3.118. Prove that $\nabla^4 U = \nabla^2(\nabla^2 U) = \dfrac{\partial^4 U}{\partial x^4} + 2\dfrac{\partial^4 U}{\partial x^2 \partial y^2} + \dfrac{\partial^4 U}{\partial y^4} = 16\dfrac{\partial^4 U}{\partial z^2 \partial \bar{z}^2}$.

3.119. Solve the partial differential equation $\dfrac{\partial^4 U}{\partial x^4} + 2\dfrac{\partial^4 U}{\partial x^2 \partial y^2} + \dfrac{\partial^4 U}{\partial y^4} = 36(x^2 + y^2)$.

ANSWERS TO SUPPLEMENTARY PROBLEMS

3.43. (a) $12 + 4i$, (b) $-5i$, (c) $3/2 + 3i/2$

3.50. (b) $2y + x^2 - y^2$, (c) $iz^2 + 2z$

3.46. (a) $-i$, $i/(z+i)^2$; (b) $-1 \pm 2i$, $(19 + 4z - 3z^2)/(z^2 + 2z + 5)^2$

3.51. (b) $x^2 - y^2 + 2xy - 3x - 2y$

3.53. (a) $v = 4xy - x^3 + 3xy^2 + c$, $f(z) = 2z^2 - iz^3 + ic$, (b) Not harmonic

(c) $ye^x \cos y + xe^x \sin y + c$, $ze^z + ic$, (d) $-e^{2xy}\cos(x^2 - y^2) + c$, $-ie^{iz^2} + ic$

3.54. (b) $-2\tan^{-1}\{(y-2)/(x-1)\}$, (c) $2i\ln(z - 1 - 2i)$

3.55. $6 + 3i$

3.56. (a) $-8\Delta z + i(\Delta z)^2 = -8\,dz = i(dz)^2$, (b) $-8\,dz$, (c) $i(dz)^2$

3.57. (a) $38 - 2i$, (b) $6 - 42i$

3.58. (a) $-4\Delta z + 3i(\Delta z)^2$, (b) $-4\,dz$, (c) $-4 + 3i\Delta z$, (d) -4

3.63. (a) $(2+8i)z - 3$, (b) $4z + i$, (c) $5i/(z+2i)^2$, (d) $4i - 8z$, (e) $-3i(iz-1)^{-4}$

3.64. (a) $-6/5 + 3i/5$, (b) $-108 - 78i$

3.67. (a) $3\sin(z/2)\cos(z/2)$, (b) $3(2z-3)\tan^2(z^2 - 3z + 4i)\sec^2(z^2 - 3z + 4i)$ (c) $\sec z$

(d) $\dfrac{-z\csc\{(z^2+1)^{1/2}\}\cot\{(z^2+1)^{1/2}\}}{(z^2+1)^{1/2}}$, (e) $(1 - z^2)\sin(z + 2i) + 2z\cos(z + 2i)$

3.71. (a) $2\sin^{-1}(2z-1)/(z-z^2)^{1/2}$, (b) $-2z/(1+z^4)\cot^{-1} z^2$, (c) $-(\sin z + \cos z)/(\sin 2z)^{1/2}$,

(d) $-1/2(z + 1 + 3i)(z + 3i)^{1/2}$, (e) $(\csc 2z)(1 - 2z\cot 2z)/(1 - z^2\csc^2 2z)$, (f) $1/\sqrt{z^2 - 3z + 2i}$

3.72. $-3[\cosh(3\zeta + 2i)]/2(2z - z^2)^{1/2}t^{1/2}$

3.73. $\sec(t - 3i)\{1 + t\tan(t - 3i)\}(t - t^2)^{1/2}$

3.74. (a) $(\cos 2z)/(1 - w)$, (b) $\{\cos^2 2z - 2(1-w)^2 \sin 2z\}/(1-w)^3$,

3.75. $-\cosh^4 \pi$

3.76. (a) $2z^{\ln z - 1}\ln z$, (b) $\{[\sin(iz-2)]^{\tan^{-1}(z+3i)}\}\{i\tan^{-1}(z+3i)\cot(iz-2) + [\ln\sin(iz-2)]/[z^2 + 6iz - 8]\}$

3.77. (a) $24\cos(4z - 2 + 2i)$, (b) $4\csc 2z^2 - 16z^2\csc 2z^2\cot 2z^2$

(c) $2\cosh(z+1)^2 + 4(z+1)^2\sinh(z+1)^2$, (d) $(1 - \ln z - \ln^2 z)/z^2(1 - \ln^2 z)^{3/2}$

(e) $-i(1+3z)/4(1+z)^2 z^{3/2}$

3.78. (a) $(16 + 12i)/25$, (b) $\left(1 - i\sqrt{3}\right)/6$, (c) $-1/4$

3.79. (a) $1/6$, (b) $e^{m\pi i}/\cosh m\pi$

3.80. 1

3.81. $e^{-1/6}$

3.82. (a) $z = -1 \pm i$; simple poles
(b) $z = -3i$; branch point, $z = 0$; pole of order 2
(c) $z = 0$; logarithmic branch point
(d) $z = 0, \pm i$; branch points
(e) $z = -i$; pole of order 3

3.85. (a) $z = \pm 1$; simple pole
(b) $z = 1/\sqrt{m\pi}$, $m = \pm 1, \pm 2, \pm 3, \ldots$; simple poles, $z = 0$; essential singularity, $z = \infty$; pole of order 2
(c) $z = 0$; branch point, $z = \infty$; branch point

3.86. (a) $x^4 - 6x^2y^2 + y^4 = \beta$, (b) $2e^{-x}\sin y + x^2 - y^2 = \beta$

3.87. $r^2 \sin 2\theta = \beta$

3.90. (a) $\pm i$, (b) Velocity: $\sqrt{5}$, $\sqrt{5}e^{-\pi/2}$. Acceleration: 4, $2e^{-\pi/2}$

3.92. (a) 3, $3\sqrt{1 + 16\pi^2}$, (b) 24, $24\sqrt{1 + 4\pi^2}$

3.93. $24\sqrt{10}$, (b) 72

3.94. (a) $(2xy - y^2) + i(x^2 - 2xy)$, (b) $2y - 2x$

3.95. (a) 8, (b) $12x$, (c) $|12y|$, (d) 0

3.96. (a) $(-4 + 5i)/\sqrt{41}$, (b) $\{2x - y + i(2y - x)\}/\sqrt{5x^2 - 8xy + 5y^2}$

3.97. $x = 8t + 3$, $y = 3t + 2$

3.104. $z^3 + 2iz^2 + 6 - 2i$,

3.117. $U = \frac{1}{2}\{\ln(x^2 + y^2)\}^2 + 2\{\tan^{-1}(y/x)\}^2 + F(x + iy) + G(x - iy)$

3.119. $U = \frac{1}{16}(x^2 + y^2)^3 + (x + iy)F_1(x - iy) + G_1(x - iy) + (x - iy)F_2(x + iy) + G_2(x + iy)$

CHAPTER 4

Complex Integration and Cauchy's Theorem

4.1 Complex Line Integrals

Let $f(z)$ be continuous at all points of a curve C [Fig. 4-1], which we shall assume has a finite length, i.e., C is a *rectifiable curve*.

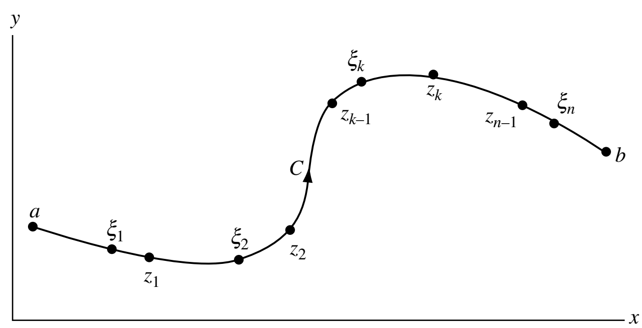

Fig. 4-1

Subdivide C into n parts by means of points $z_1, z_2, \ldots, z_{n-1}$, chosen arbitrarily, and call $a = z_0$, $b = z_n$. On each arc joining z_{k-1} to z_k [where k goes from 1 to n], choose a point ξ_k. Form the sum

$$S_n = f(\xi_1)(z_1 - a) + f(\xi_2)(z_2 - z_1) + \cdots + f(\xi_n)(b - z_{n-1}) \tag{4.1}$$

On writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta z_k \tag{4.2}$$

Let the number of subdivisions n increase in such a way that the largest of the chord lengths $|\Delta z_k|$ approaches zero. Then, since $f(z)$ is continuous, the sum S_n approaches a limit that does not depend on the mode of subdivision and we denote this limit by

$$\int_a^b f(z)\, dz \quad \text{or} \quad \int_C f(z)\, dz \tag{4.3}$$

called the *complex line integral* or simply *line integral* of $f(z)$ along curve C, or the *definite integral* of $f(z)$ from a to b along curve C. In such a case, $f(z)$ is said to be *integrable* along C. If $f(z)$ is analytic at all points of a region $\mathcal{R}$ and if C is a curve lying in $\mathcal{R}$, then $f(z)$ is continuous and therefore integrable along C.

4.2 Real Line Integrals

Let $P(x, y)$ and $Q(x, y)$ be real functions of x and y continuous at all points of curve C. Then the *real line integral* of $P\,dx + Q\,dy$ along curve C can be defined in a manner similar to that given above and is denoted by

$$\int_C [P(x, y)\,dx + Q(x, y)\,dy] \quad \text{or} \quad \int_C P\,dx + Q\,dy \tag{4.4}$$

the second notation being used for brevity. If C is smooth and has parametric equations $x = \phi(t)$, $y = \psi(t)$ where $t_1 \le t \le t_2$, it can be shown that the value of (4) is given by

$$\int_{t_1}^{t_2} [P\{\phi(t), \psi(t)\}\phi'(t)\,dt + Q\{\phi(t), \psi(t)\}\psi'(t)\,dt]$$

Suitable modifications can be made if C is piecewise smooth (see Problem 4.1).

4.3 Connection Between Real and Complex Line Integrals

Suppose $f(z) = u(x, y) + iv(x, y) = u + iv$. Then the complex line integral (3) can be expressed in terms of real line integrals as follows:

$$\begin{aligned}\int_C f(z)\,dz &= \int_C (u + iv)(dx + i\,dy)\\ &= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy \end{aligned} \tag{4.5}$$

For this reason, (4.5) is sometimes taken as a definition of a complex line integral.

4.4 Properties of Integrals

Suppose $f(z)$ and $g(z)$ are integrable along C. Then the following hold:

(a) $\displaystyle\int_C f(z) + g(z)\}\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

(b) $\displaystyle\int_C Af(z)\,dz = A\int_C f(z)\,dz$ where $A =$ any constant

(c) $\displaystyle\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$

(d) $\displaystyle\int_a^b f(z)\,dz = \int_a^m f(z)\,dz + \int_m^b f(z)\,dz$ where points a, b, m are on C

(e) $\displaystyle\left|\int_C f(z)\,dz\right| \le ML$

where $|f(z)| \le M$, i.e., M is an *upper bound* of $|f(z)|$ on C, and L is the *length* of C.

There are various other ways in which the above properties can be described. For example, if T, U, and V are successive points on a curve, property (c) can be written $\int_{TUV} f(z)\,dz = -\int_{VUT} f(z)\,dz$.

Similarly, if C, C_1, and C_2 represent curves from a to b, a to m, and m to b, respectively, it is natural for us to consider $C = C_1 + C_2$ and to write property (d) as

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

4.5 Change of Variables

Let $z = g(\zeta)$ be a continuous function of a complex variable $\zeta = u + iv$. Suppose that curve C in the z plane corresponds to curve C' in the ζ plane and that the derivative $g'(\zeta)$ is continuous on C'. Then

$$\int_C f(z)\,dz = \int_{C'} f\{g(\zeta)\}g'(\zeta)\,d\zeta \tag{4.6}$$

These conditions are certainly satisfied if g is analytic in a region containing curve C'.

4.6 Simply and Multiply Connected Regions

A region $\mathcal{R}$ is called *simply-connected* if any simple closed curve [Section 3.13], which lies in $\mathcal{R}$, can be shrunk to a point without leaving $\mathcal{R}$. A region $\mathcal{R}$, which is not simply-connected, is called *multiply-connected*.

