

5.47. Discuss Cauchy's inequality for the function $f(z) = e^{-1/z^2}$ in the neighborhood of $z = 0$.

Liouville's Theorem

5.48. The function of a real variable defined by $f(x) = \sin x$ is (a) analytic everywhere and (b) bounded, i.e., $|\sin x| \leq 1$ for all x but it is certainly not a constant. Does this contradict Liouville's theorem? Explain.

5.49. Suppose $a > 0$ and $b > 0$ are constants and a non-constant function $F(z)$ is such that $F(z + a) = F(z)$, and $F(z + bi) = F(z)$. Prove that $F(z)$ cannot be analytic in the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$.

Fundamental Theorem of Algebra

5.50. (a) Carry out the details of proof of the fundamental theorem of algebra to show that the particular function $f(z) = z^4 - z^2 - 2z + 2$ has exactly four zeros. (b) Determine the zeros of $f(z)$.

5.51. Determine all the roots of the equations: (a) $z^3 - 3z + 4i = 0$, (b) $z^4 + z^2 + 1 = 0$.

Gauss' Mean Value Theorem

5.52. Evaluate $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sin^2(\pi/6 + 2e^{i\theta})\, d\theta$.

5.53. Show that the mean value of any harmonic function over a circle is equal to the value of the function at the center.

5.54. Find the mean value of $x^2 - y^2 + 2y$ over the circle $|z - 5 + 2i| = 3$.

5.55. Prove that $\int_0^{\pi} \ln \sin \theta \, d\theta = -\pi \ln 2$. [*Hint.* Consider $f(z) = \ln(1 + z)$.]

Maximum Modulus Theorem

5.56. Find the maximum of $|f(z)|$ in $|z| \leq 1$ for the functions $f(z)$ given by: (a) $z^2 - 3z + 2$, (b) $z^4 + z^2 + 1$, (c) $\cos 3z$, (d) $(2z + 1)/(2z - 1)$.

5.57. (a) Let $f(z)$ be analytic inside and on the simple closed curve C enclosing $z = a$, prove that

$$\{f(a)\}^n = \frac{1}{2\pi i}\oint_C \frac{\{f(z)\}^n}{z - a}\,dz \quad n = 0, 1, 2, \ldots$$

(b) Use (a) to prove that $|f(a)|^n \leq M^n / 2\pi D$ where D is the minimum distance from a to the curve C and M is the maximum value of $|f(z)|$ on C.

(c) By taking the nth root of both sides of the inequality in (b) and letting $n \to \infty$, prove the maximum modulus theorem.

5.58. Let $U(x, y)$ be harmonic inside and on a simple closed curve C. Prove that the (a) maximum and (b) minimum values of $U(x, y)$ are attained on C. Are there other restrictions on $U(x, y)$?

5.59. Given C is the circle $|z| = 1$. Verify Problem 5.58 for the functions (a) $x^2 - y^2$ and (b) $x^3 - 3xy^2$.

5.60. Is the maximum modulus theorem valid for multiply-connected regions? Justify your answer.

The Argument Theorem

5.61. Let $f(z) = z^5 - 3iz^2 + 2z - 1 + i$. Evaluate $\oint_C \frac{f'(z)}{f(z)} dz$ where C encloses all the zeros of $f(z)$.

5.62. Let $f(z) = \dfrac{(z^2+1)^2}{(z^2+2z+2)^3}$. Evaluate $\dfrac{1}{2\pi i} \oint_C \dfrac{f'(z)}{f(z)} dz$ where C is the circle $|z| = 4$.

5.63. Evaluate $\oint_C \frac{f'(z)}{f(z)} dz$ if C is the circle $|z| = \pi$ and (a) $f(z) = \sin \pi z$, (b) $f(z) = \cos \pi z$, (c) $f(z) = \tan \pi z$.

5.64. Let $f(z) = z^4 - 2z^3 + z^2 - 12z + 20$ and C is the circle $|z| = 5$. Evaluate $\oint_C \frac{zf'(z)}{f(z)} dz$.

Rouché's Theorem

5.65. If $a > e$, prove that the equation $az^n = e^z$ has n roots inside $|z| = 1$.

5.66. Prove that $ze^z = a$ where $a \neq 0$ is real has infinitely many roots.

5.67. Prove that $\tan z = az$, $a > 0$ has (a) infinitely many real roots, (b) only two pure imaginary roots if $0 < a < 1$, (c) all real roots if $a \geq 1$.