For example, suppose $\mathcal{R}$ is the region defined by $|z| < 2$ shown shaded in Fig. 4-2. If Γ is any simple closed curve lying in $\mathcal{R}$ [i.e., whose points are in $\mathcal{R}$], we see that it can be shrunk to a point that lies in $\mathcal{R}$, and thus does not leave $\mathcal{R}$, so that $\mathcal{R}$ is simply-connected. On the other hand, if $\mathcal{R}$ is the region defined by $1 < |z| < 2$, shown shaded in Fig. 4-3, then there is a simple closed curve Γ lying in $\mathcal{R}$ that cannot possibly be shrunk to a point without leaving $\mathcal{R}$, so that $\mathcal{R}$ is multiply-connected.

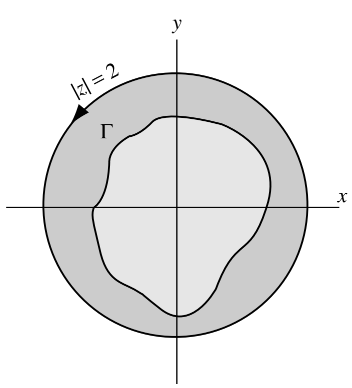

Fig. 4-2

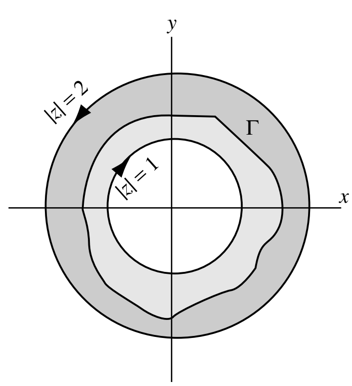

Fig. 4-3

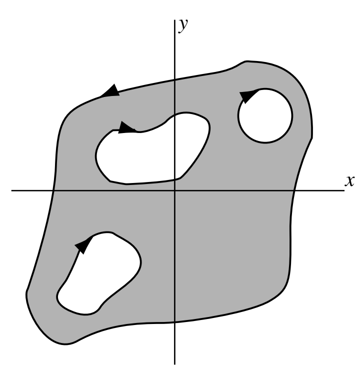

Fig. 4-4

Intuitively, a simply-connected region is one that does not have any "holes" in it, while a multiply-connected region is one that does. The multiply-connected regions of Figs. 4-3 and 4-4 have, respectively, one and three holes in them.

4.7 Jordan Curve Theorem

Any continuous, closed curve that does not intersect itself and may or may not have a finite length, is called a *Jordan curve* [see Problem 4.30]. An important theorem that, although very difficult to prove, seems intuitively obvious is the following.

Jordan Curve Theorem. A Jordan curve divides the plane into two regions having the curve as a common boundary. That region, which is bounded [i.e., is such that all points of it satisfy $|z| < M$ where M is some positive constant], is called the *interior* or *inside* of the curve, while the other region is called the *exterior* or *outside* of the curve.

Using the Jordan curve theorem, it can be shown that the region inside a simple closed curve is a simply-connected region whose boundary is the simple closed curve.

4.8 Convention Regarding Traversal of a Closed Path

The boundary C of a region is said to be traversed in the *positive sense* or *direction* if an observer travelling in this direction [and perpendicular to the plane] has the region to the left. This convention leads to the directions indicated by the arrows in Figs. 4-2, 4-3, and 4-4. We use the special symbol

$$\oint_C f(z)\,dz$$

to denote integration of $f(z)$ around the boundary C in the positive sense. In the case of a circle [Fig. 4-2], the positive direction is the *counterclockwise direction*. The integral around C is often called a *contour integral*.

4.9 Green's Theorem in the Plane

Let $P(x, y)$ and $Q(x, y)$ be continuous and have continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C. *Green's theorem* states that

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \tag{4.7}$$

The theorem is valid for both simply- and multiply-connected regions.

4.10 Complex Form of Green's Theorem

Let $F(z, \bar{z})$ be continuous and have continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C, where $z = x + iy$, $\bar{z} = x - iy$ are complex conjugate coordinates [see page 7]. Then Green's theorem can be written in the complex form

$$\oint_C F(z, \bar{z})\,dz = 2i \iint_{\mathcal{R}} \frac{\partial F}{\partial \bar{z}}\,dA \tag{4.8}$$

where dA represents the element of area $dx\,dy$.

For a generalization of (4.8), see Problem 4.56.

4.11 Cauchy's Theorem. The Cauchy–Goursat Theorem

Let $f(z)$ be analytic in a region $\mathcal{R}$ and on its boundary C. Then

$$\oint_C f(z)\,dz = 0 \tag{4.9}$$

This fundamental theorem, often called *Cauchy's integral theorem* or simply *Cauchy's theorem*, is valid for both simply- and multiply-connected regions. It was first proved by use of Green's theorem with the added restriction that $f'(z)$ be continuous in $\mathcal{R}$ [see Problem 4.11]. However, *Goursat* gave a proof which removed this restriction. For this reason, the theorem is sometimes called the *Cauchy–Goursat theorem* [see Problems 4.13–4.16] when one desires to emphasize the removal of this restriction.

4.12 Morera's Theorem

Let $f(z)$ be continuous in a simply-connected region $\mathcal{R}$ and suppose that

$$\oint_C f(z)\,dz = 0 \tag{4.10}$$

around *every* simple closed curve C in $\mathcal{R}$. Then $f(z)$ is analytic in $\mathcal{R}$.

This theorem, due to *Morera*, is often called the *converse of Cauchy's theorem*. It can be extended to multiply-connected regions. For a proof, which assumes that $f'(z)$ is continuous in $\mathcal{R}$, see Problem 4.27. For a proof, which eliminates this restriction, see Problem 5.7, Chapter 5.

4.13 Indefinite Integrals

Suppose $f(z)$ and $F(z)$ are analytic in a region $\mathcal{R}$ and such that $F'(z) = f(z)$. Then $F(z)$ is called an *indefinite integral* or *anti-derivative* of $f(z)$ denoted by

$$F(z) = \int f(z)\,dz \tag{4.11}$$

Just as in real variables, any two indefinite integrals differ by a constant. For this reason, an arbitrary constant c is often added to the right of (11).

EXAMPLE 4.1: Since $\dfrac{d}{dz}\left(3z^2 - 4\sin z\right) = 6z - 4\cos z$, we can write

$$\int (6z - 4\cos z)\,dz = 3z^2 - 4\sin z + c$$

4.14 Integrals of Special Functions

Using results on page 80 [or by direct differentiation], we can arrive at the results in Fig. 4-5 (omitting a constant of integration).

1. $\int z^n\,dz = \dfrac{z^{n+1}}{n+1} \quad n \neq -1$

2. $\int \dfrac{dz}{z} = \ln z$

3. $\int e^z\,dz = e^z$

4. $\int a^z\,dz = \dfrac{a^z}{\ln a}$

5. $\int \sin z\,dz = -\cos z$

6. $\int \cos z\,dz = \sin z$

7. $\int \tan z\,dz = \ln \sec z = -\ln \cos z$

8. $\int \cot z\,dz = \ln \sin z$

9. $\int \sec z\,dz = \ln(\sec z + \tan z) = \ln \tan(z/2 + \pi/4)$

10. $\int \csc z\,dz = \ln(\csc z - \cot z) = \ln \tan(z/2)$

11. $\int \sec^2 z\,dz = \tan z$

12. $\int \csc^2 z\,dz = -\cot z$

13. $\int \sec z \tan z\,dz = \sec z$

14. $\int \csc z \cot z\,dz = -\csc z$

15. $\int \sinh z\,dz = \cosh z$

16. $\int \cosh z\,dz = \sinh z$

17. $\int \tanh z\,dz = \ln \cosh z$

18. $\int \coth z\,dz = \ln \sinh z$

19. $\int \operatorname{sech} z\,dz = \tan^{-1}(\sinh z)$

20. $\int \operatorname{csch} z\,dz = -\coth^{-1}(\cosh z)$

21. $\int \operatorname{sech}^2 z\,dz = \tanh z$

22. $\int \operatorname{csch}^2 z\,dz = -\coth z$

23. $\int \operatorname{sech} z \tanh z\,dz = -\operatorname{sech} z$

24. $\int \operatorname{csch} z \coth z\,dz = -\operatorname{csch} z$

25. $\int \dfrac{dz}{\sqrt{z^2 \pm a^2}} = \ln\left(z + \sqrt{z^2 \pm a^2}\right)$

26. $\int \dfrac{dz}{z^2 + a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{z}{a} \quad \text{or} \quad -\dfrac{1}{a}\cot^{-1}\dfrac{z}{a}$

27. $\int \dfrac{dz}{z^2 - a^2} = \dfrac{1}{2a}\ln\left(\dfrac{z-a}{z+a}\right)$

28. $\int \dfrac{dz}{\sqrt{a^2 - z^2}} = \sin^{-1}\dfrac{z}{a} \quad \text{or} \quad -\cos^{-1}\dfrac{z}{a}$

29. $\int \dfrac{dz}{z\sqrt{a^2 \pm z^2}} = \dfrac{1}{a}\ln\left(\dfrac{z}{a + \sqrt{a^2 \pm z^2}}\right)$

30. $\int \dfrac{dz}{z\sqrt{z^2 - a^2}} = \dfrac{1}{a}\cos^{-1}\dfrac{a}{z} \quad \text{or} \quad \dfrac{1}{a}\sec^{-1}\dfrac{z}{a}$

31. $\int \sqrt{z^2 \pm a^2}\,dz = \dfrac{z}{2}\sqrt{z^2 \pm a^2} \pm \dfrac{a^2}{2}\ln\left(z + \sqrt{z^2 \pm a^2}\right)$

32. $\int \sqrt{a^2 - z^2}\,dz = \dfrac{z}{2}\sqrt{a^2 - z^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{z}{a}$

33. $\int e^{ax}\sin bz\,dz = \dfrac{e^{az}(a\sin bz - b\cos bz)}{a^2 + b^2}$

34. $\int e^{ax}\cos bz\,dz = \dfrac{e^{ax}(a\cos bz + b\sin bz)}{a^2 + b^2}$

Fig. 4-5

4.15 Some Consequences of Cauchy's Theorem

Let $f(z)$ be analytic in a simply-connected region $\mathcal{R}$. Then the following theorems hold.

THEOREM 4.1. Suppose a and z are any two points in $\mathcal{R}$. Then

$$\int_a^z f(z)\,dz$$

is *independent of the path* in $\mathcal{R}$ joining a and z.

THEOREM 4.2. Suppose a and z are any two points in $\mathcal{R}$ and

$$G(z) = \int_a^z f(z)\,dz \tag{4.12}$$

Then $G(z)$ is analytic in $\mathcal{R}$ and $G'(z) = f(z)$.

Occasionally, confusion may arise because the variable of integration z in (4.12) is the same as the upper limit of integration. Since a definite integral depends only on the curve and limits of integration, any symbol can be used for the variable of integration and, for this reason, we call it a *dummy variable* or *dummy symbol*. Thus (4.12) can be equivalently written

$$G(z) = \int_a^z f(\zeta)\,d\zeta \tag{4.13}$$

THEOREM 4.3. Suppose a and b are any two points in $\mathcal{R}$ and $F'(z) = f(z)$. Then

$$\int_a^b f(z)\,dz = F(b) - F(a) \tag{4.14}$$

This can also be written in the form, familiar from elementary calculus,

$$\int_a^b F'(z)\,dz = F(z)\Big|_a^b \quad \text{or} \quad [F(z)]_a^b = F(b) - F(a) \tag{4.15}$$

EXAMPLE 4.2: $\displaystyle\int_{3i}^{1-i} 4z\,dz = 2z^2\Big|_{3i}^{1-i} = 2(1-i)^2 - 2(3i)^2 = 18 - 4i$

THEOREM 4.4. Let $f(z)$ be analytic in a region bounded by two simple closed curves C and C_1 [where C_1 lies inside C as in Fig. 4-6(a)] and on these curves. Then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz \tag{4.16}$$

where C and C_1 are both traversed in the positive sense relative to their interiors [counter-clockwise in Fig. 4-6(a)].

The result shows that if we wish to integrate $f(z)$ along curve C, we can equivalently replace C by any curve C_1 so long as $f(z)$ is analytic in the region between C and C_1 as in Fig. 4-6(a).

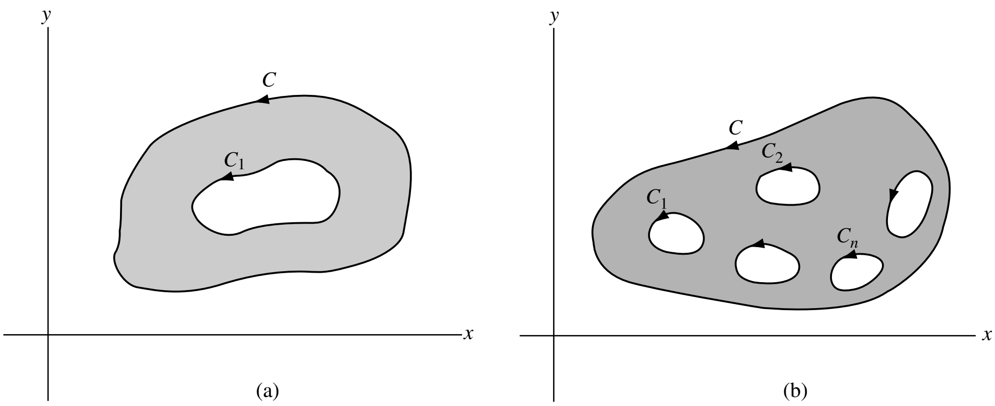

Fig. 4-6

THEOREM 4.5. Let $f(z)$ be analytic in a region bounded by the non-overlapping simple closed curves $C, C_1, C_2, C_3, \ldots, C_n$ where $C_1, C_2, \ldots, C_n$ are inside C [as in Fig. 4-6(b)] and on these curves. Then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz \tag{4.17}$$

This is a generalization of Theorem 4.4.

SOLVED PROBLEMS

Line Integrals

4.1. Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along: (a) the parabola $x = 2t$, $y = t^2 + 3$; (b) straight lines from (0, 3) to (2, 3) and then from (2, 3) to (2, 4); (c) a straight line from (0, 3) to (2, 4).

Solution

(a) The points (0, 3) and (2, 4) on the parabola correspond to $t = 0$ and $t = 1$, respectively. Then, the given integral equals

$$\int_{t=0}^{1} [2(t^2 + 3) + (2t)^2]2\,dt + [3(2t) - (t^2 + 3)]2t\,dt = \int_0^1 (24t^2 + 12 - 2t^3 - 6t)\,dt = \frac{33}{2}$$

(b) Along the straight line from (0, 3) to (2, 3), $y = 3$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{2} (6 + x^2)\,dx + (3x - 3)0 = \int_{x=0}^{2} (6 + x^2)\,dx = \frac{44}{3}$$

Along the straight line from (2, 3) to (2, 4), $x = 2$, $dx = 0$ and the line integral equals

$$\int_{y=3}^{4} (2y + 4)0 + (6 - y)\,dy = \int_{y=3}^{4} (6 - y)\,dy = \frac{5}{2}$$

Then, the required value $= 44/3 + 5/2 = 103/6$.

(c) An equation for the line joining (0, 3) and (2, 4) is $2y - x = 6$. Solving for x, we have $x = 2y - 6$. Then, the line integral equals

$$\int_{y=3}^{4} \left[2y + (2y-6)^2\right] 2\,dy + [3(2y-6) - y]\,dy = \int_{3}^{4} (8y^2 - 39y + 54)\,dy = \frac{97}{6}$$

The result can also be obtained by using $y = \frac{1}{2}(x+6)$.

4.2. Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by: (a) $z = t^2 + it$, (b) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

Solution

(a) The points $z = 0$ and $z = 4 + 2i$ on C correspond to $t = 0$ and $t = 2$, respectively. Then, the line integral equals

$$\int_{t=0}^{2} (\overline{t^2 + it})\, d(t^2 + it) = \int_{0}^{2} (t^2 - it)(2t + i)\,dt = \int_{0}^{2} (2t^3 - it^2 + t)\,dt = 10 - \frac{8i}{3}$$

Another Method. The given integral equals

$$\int_C (x - iy)(dx + i\,dy) = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

The parametric equations of C are $x = t^2$, $y = t$ from $t = 0$ to $t = 2$. Then, the line integral equals

$$\int_{t=0}^{2} (t^2)(2t\,dt) + (t)(dt) + i\int_{t=0}^{2} (t^2)(dt) - (t)(2t\,dt)$$

$$= \int_{0}^{2} (2t^3 + t)\,dt + i\int_{0}^{2} (-t^2)\,dt = 10 - \frac{8i}{3}$$

(b) The given line integral equals

$$\int_C (x - iy)(dx + i\,dy) = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

The line from $z = 0$ to $z = 2i$ is the same as the line from (0, 0) to (0, 2) for which $x = 0$, $dx = 0$ and the line integral equals

$$\int_{y=0}^{2} (0)(0) + y\,dy + i\int_{y=0}^{2} (0)(dy) - y(0) = \int_{y=0}^{2} y\,dy = 2$$

The line from $z = 2i$ to $z = 4 + 2i$ is the same as the line from (0, 2) to (4, 2) for which $y = 2$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{4} x\,dx + 2\cdot 0 + i\int_{x=0}^{4} x\cdot 0 - 2\,dx = \int_{0}^{4} x\,dx + i\int_{0}^{4} -2\,dx = 8 - 8i$$

Then, the required value $= 2 + (8 - 8i) = 10 - 8i$.

4.3. Suppose $f(z)$ is integrable along a curve C having finite length L and suppose there exists a positive number M such that $|f(z)| \leq M$ on C. Prove that

$$\left| \int_C f(z)\,dz \right| \leq ML$$

Solution

By definition, we have on using the notation of page 111,

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta z_k \tag{1}$$

Now

$$\begin{aligned} \left| \sum_{k=1}^{n} f(\xi_k)\Delta z_k \right| &\leq \sum_{k=1}^{n} |f(\xi_k)|\,|\Delta z_k| \\ &\leq M \sum_{k=1}^{n} |\Delta z_k| \\ &\leq ML \end{aligned} \tag{2}$$

where we have used the facts that $|f(z)| \leq M$ for all points z on C and that $\sum_{k=1}^{n} |\Delta z_k|$ represents the sum of all the chord lengths joining points z_{k-1} and z_k, where $k = 1, 2, \ldots, n$, and that this sum is not greater than the length of C.

Taking the limit of both sides of (2), using (1), the required result follows. It is possible to show, more generally, that

$$\left| \int_C f(z)\,dz \right| \leq \int_C |f(z)|\,|dz|$$

Green's Theorem in the Plane

4.4. Prove Green's theorem in the plane if C is a simple closed curve which has the property that any straight line parallel to the coordinate axes cuts C in at most two points.

Solution

Let the equations of the curves EGF and EHF (see Fig. 4-7) be $y = Y_1(x)$ and $y = Y_2(x)$, respectively. If $\mathcal{R}$ is the region bounded by C, we have

$$\begin{aligned} \iint_{\mathcal{R}} \frac{\partial P}{\partial y}\,dx\,dy &= \int_{x=e}^{f} \left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial P}{\partial y}\,dy \right] dx \\ &= \int_{x=e}^{f} P(x, y)\Big|_{y=Y_1(x)}^{Y_2(x)}\,dx = \int_e^f [P(x, Y_2) - P(x, Y_1)]\,dx \\ &= -\int_e^f P(x, Y_1)\,dx - \int_f^e P(x, Y_2)\,dx = -\oint_C P\,dx \end{aligned}$$

Then

$$\oint_C P\,dx = -\iint_{\mathcal{R}} \frac{\partial P}{\partial y}\,dx\,dy \tag{1}$$

Similarly, let the equations of curves *GEH* and *GFH* be $x = X_1(y)$ and $x = X_2(y)$, respectively. Then

$$\iint_{\mathcal{R}} \frac{\partial Q}{\partial x}\,dx\,dy = \int_{y=g}^{h}\left[\int_{x=X_1(y)}^{X_2(y)} \frac{\partial Q}{\partial x}\,dx\right]dy = \int_{g}^{h}[Q(X_2, y) - Q(X_1, y)]\,dy$$

$$= \int_{h}^{g} Q(X_1, y)\,dy + \int_{g}^{h} Q(X_2, y)\,dy = \oint_C Q\,dy$$

Then

$$\oint_C Q\,dy = \iint_{\mathcal{R}} \frac{\partial Q}{\partial x}\,dx\,dy \tag{2}$$

Adding (1) and (2),

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

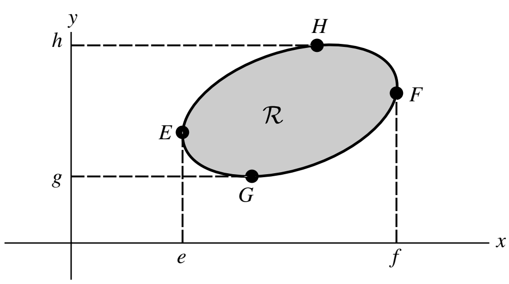

Fig. 4-7

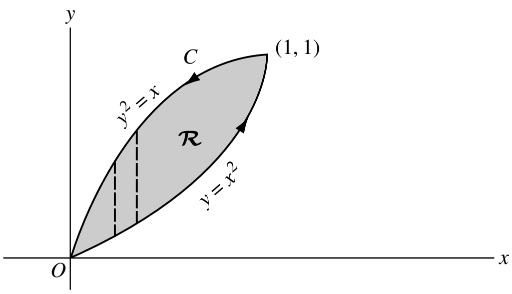

Fig. 4-8

4.5. Verify Green's theorem in the plane for

$$\oint_C (2xy - x^2)\,dx + (x + y^2)\,dy$$

where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

Solution

The plane curves $y = x^2$ and $y^2 = x$ intersect at (0, 0) and (1, 1). The positive direction in traversing C is as shown in Fig. 4-8.

Along $y = x^2$, the line integral equals

$$\int_{x=0}^{1} \left\{(2x)(x^2) - x^2\right\}dx + \left\{x + (x^2)^2\right\}d(x^2) = \int_0^1 (2x^3 + x^2 + 2x^5)\,dx = \frac{7}{6}$$

Along $y^2 = x$, the line integral equals

$$\int_{y=1}^{0} \{2(y^2)(y) - (y^2)^2\}\,d(y^2) + \{y^2 + y^2\}\,dy = \int_1^0 (4y^4 - 2y^5 + 2y^2)\,dy = -\frac{17}{15}$$

Then the required integral $= 7/6 - 17/15 = 1/30$. On the other hand,

$$\iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{\mathcal{R}} \left\{\frac{\partial}{\partial x}(x + y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right\} dx\,dy$$

$$= \iint_{\mathcal{R}} (1 - 2x)\,dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (1 - 2x)\,dy\,dx$$

$$= \int_{x=0}^{1} (y - 2xy)\Big|_{y=x^2}^{\sqrt{x}} dx = \int_{0}^{1} (x^{1/2} - 2x^{3/2} - x^2 + 2x^3)\,dx = \frac{1}{30}$$

Hence, Green's theorem is verified.

4.6. Extend the proof of Green's theorem in the plane given in Problem 4.4 to curves C for which lines parallel to the coordinate axes may cut C in more than two points.

Solution

Consider a simple closed curve C such as shown in Fig. 4-9 in which lines parallel to the axes may meet C in more than two points. By constructing line ST, the region is divided into two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ which are of the type considered in Problem 4.4 and for which Green's theorem applies, i.e.,

$$\int_{STUS} P\,dx + Q\,dy = \iint_{\mathcal{R}_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \tag{1}$$

$$\int_{SVTS} P\,dx + Q\,dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy \tag{2}$$

Adding the left-hand sides of (1) and (2), we have, omitting the integrand $P\,dx + Q\,dy$ in each case,

$$\int_{STUS} + \int_{SVTS} = \int_{ST} + \int_{TUS} + \int_{SVT} + \int_{TS} = \int_{TUS} + \int_{SVT} = \int_{TUSVT}$$

using the fact that $\int_{ST} = -\int_{TS}$.

Adding the right-hand sides of (1) and (2), omitting the integrand,

$$\iint_{\mathcal{R}_1} + \iint_{\mathcal{R}_2} = \iint_{\mathcal{R}}$$

Then

$$\int_{TUSVT} P\,dx + Q\,dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

and the theorem is proved. We have proved Green's theorem for the simply-connected region of Fig. 4-9 bounded by the simple closed curve C. For more complicated regions, it may be necessary to construct more lines, such as ST, to establish the theorem.

Green's theorem is also true for multiply-connected regions, as shown in Problem 4.7.