5.68. Prove that $z \tan z = a$, $a > 0$ has infinitely many real roots but no imaginary roots.

Poisson's Integral Formulas for a Circle

5.69. Show that $$\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi = 2\pi$$

(a) with, (b) without Poisson's integral formula for a circle.

5.70. Show that:

(a) $\displaystyle\int_0^{2\pi} \frac{e^{\cos\phi}\cos(\sin\phi)}{5 - 4\cos(\theta - \phi)} d\phi = \frac{2\pi}{3} e^{\cos\theta} \cos(\sin\theta)$, (b) $\displaystyle\int_0^{2\pi} \frac{e^{\cos\phi}\sin(\sin\phi)}{5 - 4\cos(\theta - \phi)} d\phi = \frac{2\pi}{3} e^{\cos\theta} \sin(\sin\theta)$.

5.71. (a) Prove that the function

$$U(r, \theta) = \frac{2}{\pi} \tan^{-1}\left(\frac{2r\sin\theta}{1 - r^2}\right), \qquad 0 < r < 1,\ 0 \leq \theta < 2\pi$$

is harmonic inside the circle $|z| = 1$.

(b) Show that $\lim_{r \to 1-} U(r, \theta) = \begin{cases} 1 & 0 < \theta < \pi \\ -1 & \pi < \theta < 2\pi. \end{cases}$

(c) Can you derive the expression for $U(r, \theta)$ from Poisson's integral formula for a circle?

5.72. Suppose $f(z)$ is analytic inside and on the circle C defined by $|z| = R$ and suppose $z = re^{i\theta}$ is any point inside C. Show that

$$f'(re^{i\theta}) = \frac{i}{2\pi} \int_0^{2\pi} \frac{R(R^2 - r^2) f(Re^{i\phi}) \sin(\theta - \phi)}{[R^2 - 2Rr\cos(\theta - \phi) + r^2]^2} d\phi$$

5.73. Verify that the functions u and v of equations (5.7) and (5.8), page 146, satisfy Laplace's equation.

Poisson's Integral Formulas for a Half Plane

5.74. Find a function that is harmonic in the upper half plane $y > 0$ and that on the x axis takes the values -1 if $x < 0$ and 1 if $x > 0$.

5.75. Work Problem 5.74 if the function takes the values -1 if $x < -1$, 0 if $-1 < x < 1$, and 1 if $x > 1$.

5.76. Prove the statement made in Problem 5.22 that the integral over Γ approaches zero as $R \to \infty$.

5.77. Prove that under suitable restrictions on $f(x)$,

$$\lim_{\eta \to 0+} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta f(x)}{(x-\xi)^2 + \eta^2} dx = f(\xi)$$

and state these restrictions.

5.78. Verify that the functions u and v of equations (5.10) and (5.11), page 146, satisfy Laplace's equation.

Miscellaneous Problems

5.79. Evaluate $\dfrac{1}{2\pi i} \oint_C \dfrac{z^2\,dz}{z^2+4}$ where C is the square with vertices at ± 2, $\pm 2 + 4i$.

5.80. Evaluate $\oint_C \dfrac{\cos^2 tz}{z^3} dz$ where C is the circle $|z| = 1$ and $t > 0$.

5.81. (a) Show that $\oint_C \dfrac{dz}{z+1} = 2\pi i$ if C is the circle $|z| = 2$.

(b) Use (a) to show that

$$\oint_C \frac{(x+1)\,dx + y\,dy}{(x+1)^2 + y^2} = 0, \qquad \oint_C \frac{(x+1)\,dy - y\,dx}{(x+1)^2 + y^2} = 2\pi$$

and verify these results directly.

5.82. Find all functions $f(z)$ that are analytic everywhere in the entire complex plane and that satisfy the conditions (a) $f(2-i) = 4i$ and (b) $|f(z)| < e^2$ for all z.

5.83. Let $f(z)$ be analytic inside and on a simple closed curve C. Prove that

(a) $f'(a) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} e^{-i\theta} f(a + e^{i\theta})\,d\theta$ (b) $\dfrac{f^{(n)}(a)}{n!} = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} e^{-ni\theta} f(a + e^{i\theta})\,d\theta$

5.84. Prove that $8z^4 - 6z + 5 = 0$ has one root in each quadrant.

5.85. Show that (a) $\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta)\,d\theta = 0$, (b) $\int_0^{2\pi} e^{\cos\theta} \sin(\sin\theta)\,d\theta = 2\pi$.

5.86. Extend the result of Problem 5.23 so as to obtain formulas for the derivatives of $f(z)$ at any point in $\mathcal{R}$.