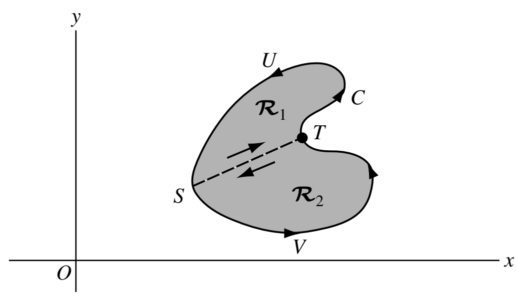

Fig. 4-9

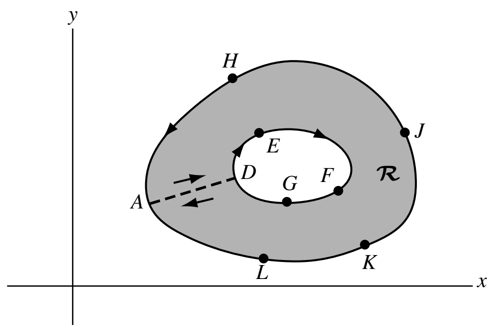

Fig. 4-10

4.7. Show that Green's theorem in the plane is also valid for a multiply-connected region $\mathcal{R}$ such as shown shaded in Fig. 4-10.

Solution

The boundary of $\mathcal{R}$, which consists of the exterior boundary *AHJKLA* and the interior boundary *DEFGD*, is to be traversed in the positive direction so that a person traveling in this direction always has the region on his/her left. It is seen that the positive directions are as indicated in the figure.

In order to establish the theorem, construct a line, such as *AD*, called a *cross-out*, connecting the exterior and interior boundaries. The region bounded by *ADEFGDALKJHA* is simply-connected, and so Green's theorem is valid. Then

$$\oint_{ADEFGDALKJHA} P\,dx + Q\,dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

But the integral on the left, leaving out the integrand, is equal to

$$\int_{AD} + \int_{DEFGD} + \int_{DA} + \int_{ALKJHA} = \int_{DEFGD} + \int_{ALKJHA}$$

since $\int_{AD} = -\int_{DA}$. Thus, if C_1 is the curve *ALKJHA*, C_2 is the curve *DEFGD* and C is the boundary of $\mathcal{R}$ consisting of C_1 and C_2 (traversed in the positive directions with respect to $\mathcal{R}$), then $\int_{C_1} + \int_{C_2} = \oint_C$ and so

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

4.8. Let $P(x, y)$ and $Q(x, y)$ be continuous and have continuous first partial derivatives at each point of a simply-connected region $\mathcal{R}$. Prove that a necessary and sufficient condition that $\oint_C P\,dx + Q\,dy = 0$ around every closed path C in $\mathcal{R}$ is that $\partial P/\partial y = \partial Q/\partial x$ identically in $\mathcal{R}$.

Solution

Sufficiency. Suppose $\partial P/\partial y = \partial Q/\partial x$. Then, by Green's theorem,

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = 0$$

where $\mathcal{R}$ is the region bounded by C.

Necessity. Suppose $\oint_C P\,dx + Q\,dy = 0$ around every closed path C in $\mathcal{R}$ and that $\partial P/\partial y \neq \partial Q/\partial x$ at some point of $\mathcal{R}$. In particular, suppose $\partial P/\partial y - \partial Q/\partial x > 0$ at the point (x_0, y_0).

By hypothesis, $\partial P/\partial y$ and $\partial Q/\partial x$ are continuous in $\mathcal{R}$ so that there must be some region τ containing (x_0, y_0) as an interior point for which $\partial P/\partial y - \partial Q/\partial x > 0$. If Γ is the boundary of τ, then by Green's theorem

$$\oint_\Gamma P\,dx + Q\,dy = \iint_\tau \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy > 0$$

contradicting the hypothesis that $\oint_C P\,dx + Q\,dy = 0$ for *all* closed curves in $\mathcal{R}$. Thus, $\partial Q/\partial x - \partial P/\partial y$ cannot be positive.

Similarly, we can show that $\partial Q/\partial x - \partial P/\partial y$ cannot be negative and it follows that it must be identically zero, i.e., $\partial P/\partial y = \partial Q/\partial x$ identically in $\mathcal{R}$.

The results can be extended to multiply-connected regions.

4.9. Let P and Q be defined as in Problem 4.8. Prove that a necessary and sufficient condition that $\int_A^B P\,dx + Q\,dy$ be independent of the path in $\mathcal{R}$ joining points A and B is that $\partial P/\partial y = \partial Q/\partial x$ identically in $\mathcal{R}$.

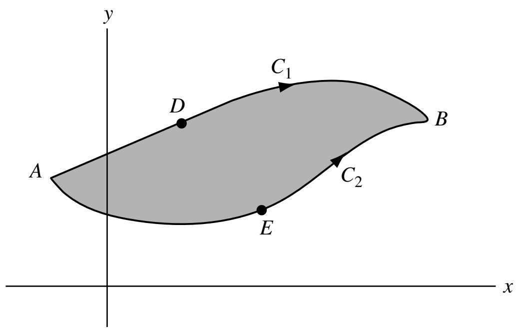

Fig. 4-11

Solution

Sufficiency. If $\partial P/\partial y = \partial Q/\partial x$, then by Problem 4.8

$$\int_{ADBEA} P\,dx + Q\,dy = 0$$

[see Fig. 4-11]. From this, omitting for brevity the integrand $P\,dx + Q\,dy$, we have

$$\int_{ADB} + \int_{BEA} = 0, \quad \int_{ADB} = -\int_{BEA} = \int_{AEB} \quad \text{and so} \quad \int_{C_1} = \int_{C_2}$$

i.e., the integral is independent of the path.

Necessity. If the integral is independent of the path, then for all paths C_1 and C_2 in $\mathcal{R}$, we have

$$\int_{C_1} = \int_{C_2}, \quad \int_{ADB} = \int_{AEB} \quad \text{and} \quad \int_{ADBEA} = 0$$

From this, it follows that the line integral around any closed path in $\mathcal{R}$ is zero and hence, by Problem 4.8, that $\partial P/\partial y = \partial Q/\partial x$.

The results can be extended to multiply-connected regions.

Complex Form of Green's Theorem

4.10. Suppose $B(z, \bar{z})$ is continuous and has continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C, where $z = x + iy$ and $\bar{z} = x - iy$. Prove that Green's theorem can be written in complex form as

$$\oint_C B(z, \bar{z})\,dz = 2i \iint_{\mathcal{R}} \frac{\partial B}{\partial \bar{z}}\,dx\,dy$$

Solution

Let $B(z, \bar{z}) = P(x, y) + iQ(x, y)$. Then, using Green's theorem, we have

$$\begin{aligned}\oint_C B(z, \bar{z})\,dz &= \oint_C (P + iQ)(dx + i\,dy) = \oint_C P\,dx - Q\,dy + i\oint_C Q\,dx + P\,dy \\ &= -\iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\,dy + i\iint_{\mathcal{R}} \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy \\ &= i\iint_{\mathcal{R}} \left[\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\right] dx\,dy \\ &= 2i\iint_{\mathcal{R}} \frac{\partial B}{\partial \bar{z}}\,dx\,dy\end{aligned}$$

from Problem 3.34, page 101. The result can also be written in terms of curl B [see page 85].

Cauchy's Theorem and the Cauchy–Goursat Theorem

4.11. Prove Cauchy's theorem $\oint_C f(z)\,dz = 0$ if $f(z)$ is analytic with derivative $f'(z)$ which is continuous at all points inside and on a simple closed curve C.

Solution

Since $f(z) = u + iv$ is analytic and has a continuous derivative

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

it follows that the partial derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{2}$$

are continuous inside and on C. Thus, Green's theorem can be applied and we have

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C (u + iv)(dx + i\,dy) = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy \\ &= \iint_{\mathcal{R}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_{\mathcal{R}} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0\end{aligned}$$

using the Cauchy–Riemann equations (1) and (2).

By using the fact that Green's theorem is applicable to multiply-connected regions, we can extend the result to multiply-connected regions under the given conditions on $f(z)$.

The *Cauchy–Goursat theorem* [see Problems 4.13–4.16] removes the restriction that $f'(z)$ be continuous.

Another Method.
The result can be obtained from the complex form of Green's theorem [Problem 4.10] by noting that if $B(z, \bar{z}) = f(z)$ is independent of $\bar{z}$, then $\partial B/\partial \bar{z} = 0$ and so $\oint_C f(z)\,dz = 0$.

4.12. Prove (a) $\oint_C dz = 0$, (b) $\oint_C z\,dz = 0$, (c) $\oint_C (z - z_0)\,dz = 0$ where C is any simple closed curve and z_0 is a constant.

Solution

These follow at once from Cauchy's theorem since the functions 1, z, and $z - z_0$ are analytic inside C and have continuous derivatives.

The results can also be established directly from the definition of an integral (see Problem 4.90).

4.13. Prove the *Cauchy–Goursat* theorem for the case of a triangle.

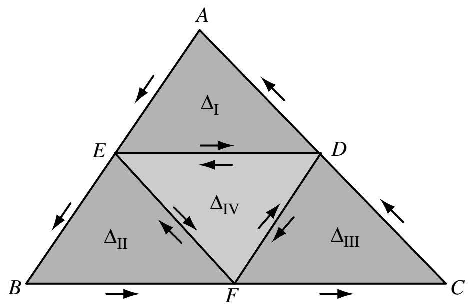

Fig. 4-12

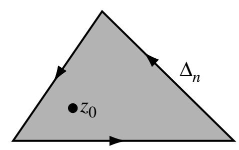

Fig. 4-13

Solution

Consider any triangle in the z plane such as ABC, denoted briefly by Δ, in Fig. 4-12. Join the midpoints D, E, and F of sides AB, AC, and BC, respectively, to form four triangles ($\Delta_{\rm I}$, $\Delta_{\rm II}$, $\Delta_{\rm III}$, and $\Delta_{\rm IV}$).

If $f(z)$ is analytic inside and on triangle ABC, we have, omitting the integrand on the right,

$$\begin{aligned}\oint_{ABCA} f(z)\,dz &= \int_{DAE} + \int_{EBF} + \int_{FCD} \\ &= \left\{\int_{DAE} + \int_{ED}\right\} + \left\{\int_{EBF} + \int_{FE}\right\} + \left\{\int_{FCD} + \int_{DF}\right\} + \left\{\int_{DE} + \int_{EF} + \int_{FD}\right\} \\ &= \int_{DAED} + \int_{EBFE} + \int_{FCDF} + \int_{DEFD} \\ &= \oint_{\Delta_{\rm I}} f(z)\,dz + \oint_{\Delta_{\rm II}} f(z)\,dz + \oint_{\Delta_{\rm III}} f(z)\,dz + \oint_{\Delta_{\rm IV}} f(z)\,dz\end{aligned}$$

where, in the second line, we have made use of the fact that

$$\int_{ED} = -\int_{DE}, \qquad \int_{FE} = -\int_{EF}, \qquad \int_{DF} = -\int_{FD}$$

Then

$$\left|\oint_{\Delta} f(z)\,dz\right| \le \left|\oint_{\Delta_{\rm I}} f(z)\,dz\right| + \left|\oint_{\Delta_{\rm II}} f(z)\,dz\right| + \left|\oint_{\Delta_{\rm III}} f(z)\,dz\right| + \left|\oint_{\Delta_{\rm IV}} f(z)\,dz\right| \tag{1}$$

Let Δ_1 be the triangle corresponding to that term on the right of (1) having largest value (if there are two or more such terms, then Δ_1 is any of the associated triangles). Then

$$\left|\oint_{\Delta} f(z)\,dz\right| \le 4\left|\oint_{\Delta_1} f(z)\,dz\right| \tag{2}$$

By joining midpoints of the sides of triangle Δ_1, we obtain similarly a triangle Δ_2 such that

$$\left|\oint_{\Delta_1} f(z)\,dz\right| \leq 4\left|\oint_{\Delta_2} f(z)\,dz\right| \tag{3}$$

so that

$$\left|\oint_{\Delta} f(z)\,dz\right| \leq 4^2\left|\oint_{\Delta_2} f(z)\,dz\right| \tag{4}$$

After n steps, we obtain a triangle Δ_n such that

$$\left|\oint_{\Delta} f(z)\,dz\right| \leq 4^n\left|\oint_{\Delta_n} f(z)\,dz\right| \tag{5}$$

Now $\Delta, \Delta_1, \Delta_2, \Delta_3, \ldots$ is a sequence of triangles, each of which is contained in the preceding (i.e., a sequence of *nested triangles*), and there exists a point z_0 which lies in every triangle of the sequence.

Since z_0 lies inside or on the boundary of Δ, it follows that $f(z)$ is analytic at z_0. Then, by Problem 3.21, page 95,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z - z_0) \tag{6}$$

where, for any $\epsilon > 0$, we can find δ such that $|\eta| < \epsilon$ whenever $|z - z_0| < \delta$.

Thus, by integration of both sides of (6) and using Problem 4.12,

$$\oint_{\Delta_n} f(z)\,dz = \oint_{\Delta_n} \eta(z - z_0)\,dz \tag{7}$$

Now, if P is the perimeter of Δ, then the perimeter of Δ_n is $P_n = P/2^n$. If z is any point on Δ_n, then as seen from Fig. 4-13, we must have $|z - z_0| < P/2^n < \delta$. Hence, from (7) and Property e, page 112, we have

$$\left|\oint_{\Delta_n} f(z)\,dz\right| = \left|\oint_{\Delta_n} \eta(z - z_0)\,dz\right| \leq \epsilon \cdot \frac{P}{2^n} \cdot \frac{P}{2^n} = \frac{\epsilon P^2}{4^n}$$

Then (5) becomes

$$\left|\oint_{\Delta} f(z)\,dz\right| \leq 4^n \cdot \frac{\epsilon P^2}{4^n} = \epsilon P^2$$

Since ϵ can be made arbitrarily small, it follows that, as required,

$$\oint_{\Delta} f(z)\,dz = 0$$

4.14. Prove the Cauchy–Goursat theorem for any closed polygon.

Solution

Consider, for example, a closed polygon *ABCDEFA* such as indicated in Fig. 4-14. By constructing the lines *BF*, *CF*, and *DF*, the polygon is subdivided into triangles. Then, by Cauchy's theorem for triangles [Problem 4.13] and the fact that the integrals along *BF* and *FB*, *CF* and *FC*, and *DF* and *FD* cancel, we find as required

$$\int_{ABCDEFA} f(z)\,dz = \int_{ABFA} f(z)\,dz + \int_{BCFB} f(z)\,dz + \int_{CDFC} f(z)\,dz + \int_{DEFD} f(z)\,dz = 0$$

where we suppose that $f(z)$ is analytic inside and on the polygon.

It should be noted that we have proved the result for simple polygons whose sides do not cross. A proof can also be given for any polygon that intersects itself (see Problem 4.66).

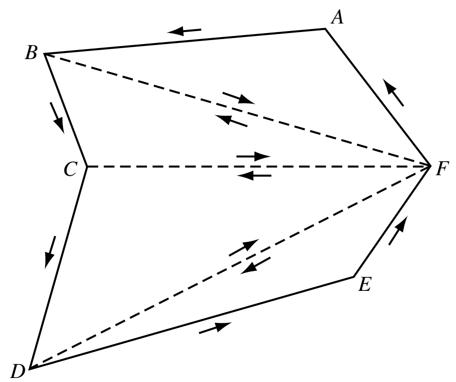

Fig. 4-14

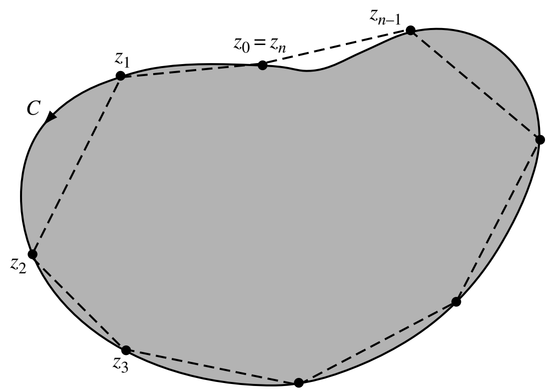

Fig. 4-15

4.15. Prove the Cauchy–Goursat theorem for any simple closed curve.

Solution

Let us assume that C is contained in a region $\mathcal{R}$ in which $f(z)$ is analytic.

Choose n points of subdivision $z_1, z_2, \ldots, z_n$ on curve C [Fig. 4-15] where, for convenience of notation, we consider $z_0 = z_n$. Construct polygon P by joining these points.

Let us define the sum

$$S_n = \sum_{k=1}^{n} f(z_k)\Delta z_k$$

where $\Delta z_k = z_k - z_{k-1}$. Since

$$\lim S_n = \oint_C f(z)\,dz$$

where the limit on the left means that $n \to \infty$ in such a way that the largest of $|\Delta z_k| \to 0$. It follows that, given any $\epsilon > 0$, we can choose N so that for $n > N$

$$\left| \oint_C f(z)\,dz - S_n \right| < \frac{\epsilon}{2} \qquad (1)$$

Consider now the integral along polygon P. Since this is zero by Problem 4.14, we have

$$\begin{aligned}
\oint_P f(z)\,dz = 0 &= \int_{z_0}^{z_1} f(z)\,dz + \int_{z_1}^{z_2} f(z)\,dz + \cdots + \int_{z_{n-1}}^{z_n} f(z)\,dz \\
&= \int_{z_0}^{z_1} \{f(z) - f(z_1) + f(z_1)\}\,dz + \cdots + \int_{z_{n-1}}^{z_n} \{f(z) - f(z_n) + f(z_n)\}\,dz \\
&= \int_{z_0}^{z_1} \{f(z) - f(z_1)\}\,dz + \cdots + \int_{z_{n-1}}^{z_n} \{f(z) - f(z_n)\}\,dz + S_n
\end{aligned}$$

so that

$$S_n = \int_{z_0}^{z_1} \{f(z_1) - f(z)\}\, dz + \cdots + \int_{z_{n-1}}^{z_n} \{f(z_n) - f(z)\}\, dz \tag{2}$$

Let us now choose N so large that on the lines joining z_0 and z_1, z_1 and $z_2, \ldots, z_{n-1}$ and z_n,

$$|f(z_1) - f(z)| < \frac{\epsilon}{2L}, \quad |f(z_2) - f(z)| < \frac{\epsilon}{2L}, \quad \ldots, \quad |f(z_n) - f(z)| < \frac{\epsilon}{2L} \tag{3}$$

where L is the length of C. Then, from (2) and (3), we have

$$|S_n| \le \left| \int_{z_0}^{z_1} \{f(z_1) - f(z)\}\, dz \right| + \left| \int_{z_1}^{z_2} \{f(z_2) - f(z)\}\, dz \right| + \cdots + \left| \int_{z_{n-1}}^{z_n} \{f(z_n) - f(z)\}\, dz \right|$$

or

$$|S_n| \le \frac{\epsilon}{2L} \{|z_1 - z_0| + |z_2 - z_1| + \cdots + |z_n - z_{n-1}|\} = \frac{\epsilon}{2} \tag{4}$$

From

$$\oint_C f(z)\, dz = \oint_C f(z)\, dz - S_n + S_n$$

we have, using (1) and (4),

$$\left| \oint_C f(z)\, dz \right| \le \left| \oint_C f(z)\, dz - S_n \right| + |S_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus, since ϵ is arbitrary, it follows that $\oint_C f(z)\, dz = 0$ as required.

4.16. Prove the Cauchy–Goursat theorem for multiply-connected regions.

Solution

We shall present a proof for the multiply-connected region $\mathcal{R}$ bounded by the simple closed curves C_1 and C_2 as indicated in Fig. 4-16. Extensions to other multiply-connected regions are easily made (see Problem 4.67).

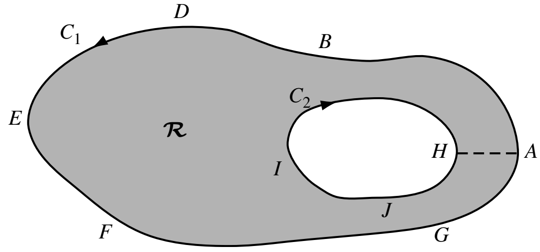

Fig. 4-16

Construct cross-cut AH. Then the region bounded by $ABDEFGAHJIHA$ is simply-connected so that by Problem 4.15,

$$\oint_{ABDEFGAHJIHA} f(z)\, dz = 0$$

Hence

$$\int_{ABDEFGA} f(z)\, dz + \int_{AH} f(z)\, dz + \int_{HJIH} f(z)\, dz + \int_{HA} f(z)\, dz = 0$$

Since $\int_{AH} f(z)\,dz = -\int_{HA} f(z)\,dz$, this becomes

$$\int_{ABDEFGA} f(z)\,dz + \int_{HJIH} f(z)\,dz = 0$$

This, however, amounts to saying that

$$\oint_C f(z)\,dz = 0$$

where C is the complete boundary of $\mathcal{R}$ (consisting of $ABDEFGA$ and $HJIH$) traversed in the sense that an observer walking on the boundary always has the region $\mathcal{R}$ on his/her left.

Consequences of Cauchy's Theorem

4.17. Suppose $f(z)$ is analytic in a simply-connected region $\mathcal{R}$. Prove that $\int_a^b f(z)\,dz$ is independent of the path in $\mathcal{R}$ joining any two points a and b in $\mathcal{R}$ [as in Fig. 4-17].

Solution

By Cauchy's theorem,

$$\int_{ADBEA} f(z)\,dz = 0$$

or

$$\int_{ADB} f(z)\,dz + \int_{BEA} f(z)\,dz = 0$$

Hence

$$\int_{ADB} f(z)\,dz = -\int_{BEA} f(z)\,dz = \int_{AEB} f(z)\,dz$$

Thus

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz = \int_a^b f(z)\,dz$$

which yields the required result.

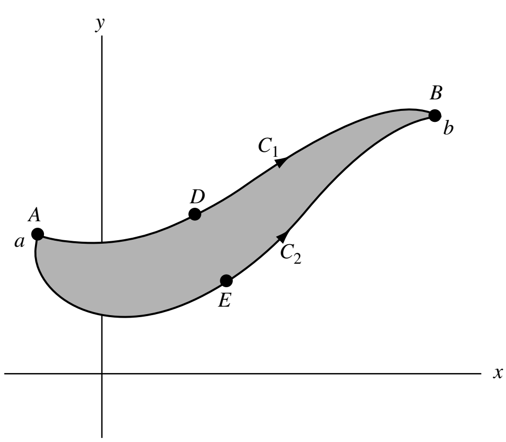

Fig. 4-17

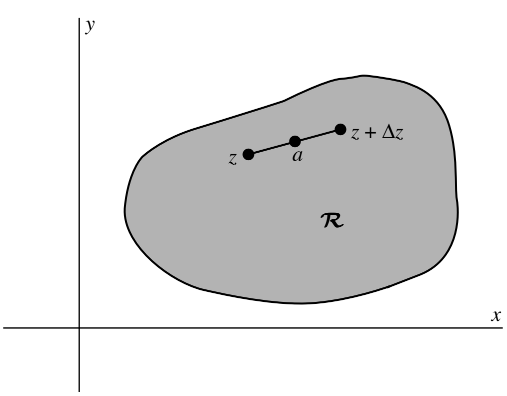

Fig. 4-18

4.18. Let $f(z)$ be analytic in a simply-connected region $\mathcal{R}$ and let a and z be points in $\mathcal{R}$. Prove that (a) $F(z) = \int_a^z f(u)\,du$ is analytic in $\mathcal{R}$ and (b) $F'(z) = f(z)$.

Solution

We have

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\left\{ \int_a^{z+\Delta z} f(u)\,du - \int_a^z f(u)\,du \right\} - f(z)$$
$$= \frac{1}{\Delta z}\int_z^{z+\Delta z} \{f(u) - f(z)\}\,du \tag{1}$$

By Cauchy's theorem, the last integral is independent of the path joining z and $z + \Delta z$ so long as the path is in $\mathcal{R}$. In particular, we can choose as a path the straight line segment joining z and $z + \Delta z$ (see Fig. 4-18) provided we choose $|\Delta z|$ small enough so that this path lies in $\mathcal{R}$.

Now, by the continuity of $f(z)$, we have for all points u on this straight line path $|f(u) - f(z)| < \epsilon$ whenever $|u - z| < \delta$, which will certainly be true if $|\Delta z| < \delta$.

Furthermore, we have

$$\left| \int_z^{z+\Delta z} \{ f(u) - f(z) \}\, du \right| < \epsilon |\Delta z| \tag{2}$$

so that from (1)

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|}\left| \int_z^{z+\Delta z} [f(u) - f(z)]\,du \right| < \epsilon$$

for $|\Delta z| < \delta$. This, however, amounts to saying that

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z),$$

i.e., $F(z)$ is analytic and $F'(z) = f(z)$.

4.19. A function $F(z)$ such that $F'(z) = f(z)$ is called an *indefinite integral* of $f(z)$ and is denoted by $\int f(z)\,dz$. Show that (a) $\int \sin z\,dz = -\cos z + c$, (b) $\int dz/z = \ln z + c$ where c is an arbitrary constant.