5.87. Prove that $z^3 e^{1-z} = 1$ has exactly two roots inside the circle $|z| = 1$.

5.88. Suppose $t > 0$ and C is any simple closed curve enclosing $z = -1$. Prove that

$$\frac{1}{2\pi i} \oint_C \frac{ze^{zt}}{(z+1)^3} dz = \left(t - \frac{t^2}{2}\right) e^{-t}$$

5.89. Find all functions $f(z)$ that are analytic in $|z| < 1$ and that satisfy the conditions (a) $f(0) = 1$ and (b) $|f(z)| \geq 1$ for $|z| < 1$.

5.90. Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C except that $f(z)$ has zeros at $a_1, a_2, \ldots, a_m$ and poles at $b_1, b_2, \ldots, b_n$ of orders (multiplicities) $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_n$, respectively. Prove that

$$\frac{1}{2\pi i}\oint_C g(z)\frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} p_k g(a_k) - \sum_{k=1}^{n} q_k g(b_k)$$

5.91. Suppose $f(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n$ where $a_0 \neq 0, a_1, \ldots, a_n$ are complex constants and C encloses all the zeros of $f(z)$. Evaluate

(a) $\displaystyle\frac{1}{2\pi i}\oint_C \frac{z\,f'(z)}{f(z)}\,dz$ (b) $\displaystyle\frac{1}{2\pi i}\oint_C \frac{z^2 f'(z)}{f(z)}\,dz$

and interpret the results.

5.92. Find all functions $f(z)$ that are analytic in the region $|z| \leq 1$ and are such that (a) $f(0) = 3$ and (b) $|f(z)| \leq 3$ for all z such that $|z| < 1$.

5.93. Prove that $z^6 + 192z + 640 = 0$ has one root in the first and fourth quadrants and two roots in the second and third quadrants.

5.94. Prove that the function $xy(x^2 - y^2)$ cannot have an absolute maximum or minimum inside the circle $|z| = 1$.

5.95. (a) If a function is analytic in a region $\mathcal{R}$, is it bounded in $\mathcal{R}$? (b) In view of your answer to (a), is it necessary to state that $f(z)$ is bounded in Liouville's theorem?

5.96. Find all functions $f(z)$ that are analytic everywhere, have a zero of order two at $z = 0$, satisfy the condition $|f'(z)| \leq 6|z|$ for all z, and are such that $f(i) = -2$.

5.97. Prove that all the roots of $z^5 + z - 16i = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

5.98. Let U be harmonic inside and on a simple closed curve C. Prove that

$$\oint_C \frac{\partial U}{\partial n}\,ds = 0$$

where n is a unit normal to C in the z plane and s is the arc length parameter.

5.99. A theorem of Cauchy states that all the roots of the equation $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = 0$, where $a_1, a_2, \ldots, a_n$ are real, lie inside the circle $|z| = 1 + \max\{a_1, a_2, \ldots, a_n\}$, i.e., $|z| = 1$ plus the maximum of the values $a_1, a_2, \ldots, a_n$. Verify this theorem for the special cases:
(a) $z^3 - z^2 + z - 1 = 0$, (b) $z^4 + z^2 + 1 = 0$, (c) $z^4 - z^2 - 2z + 2 = 0$, (d) $z^4 + 3z^2 - 6z + 10 = 0$.

5.100. Prove the theorem of Cauchy stated in Problem 5.99.

5.101. Let $P(z)$ be any polynomial. If m is any positive integer and $\omega = e^{2\pi i/m}$, prove that

$$\frac{P(1) + P(\omega) + P(\omega^2) + \cdots + P(\omega^{m-1})}{m} = P(0)$$

and give a geometric interpretation.

5.102. Is the result of Problem 5.101 valid for any function $f(z)$? Justify your answer.

5.103. Prove *Jensen's theorem*: Suppose $f(z)$ is analytic inside and on the circle $|z| = R$ except for zeros at $a_1, a_2, \ldots, a_m$ of multiplicities $p_1, p_2, \ldots, p_m$ and poles at $b_1, b_2, \ldots, b_n$ of multiplicities $q_1, q_2, \ldots, q_n$, respectively, and suppose $f(0)$ is finite and different from zero. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(Re^{i\theta})|\,d\theta = \ln|f(0)| + \sum_{k=1}^{m} p_k \ln\left(\frac{R}{|a_k|}\right) - \sum_{k=1}^{n} q_k \ln\left(\frac{R}{|b_k|}\right)$$

[*Hint.* Consider $\oint_C \ln z\{f'(z)/f(z)\}\,dz$ where C is the circle $|z| = R$.]