Solution

(a) Since $d/dz(-\cos z + c) = \sin z$, we have $\int \sin z\,dz = -\cos z + c$.
(b) Since $d/dz(\ln z + c) = 1/z$, we have $\int dz/z = \ln z + c$.

4.20. Let $f(z)$ be analytic in a region $\mathcal{R}$ bounded by two simple closed curves C_1 and C_2 [shaded in Fig. 4-19] and also on C_1 and C_2. Prove that $\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$, where C_1 and C_2 are both traversed in the positive sense relative to their interiors [counterclockwise in Fig. 4-19].

Solution

Construct cross-cut DE. Then, since $f(z)$ is analytic in the region $\mathcal{R}$, we have by Cauchy's theorem

$$\int_{DEFGEDHJKLD} f(z)\,dz = 0$$

or

$$\int_{DE} f(z)\,dz + \int_{EFGE} f(z)\,dz + \int_{ED} f(z)\,dz + \int_{DHJKLD} f(z)\,dz = 0$$

Hence since $\int_{DE} f(z)\,dz = -\int_{ED} f(z)\,dz$,

$$\int_{DHJKLD} f(z)\,dz = -\int_{EFGE} f(z)\,dz = \int_{EGFE} f(z)\,dz \quad \text{or} \quad \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

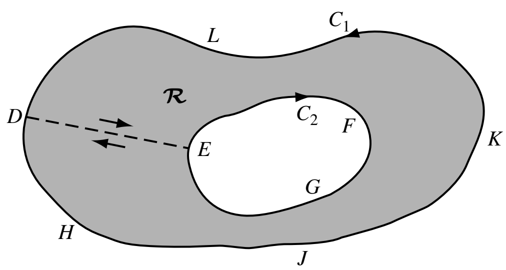

Fig. 4-19

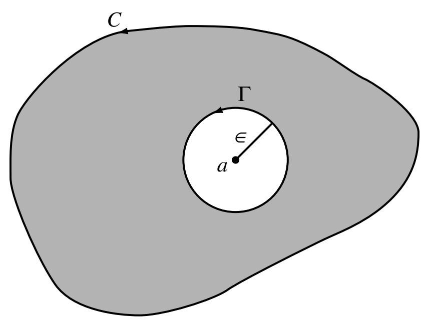

Fig. 4-20

4.21. Evaluate $\oint_C dz/z - a$ where C is any simple closed curve C and $z = a$ is (a) outside C, (b) inside C.

Solution

(a) If a is outside C, then $f(z) = 1/(z-a)$ is analytic everywhere inside and on C. Hence, by Cauchy's theorem, $\oint_C dx/z - a = 0$.

(b) Suppose a is inside C and let Γ be a circle of radius ϵ with center at $z = a$ so that Γ is inside C (this can be done since $z = a$ is an interior point).

By Problem 4.20,

$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a} \tag{1}$$

Now on Γ, $|z-a| = \epsilon$ or $z - a = \epsilon e^{i\theta}$, i.e., $z = a + \epsilon e^{i\theta}$, $0 \le \theta < 2\pi$. Thus, since $dz = i\epsilon e^{i\theta}\,d\theta$, the right side of (1) becomes

$$\int_{\theta=0}^{2\pi} \frac{i\epsilon e^{i\theta}\,d\theta}{\epsilon e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i$$

which is the required value.

4.22. Evaluate $\oint_C \dfrac{dz}{(z-a)^n}$, $n = 2, 3, 4, \ldots$ where $z = a$ is inside the simple closed curve C.

Solution

As in Problem 4.21,

$$\oint_C \frac{dz}{(z-a)^n} = \oint_\Gamma \frac{dz}{(z-a)^n}$$
$$= \int_0^{2\pi} \frac{i\epsilon e^{i\theta}\,d\theta}{\epsilon^n e^{in\theta}} = \frac{i}{\epsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta}\,d\theta$$
$$= \frac{i}{\epsilon^{n-1}} \frac{e^{(1-n)i\theta}}{(1-n)i}\bigg|_0^{2\pi} = \frac{1}{(1-n)\epsilon^{n-1}}[e^{2(1-n)\pi i} - 1] = 0$$

where $n \neq 1$.

4.23. Let C be the curve $y = x^3 - 3x^2 + 4x - 1$ joining points (1, 1) and (2, 3). Find the value of $\int_C (12z^2 - 4iz)\,dz$.

Solution

Method 1. By Problem 4.17, the integral is independent of the path joining (1, 1) and (2, 3). Hence, any path can be chosen. In particular, let us choose the straight line paths from (1, 1) to (2, 1) and then from (2, 1) to (2, 3).

Case 1. Along the path from (1, 1) to (2, 1), $y = 1$, $dy = 0$ so that $z = x + iy = x + i$, $dz = dx$. Then, the integral equals

$$\int_{x=1}^{2} \left\{12(x+i)^2 - 4i(x+i)\right\} dx = \left\{4(x+i)^3 - 2i(x+i)^2\right\}\Big|_1^2 = 20 + 30i$$

Case 2. Along the path from (2, 1) to (2, 3), $x = 2$, $dx = 0$ so that $z = x + iy = 2 + iy, dz = i\,dy$. Then, the integral equals

$$\int_{y=1}^{3} \left\{12(2+iy)^2 - 4i(2+iy)\right\} i\,dy = \left\{4(2+iy)^3 - 2i(2+iy)^2\right\}\Big|_1^3 = -176 + 8i$$

Then, adding the required value $= (20 + 30i) + (-176 + 8i) = -156 + 38i$.

Method 2. The given integral equals

$$\int_{1+i}^{2+3i} (12z^2 - 4iz)\,dz = (4z^3 - 2iz^2)\Big|_{1+i}^{2+3i} = -156 + 38i$$

It is clear that Method 2 is easier.

Integrals of Special Functions

4.24. Determine (a) $\int \sin 3z \cos 3z\,dz$, (b) $\int \cot(2z + 5)\,dz$.

Solution

(a) ***Method 1.*** Let $\sin 3z = u$. Then, $du = 3\cos 3z\,dz$ or $\cos 3z\,dz = du/3$. Then

$$\int \sin 3z \cos 3z\,dz = \int u \frac{du}{3} = \frac{1}{3}\int u\,du = \frac{1}{3}\frac{u^2}{2} + c$$
$$= \frac{1}{6}u^2 + c = \frac{1}{6}\sin^2 3z + c$$

Method 2.

$$\int \sin 3z \cos 3z\,dz = \frac{1}{3}\int \sin 3z\,d(\sin 3z) = \frac{1}{6}\sin^2 3z + c$$

Method 3. Let $\cos 3z = u$. Then, $du = -3\sin 3z\,dz$ or $\sin 3z\,dz = -du/3$. Then

$$\int \sin 3z \cos 3z\,dz = -\frac{1}{3}\int u\,du = -\frac{1}{6}u^2 + c_1 = -\frac{1}{6}\cos^2 3z + c_1$$

Note that the results of Methods 1 and 3 differ by a constant.

(b) ***Method 1.***

$$\int \cot(2x + 5)\,dz = \int \frac{\cos(2z+5)}{\sin(2z+5)}\,dz$$

Let $u = \sin(2z + 5)$. Then $du = 2\cos(2z + 5)\,dz$ and $\cos(2z + 5)\,dz = du/2$. Thus

$$\int \frac{\cos(2z+5)\,dz}{\sin(2z+5)} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + c = \frac{1}{2}\ln \sin(2z+5) + c$$

Method 2.

$$\int \cot(2z+5)\,dz = \int \frac{\cos(2z+5)}{\sin(2z+5)}\,dz = \frac{1}{2}\int \frac{d\{\sin(2z+5)\}}{\sin(2z+5)} = \frac{1}{2}\ln \sin(2z+5) + c$$

4.25. (a) Prove that $\int F(z)G'(z)\,dz = F(z)G(z) - \int F'(z)G(z)\,dz$.

(b) Find $\int ze^{2z}\,dz$ and $\int_0^1 ze^{2z}\,dz$.

(c) Find $\int z^2 \sin 4z\,dz$ and $\int_0^{2\pi} z^2 \sin 4z\,dz$.

(d) Evaluate $\int_C (z+2)e^{iz}\,dz$ along the parabola C defined by $\pi^2 y = x^2$ from $(0, 0)$ to $(\pi, 1)$.

Solution

(a) We have

$$d\{F(z)G(z)\} = F(z)G'(z)\,dz + F'(z)G(z)\,dz$$

Integrating both sides yields

$$\int d\{F(z)G(z)\} = F(z)G(z) = \int F(z)G'(z)\,dz + \int F'(z)G(z)\,dz$$

Then

$$\int F(z)G'(z)\,dz = F(z)G(z) - \int F'(z)G(z)\,dz$$

The method is often called *integration by parts*.

(b) Let $F(z) = z$, $G'(z) = e^{2z}$. Then $F'(z) = 1$ and $G(z) = \frac{1}{2}e^{2z}$, omitting the constant of integration. Thus, by part (a),

$$\int ze^{2z}\,dz = \int F(z)G'(z)\,dz = F(z)G(z) - \int F'(z)G(z)\,dz$$
$$= (z)\left(\frac{1}{2}e^{2z}\right) - \int 1\cdot\frac{1}{2}e^{2z}\,dz = \frac{1}{2}ze^{2z} - \frac{1}{4}e^{2z} + c$$

Hence

$$\int_0^1 ze^{2z}\,dz = \left(\frac{1}{2}ze^{2z} - \frac{1}{4}e^{2z} + c\right)\Bigg|_0^1 = \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}(e^2 + 1)$$

(c) Integrating by parts choosing $F(z) = z^2$, $G'(z) = \sin 4z$, we have

$$\int z^2 \sin 4z\,dz = (z^2)\left(-\frac{1}{4}\cos 4z\right) - \int (2z)\left(-\frac{1}{4}\cos 4z\right)dz$$
$$= -\frac{1}{4}z^2\cos 4z + \frac{1}{2}\int z\cos 4z\,dz$$

Integrating this last integral by parts, this time choosing $F(z) = z$ and $G'(z) = \cos 4z$, we find

$$\int z\cos 4z\,dz = (z)\left(\frac{1}{4}\sin 4z\right) - \int (1)\left(\frac{1}{4}\sin 4z\right)dz = \frac{1}{4}z\sin 4z + \frac{1}{16}\cos 4z$$

Hence

$$\int z^2 \sin 4z\,dz = -\frac{1}{4}z^2\cos 4z + \frac{1}{8}z\sin 4z + \frac{1}{32}\cos 4z + c$$

and

$$\int_0^{2\pi} z^2 \sin 4z\,dz = -\pi^2 + \frac{1}{32} - \frac{1}{32} = -\pi^3$$

The double integration by parts can be indicated in a suggestive manner by writing

$$\begin{aligned}\int z^2 \sin 4z\,dz &= (z^2)\left(-\frac{1}{4}\cos 4z\right) - (2z)\left(-\frac{1}{16}\sin 4z\right) + (2)\left(\frac{1}{64}\cos 4z\right) + c\\ &= -\frac{1}{4}z^2\cos 4z + \frac{1}{8}z\sin 4z + \frac{1}{32}\cos 4z\end{aligned}$$

where the first parentheses in each term (after the first) is obtained by differentiating z^2 successively, the second parentheses is obtained by integrating sin $4z$ successively, and the terms alternate in sign.

(d) The points (0, 0) and (π, 1) correspond to $z = 0$ and $z = \pi + i$. Since $(z+2)e^{iz}$ is analytic, we see by Problem 4.17 that the integral is independent of the path and is equal to

$$\begin{aligned}\int_0^{1+i} (z+2)e^{iz}dz &= \left\{(z+2)\left(\frac{e^{iz}}{i}\right) - (1)(-e^{iz})\right\}\Bigg|_0^{\pi+i}\\ &= (\pi+i+2)\left(\frac{e^{i(\pi+i)}}{i}\right) + e^{i(\pi+i)} - \frac{2}{i} - 1\\ &= -2e^{-1} - 1 + i(2 + \pi e^{-1} + 2e^{-1})\end{aligned}$$

4.26. Show that $\displaystyle\int \frac{dz}{z^2+a^2} = \frac{1}{a}\tan^{-1}\frac{z}{a} + c_1 = \frac{1}{2ai}\ln\left(\frac{z-ai}{z+ai}\right) + c_2.$

Solution

Let $z = a\tan u$. Then

$$\int \frac{dz}{z^2+a^2} = \int \frac{a\sec^2 u\,du}{a^2(\tan^2 u + 1)} = \frac{1}{a}\int du = \frac{1}{a}\tan^{-1}\frac{z}{a} + c_1$$

Also,

$$\frac{1}{z^2+a^2} = \frac{1}{(z-ai)(z+ai)} = \frac{1}{2ai}\left(\frac{1}{z-ai} - \frac{1}{z+ai}\right)$$

and so

$$\begin{aligned}\int \frac{dz}{z^2+a^2} &= \frac{1}{2ai}\int\frac{dz}{z-ai} - \frac{1}{2ai}\int\frac{dz}{z+ai}\\ &= \frac{1}{2ai}\ln(z-ai) - \frac{1}{2ai}\ln(z+ai) + c_2 = \frac{1}{2ai}\ln\left(\frac{z-ai}{z+ai}\right) + c_2\end{aligned}$$

Miscellaneous Problems

4.27. Prove Morera's theorem [page 115] under the assumption that $f(z)$ has a continuous derivative in $\mathcal{R}$.

Solution

If $f(z)$ has a continuous derivative in $\mathcal{R}$, then we can apply Green's theorem to obtain

$$\oint_C f(z)\,dz = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy$$
$$= \iint_{\mathcal{R}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_{\mathcal{R}} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Then, if $\oint_C f(z)\,dz = 0$ around every closed path C in $\mathcal{R}$, we must have

$$\oint_C u\,dx - v\,dy = 0, \qquad \oint_C v\,dx + u\,dy = 0$$

around every closed path C in $\mathcal{R}$. Hence, from Problem 4.8, the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

are satisfied and thus (since these partial derivatives are continuous) it follows [Problem 3.5] that $u + iv = f(z)$ is analytic.

4.28. A force field is given by $F = 3z + 5$. Find the work done in moving an object in this force field along the parabola $z = t^2 + it$ from $z = 0$ to $z = 4 + 2i$.

Solution

$$\text{Total work done} = \int_C F \cdot dz = \text{Re}\int_C \overline{F}\,dz = \text{Re}\left\{\int_C (3\bar{z} + 5)\,dz\right\}$$
$$= \text{Re}\left\{3\int_C \bar{z}\,dz + 5\int_C dz\right\} = \text{Re}\left\{3\left(10 - \frac{1}{2}i\right) + 5(4 + 2i)\right\} = 50$$

using the result of Problem 4.2.

4.29. Find: (a) $\int e^{ax}\sin bx\,dx$, (b) $\int e^{ax}\cos bx\,dx$.

Solution

Omitting the constant of integration, we have

$$\int e^{(a+ib)x}\,dx = \frac{e^{(a+ib)x}}{a + ib}$$

which can be written

$$\int e^{ax}(\cos bx + i\sin bx)\,dx = \frac{e^{ax}(\cos bx + i\sin bx)}{a + ib} = \frac{e^{ax}(\cos bx + i\sin bx)(a - ib)}{a^2 + b^2}$$

Then equating real and imaginary parts,

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$
$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

4.30. Give an example of a continuous, closed, non-intersecting curve that lies in a bounded region $\mathcal{R}$ but which has an infinite length.

Solution

Consider equilateral triangle *ABC* [Fig. 4-21] with sides of unit length. By trisecting each side, construct equilateral triangles *DEF*, *GHJ*, and *KLM*. Then omitting sides *DF*, *GJ*, and *KM*, we obtain the closed non-intersecting curve *ADEFBGHJCKLMA* of Fig. 4-22.

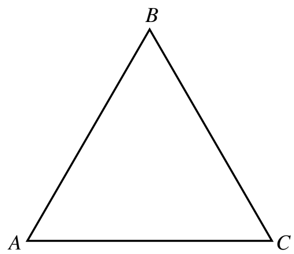

Fig. 4-21

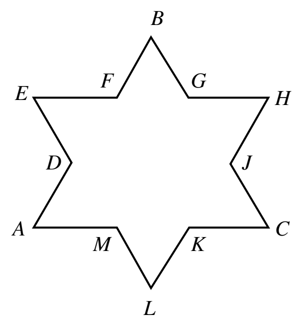

Fig. 4-22

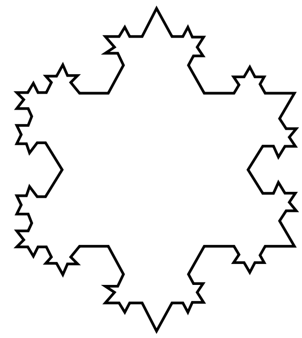

Fig. 4-23

The process can now be continued by trisecting sides *DE*, *EF*, *FB*, *BG*, *GH*, etc., and constructing equilateral triangles as before. By repeating the process indefinitely [see Fig. 4-23], we obtain a continuous closed non-intersecting curve that is the boundary of a region with finite area equal to

$$\frac{1}{4}\sqrt{3} + (3)\left(\frac{1}{3}\right)^2\frac{\sqrt{3}}{4} + (9)\left(\frac{1}{9}\right)^2\frac{\sqrt{3}}{4} + (27)\left(\frac{1}{27}\right)^2\frac{\sqrt{3}}{4} + \cdots$$
$$= \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right) = \frac{\sqrt{3}}{4}\,\frac{1}{1-1/3} = \frac{3\sqrt{3}}{8}$$

or 1.5 times the area of triangle *ABC*, and which has infinite length (see Problem 4.91).

4.31. Let $F(x, y)$ and $G(x, y)$ be continuous and have continuous first and second partial derivatives in a simply-connected region $\mathcal{R}$ bounded by a simple closed curve C. Prove that

$$\oint_C F\left(\frac{\partial G}{\partial y}dx - \frac{\partial G}{\partial x}dy\right) = -\iint_{\mathcal{R}}\left[F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + \left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y}\right)\right]dx\,dy$$

Solution

Let $P = F\dfrac{\partial G}{\partial y}$, $Q = -F\dfrac{\partial G}{\partial x}$ in Green's theorem so

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Then as required

$$\oint_C F\left(\frac{\partial G}{\partial y}\,dx - \frac{\partial G}{\partial x}\,dy\right) = \iint_{\mathcal{R}} \left(\frac{\partial}{\partial x}\left\{-F\frac{\partial G}{\partial x}\right\} - \frac{\partial}{\partial y}\left\{F\frac{\partial G}{\partial y}\right\}\right) dx\,dy$$

$$= -\iint_{\mathcal{R}} \left[F\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2}\right) + \left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y}\right)\right] dx\,dy$$

SUPPLEMENTARY PROBLEMS

Line Integrals

4.32. Evaluate $\int_{(0,1)}^{(2,5)} (3x + y)\,dx + (2y - x)\,dy$ along (a) the curve $y = x^2 + 1$, (b) the straight line joining (0, 1) and (2, 5), (c) the straight lines from (0, 1) to (0, 5) and then from (0, 5) to (2, 5), (d) the straight lines from (0, 1) to (2, 1) and then from (2, 1) to (2, 5).

4.33. (a) Evaluate $\oint_C (x + 2y)\,dx + (y - 2x)\,dy$ around the ellipse C defined by $x = 4\cos\theta$, $y = 3\sin\theta$, $0 \le \theta < 2\pi$ if C is described in a counterclockwise direction.
(b) What is the answer to (a) if C is described in a clockwise direction?

4.34. Evaluate $\int_C (x^2 - iy^2)\,dz$ along (a) the parabola $y = 2x^2$ from (1, 2) to (2, 8), (b) the straight lines from (1, 1) to (1, 8) and then from (1, 8) to (2, 8), (c) the straight line from (1, 1) to (2, 8).

4.35. Evaluate $\oint_C |z|^2\,dz$ around the square with vertices at (0, 0), (1, 0), (1, 1), (0, 1).

4.36. Evaluate $\int_C (z^2 + 3z)\,dz$ along (a) the circle $|z| = 2$ from (2, 0) to (0, 2) in a counterclockwise direction, (b) the straight line from (2, 0) to (0, 2), (c) the straight lines from (2, 0) to (2, 2) and then from (2, 2) to (0, 2).

4.37. Suppose $f(z)$ and $g(z)$ are integrable. Prove that

$$\text{(a)}\ \int_a^b f(z)\,dz = -\int_b^a f(z)\,dz, \quad \text{(b)}\ \int_C \{2f(z) - 3ig(z)\}\,dz = 2\int_C f(z)\,dz - 3i\int_C g(z)\,dz.$$

4.38. Evaluate $\int_i^{2-i} (3xy + iy^2)\,dz$ (a) along the straight line joining $z = i$ and $z = 2 - i$,
(b) along the curve $x = 2t - 2$, $y = 1 + t - t^2$.

4.39. Evaluate $\oint_C \bar{z}^2\,dz$ around the circles (a) $|z| = 1$, (b) $|z - 1| = 1$.

4.40. Evaluate $\oint_C (5z^4 - z^3 + 2)\,dz$ around (a) the circle $|z| = 1$, (b) the square with vertices at (0, 0), (1, 0), (1, 1), and (0, 1), (c) the curve consisting of the parabolas $y = x^2$ from (0, 0) to (1, 1) and $y^2 = x$ from (1, 1) to (0, 0).

4.41. Evaluate $\int_C (z^2 + 1)^2\,dz$ along the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ from the point where $\theta = 0$ to the point where $\theta = 2\pi$.

4.42. Evaluate $\int_C \bar{z}^2\,dz + z^2\,d\bar{z}$ along the curve C defined by $z^2 + 2z\bar{z} + \bar{z}^2 = (2 - 2i)z + (2 + 2i)\bar{z}$ from the point $z = 1$ to $z = 2 + 2i$.

4.43. Evaluate $\oint_C dz/z - 2$ around
(a) the circle $|z - 2| = 4$, (b) the circle $|z - 1| = 5$, (c) the square with vertices at $3 \pm 3i$, $-3 \pm 3i$.

4.44. Evaluate $\oint_C (x^2 + iy^2)\,ds$ around the circle $|z| = 2$ where s is the arc length.

Green's Theorem in the Plane

4.45. Verify Green's theorem in the plane for $\oint_C (x^2 - 2xy)\,dx + (y^2 - x^3 y)\,dy$ where C is a square with vertices at (0, 0), (2, 0), (2, 2), and (0, 2).

4.46. Evaluate $\oint_C (5x + 6y - 3)\,dx + (3x - 4y + 2)\,dy$ around a triangle in the xy plane with vertices at (0, 0), (4, 0), and (4, 3).

4.47. Let C be any simple closed curve bounding a region having area A. Prove that

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

4.48. Use the result of Problem 4.47 to find the area bounded by the ellipse $x = a\cos\theta,\ y = b\sin\theta,\ 0 \le \theta < 2\pi$.

4.49. Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ shown shaded in Fig. 4-24. [*Hint*. Parametric equations are $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta < 2\pi$.]

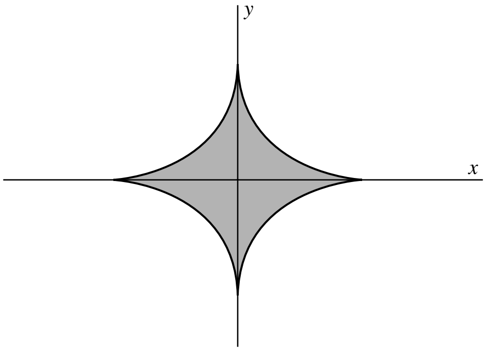

Fig. 4-24

4.50. Verify Green's theorem in the plane for $\oint_C x^2y\,dx + (y^3 - xy^2)\,dy$ where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 4$, $x^2 + y^2 = 16$.

4.51. (a) Prove that $\oint_C (y^2\cos x - 2e^y)\,dx + (2y\sin x - 2xe^y)\,dy = 0$ around any simple closed curve C.

(b) Evaluate the integral in (a) along the parabola $y = x^2$ from (0, 0) to (π, π^2).

4.52. (a) Show that $\int_{(2,1)}^{(3,2)} (2xy^3 - 2y^2 - 6y)\,dx + (3x^2y^2 - 4xy - 6x)\,dy$ is independent of the path joining points (2, 1) and (3, 2). (b) Evaluate the integral in (a).

Complex Form of Green's Theorem

4.53. If C is a simple closed curve enclosing a region of area A, prove that $A = \dfrac{1}{2i}\oint_C \bar{z}\,dz$.

4.54. Evaluate $\oint_C \bar{z}\,dz$ around (a) the circle $|z - 2| = 3$, (b) the square with vertices at $z = 0, 2, 2i$, and $2 + 2i$, (c) the ellipse $|z - 3| + |z + 3| = 10$.

4.55. Evaluate $\oint_C (8\bar{z} + 3z)\,dz$ around the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.

4.56. Let $P(z, \bar{z})$ and $Q(z, \bar{z})$ be continuous and have continuous partial derivatives in a region $\mathcal{R}$ and on its boundary C. Prove that

$$\oint_C P(z, \bar{z})\,dz + Q(z, \bar{z})\,d\bar{z} = 2i\iint_{\mathcal{R}} \left(\frac{\partial P}{\partial \bar{z}} - \frac{\partial Q}{\partial z}\right) dA$$

4.57. Show that the area in Problem 4.53 can be written in the form $A = \dfrac{1}{4i}\oint_C \bar{z}\,dz - z\,d\bar{z}$.

4.58. Show that the centroid of the region of Problem 4.53 is given in conjugate coordinates by $(\hat{z}, \hat{\bar{z}})$ where

$$\hat{z} = -\frac{1}{4Ai}\oint_C z^2\,d\bar{z}, \qquad \hat{\bar{z}} = \frac{1}{4Ai}\oint_C \bar{z}^2\,dz$$

4.59. Find the centroid of the region bounded above by $|z| = a > 0$ and below by $\operatorname{Im} z = 0$.

Cauchy's Theorem and the Cauchy–Goursat Theorem

4.60. Verify Cauchy's theorem for the functions (a) $3z^2 + iz - 4$, (b) $5\sin 2z$, (c) $3\cosh(z + 2)$ where C is the square with vertices at $1 \pm i, -1 \pm i$.

4.61. Verify Cauchy's theorem for the function $z^3 - iz^2 - 5z + 2i$ if C is
(a) the circle $|z| = 1$, (b) the circle $|z - 1| = 2$, (c) the ellipse $|z - 3i| + |z + 3i| = 20$.

4.62. Let C be the circle $|z - 2| = 5$. (a) Determine whether $\oint_C \frac{dz}{z-3} = 0$. (b) Does your answer to (a) contradict Cauchy's theorem?

4.63. For any simple closed curve C, explain clearly the relationship between the observations

$$\oint_C (x^2 - y^2 + 2y)\,dx + (2x - 2xy)\,dy = 0 \quad \text{and} \quad \oint_C (z^2 - 2iz)\,dz = 0$$

4.64. By evaluating $\oint_C e^z\,dz$ around the circle $|z| = 1$, show that

$$\int_0^{2\pi} e^{\cos\theta}\cos(\theta + \sin\theta)\,d\theta = \int_0^{2\pi} e^{\cos\theta}\sin(\theta + \sin\theta)\,d\theta = 0$$

4.65. State and prove Cauchy's theorem for multiply-connected regions.

4.66. Prove the Cauchy–Goursat theorem for a polygon, such as $ABCDEFGA$ shown in Fig. 4-25, which may intersect itself.

4.67. Prove the Cauchy–Goursat theorem for the multiply-connected region $\mathcal{R}$ shown shaded in Fig. 4-26.

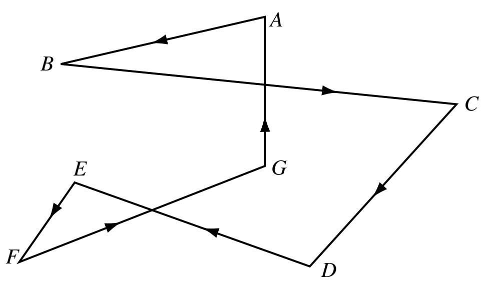

Fig. 4-25

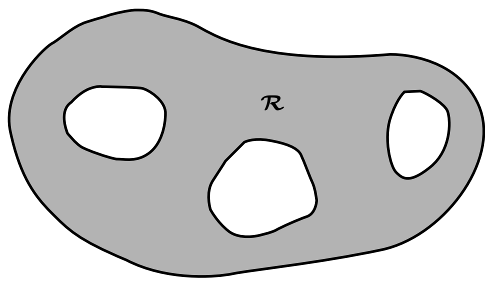

Fig. 4-26

4.68. (a) Prove the Cauchy–Goursat theorem for a rectangle and (b) show how the result of (a) can be used to prove the theorem for any simple closed curve C.

4.69. Let P and Q be continuous and have continuous first partial derivatives in a region $\mathcal{R}$. Let C be any simple closed curve in $\mathcal{R}$ and suppose that for any such curve

$$\oint_C P\,dx + Q\,dy = 0$$

(a) Prove that there exists an analytic function $f(z)$ such that $\text{Re}\{f(z)\,dz\} = P\,dx + Q\,dy$ is an exact differential.

(b) Determine p and q in terms of P and Q such that $\text{Im}\{f(z)\,dz\} = p\,dx + q\,dy$ and verify that $\oint_C p\,dx + q\,dy = 0$.

(c) Discuss the connection between (a) and (b) and Cauchy's theorem.

4.70. Illustrate the results of Problem 4.69 if $P = 2x + y - 2xy$, $Q = x - 2y - x^2 + y^2$ by finding p, q, and $f(z)$.

4.71. Let P and Q be continuous and have continuous partial derivatives in a region $\mathcal{R}$. Suppose that for any simple closed curve C in $\mathcal{R}$, we have $\oint_C P\,dx + Q\,dy = 0$.
(a) Prove that $\oint_C Q\,dx - P\,dy = 0$. (b) Discuss the relationship of (a) with Cauchy's theorem.

Consequences of Cauchy's Theorem

4.72. Show directly that $\int_{3+4i}^{4-3i}(6z^2+8iz)\,dz$ has the same value along the following paths C joining the points $3+4i$ and $4-3i$: (a) a straight line, (b) the straight lines from $3+4i$ to $4+4i$ and then from $4+4i$ to $4-3i$, (c) the circle $|z|=5$. Determine this value.

4.73. Show that $\int_C e^{-2z}\,dz$ is independent of the path C joining the points $1-\pi i$ and $2+3\pi i$ and determine its value.

4.74. Given $G(z)=\int_{\pi-\pi i}^{z}\cos 3\zeta\,d\zeta$. (a) Prove that $G(z)$ is independent of the path joining $\pi-\pi i$ and the arbitrary point z. (b) Determine $G(\pi i)$. (c) Prove that $G'(z)=\cos 3z$.

4.75. Given $G(z)=\int_{1+i}^{z}\sin\zeta^2\,d\zeta$. (a) Prove that $G(z)$ is an analytic function of z. (b) Prove that $G'(z)=\sin z^2$.

4.76. For the real line integral $\int_C P\,dx+Q\,dy$, state and prove a theorem corresponding to:
(a) Problem 4.17, (b) Problem 4.18, (c) Problem 4.20.

4.77. Prove Theorem 4.5, page 118 for the region of Fig. 4-26.

4.78. (a) If C is the circle $|z|=R$, show that $\lim_{R\to\infty}\oint_C \dfrac{z^2+2z-5}{(z^2+4)(z^2+2z+2)}\,dz=0$

(b) Use the result of (a) to deduce that if C_1 is the circle $|z-2|=5$, then

$$\oint_{C_1}\frac{z^2+2z-5}{(z^2+4)(z^2+2z+2)}\,dz=0$$

(c) Is the result in (b) true if C_1 is the circle $|z+1|=2$? Explain.

Integrals of Special Functions

4.79. Find each of the following integrals:

(a) $\int e^{-2z}\,dz$, (b) $\int z\sin z^2\,dz$, (c) $\int \dfrac{z^2+1}{z^3+3z+2}\,dz$, (d) $\int \sin^4 2z\cos 2z\,dz$, (e) $\int z^2\tanh(4z^3)\,dz$

4.80. Find each of the following integrals:

(a) $\int z\cos 2z\,dz$, (b) $\int z^2e^{-z}\,dz$, (c) $\int z\ln z\,dz$, (d) $\int z^3\sinh z\,dz$.

4.81. Evaluate each of the following: (a) $\int_{\pi i}^{2\pi i} e^{3z}\,dz$, (b) $\int_0^{\pi i}\sinh 5z\,dz$, (c) $\int_0^{\pi+i} z\cos 2z\,dz$.

4.82. Show that $\int_0^{\pi/2}\sin^2 z\,dz=\int_0^{\pi/2}\cos^2 z\,dz=\pi/4$.

4.83. Show that $\int\dfrac{dz}{z^2-a^2}=\dfrac{1}{2a}\ln\left(\dfrac{z-a}{z+a}\right)+c_1=\dfrac{1}{a}\coth^{-1}\dfrac{z}{a}+c_2$.

4.84. Show that if we restrict ourselves to the same branch of the square root,

$$\int z\sqrt{2z+5}\,dz=\frac{1}{20}(2z+5)^{5/2}-\frac{5}{6}(2z+5)^{3/2}+c$$

4.85. Evaluate $\int\sqrt{1+\sqrt{z+1}}\,dz$, stating conditions under which your result is valid.

Miscellaneous Problems

4.86. Use the definition of an integral to prove that along any arbitrary path joining points a and b,

(a) $\int_a^b dz=b-a$, (b) $\int_a^b z\,dz=\dfrac{1}{2}(b^2-a^2)$.

4.87. Prove the theorem concerning change of variable on page XX. [*Hint.* Express each side as two real line integrals and use the Cauchy–Riemann equations.]

4.88. Let $u(x, y)$ be harmonic and have continuous derivatives, of order two at least, in a region $\mathcal{R}$.

(a) Show that the following integral is independent of the path in $\mathcal{R}$ joining (a, b) to (x, y):

$$v(x, y) = \int_{(a,b)}^{(x,y)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

(b) Prove that $u + iv$ is an analytic function of $z = x + iy$ in $\mathcal{R}$.

(c) Prove that v is harmonic in $\mathcal{R}$.

4.89. Work Problem 4.88 for the special cases (a) $u = 3x^2y + 2x^2 - y^3 - 2y^2$, (b) $u = xe^x \cos y - ye^x \sin y$. [See Problem 4.53(a) and (c), page XX.]

4.90. Using the definition of an integral, verify directly that when C is a simple closed curve and z_0 is any constant.

(a) $\oint_C dz = 0$, (b) $\oint_C z\,dz = 0$, (c) $\oint_C (z - z_0)\,dz = 0$

4.91. Find the length of the closed curve of Problem 4.30 after n steps and verify that as $n \to \infty$, the length of the curve becomes infinite.

4.92. Evaluate $\int_C \frac{dz}{z^2+4}$ along the line $x + y = 1$ in the direction of increasing x.

4.93. Show that $\int_0^\infty xe^{-x} \sin x\, dx = \frac{1}{2}$.

4.94. Evaluate $\int_{-2-2\sqrt{3}i}^{-2+2\sqrt{3}i} z^{1/2}\,dz$ along a straight line path if we choose that branch of $z^{1/2}$ such that $z^{1/2} = 1$ for $z = 1$.

4.95. Does Cauchy's theorem hold for the function $f(z) = z^{1/2}$ where C is the circle $|z| = 1$? Explain.

4.96. Does Cauchy's theorem hold for a curve, such as *EFGHFJE* in Fig. 4-27, which intersects itself? Justify your answers.

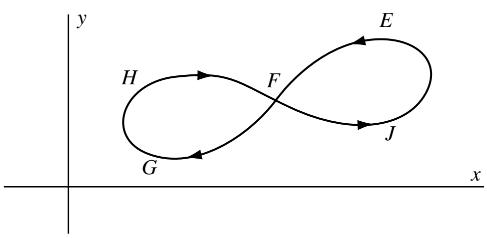

Fig. 4-27

4.97. If n is the direction of the outward drawn normal to a simple closed curve C, s is the arc length parameter and U is any continuously differentiable function, prove that

$$\frac{\partial U}{\partial n} = \frac{\partial U}{\partial x}\frac{dx}{ds} + \frac{\partial U}{\partial y}\frac{dy}{ds}$$

4.98. Prove *Green's first identity*,

$$\iint_{\mathcal{R}} U\nabla^2 V\,dx\,dy + \iint_{\mathcal{R}} \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y}\right) dx\,dy = \oint_C U\frac{\partial V}{\partial n}ds$$

where $\mathcal{R}$ is the region bounded by the simple closed curve C, $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, while n and s are as in Problem 4.97.

4.99. Use Problem 4.98 to prove *Green's second identity*

$$\iint_{\mathcal{R}} (U\nabla^2 V - V\nabla^2 U)\,dA = \oint_C \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)ds$$

where dA is an element of area of $\mathcal{R}$.

4.100. Write the result of Problem 4.31 in terms of the operator ∇.