2.55. (a) $e^{2\pi i/3}$, (b) $e^{4\pi i/3}$, 1, $e^{2\pi i/3}$

2.61. (a) $2k\pi i/3$, (b) $(1/8)\pi i + (1/2)k\pi i$, where $k = \pm 1, \pm 2, \ldots$

2.64. (a) 7, (b) 26

2.68. (a) $u = e^{-3y}\cos 3x$, $v = e^{-3y}\sin 3x$. (b) $u = \cos x \cosh y$, $v = -\sin x \sinh y$. (c) $u = \sin 2x \cosh 2y$, $v = \cos 2x \sinh 2y$. (d) $u = e^{2x}\{(x^2 - y^2)\cos 2y - 2xy \sin 2y\}$, $v = e^{2x}\{2xy \cos 2y + (x^2 - y^2)\sin 2y\}$.

2.72. (a) $u = \sinh 2x \cos 2y$, $v = \cosh 2x \sin 2y$

(b) $u = x \cosh x \cos y - y \sinh x \sin y$, $v = y \cosh x \cos y + x \sinh x \sin y$

2.73. (a) $2i\sqrt{3}$, (b) 0, (c) i

2.74. (b) $4\pi i/3$

2.75. (a) $2\ln 2 + (\pi + 2k\pi)i$, $2\ln 2 + \pi i$. (b) $\ln 3 + (\pi/2 + 2k\pi)i$, $\ln 3 + \pi i/2$. (c) $\ln 2 + (11\pi/6 + 2k\pi)i$, $\ln 2 + 11\pi i/6$

2.79. (a) $\pm\ln(2+\sqrt{3}) + \pi/2 + 2k\pi$ (b) $-i\ln(\sqrt{2}+1) + \pi/2 + 2k\pi$, $-i\ln(\sqrt{2}-1) + 3\pi/2 + 2k\pi$

2.80. (a) $\ln(\sqrt{2}+1) + \pi i/2 + 2k\pi i$, $\ln(\sqrt{2}-1) + 3\pi i/2 + 2k\pi i$

(b) $\ln\left[(2k+1)\pi + \sqrt{(2k+1)^2\pi^2 - 1}\right] + \pi i/2 + 2m\pi i$,

$\ln\left[\sqrt{(2k+1)^2\pi^2 - 1} - (2k+1)\pi\right] + 3\pi i/2 + 2m\pi i$, $k, m = 0, \pm 1, \pm 2, \ldots$

2.81. (a) $e^{-\pi/4 + 2k\pi}\left\{\cos\left(\frac{1}{2}\ln 2\right) + i\sin\left(\frac{1}{2}\ln 2\right)\right\}$, (b) $\cos(2\sqrt{2}k\pi) + i\sin(2\sqrt{2}k\pi)$

2.82. (a) $e^{1/2\ln 2 - 7\pi/4 - 2k\pi}\cos\left(7\pi/4 + \frac{1}{2}\ln 2\right)$, (b) $e^{3\pi/2 + 2k\pi}$

2.94. (a) $-12 + 6i$, (b) $\sqrt{2}(1+i)/2$, (c) $-4/3 - 4i$, (d) $1/3$, (e) $-1/4$

2.95. $1/6 - i\sqrt{3}/6$

2.104. $e^{(2k+1)\pi i/4}$, $k = 0, 1, 2, 3$

2.107. (a) $-1 \pm i$ (b) $\pm 2, \pm 2i$ (c) $k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ (d) 0, $\left(k + \dfrac{1}{2}\right)\pi$, $k = 0, \pm 1, \pm 2, \ldots$ (e) $\pm i$, $\left(k + \frac{1}{2}\right)\pi i$, $k = 0, \pm 1, \pm 2, \ldots$

2.123. (a) $\frac{1}{2}i$, (b) 1, (c) 0, (d) $\frac{1}{2}i$

2.125. $(9 + 3i)/10$

2.128. Converges

2.141. (a) 0, (b) $(2k+1)\pi i/2$, $k = 0, \pm 1, \pm 2, \ldots$

CHAPTER 3

Complex Differentiation and the Cauchy–Riemann Equations

3.1 Derivatives

If $f(z)$ is single-valued in some region $\mathcal{R}$ of the z plane, the *derivative* of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{3.1}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such a case, we say that $f(z)$ is *differentiable* at z. In the definition (3.1), we sometimes use h instead of Δz. Although differentiability implies continuity, the reverse is not true (see Problem 3.4).

3.2 Analytic Functions

If the derivative $f'(z)$ exists at all points z of a region $\mathcal{R}$, then $f(z)$ is said to be *analytic in* $\mathcal{R}$ and is referred to as an *analytic function in* $\mathcal{R}$ or a function *analytic in* $\mathcal{R}$. The terms *regular* and *holomorphic* are sometimes used as synonyms for analytic.

A function $f(z)$ is said to be *analytic at a point* z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

3.3 Cauchy–Riemann Equations

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\mathcal{R}$ is that, in $\mathcal{R}$, u and v satisfy the *Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.2}$$

If the partial derivatives in (3.2) are continuous in $\mathcal{R}$, then the Cauchy–Riemann equations are sufficient conditions that $f(z)$ be analytic in $\mathcal{R}$. See Problem 3.5.

The functions $u(x, y)$ and $v(x, y)$ are sometimes called *conjugate functions*. Given u having continuous first partials on a simply connected region $\mathcal{R}$ (see Section 4.6), we can find v (within an arbitrary additive constant) so that $u + iv = f(z)$ is analytic (see Problems 3.7 and 3.8).

3.4 Harmonic Functions

If the second partial derivatives of u and v with respect to x and y exist and are continuous in a region $\mathcal{R}$, then we find from (3.2) that (see Problem 3.6)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \tag{3.3}$$

It follows that under these conditions, the real and imaginary parts of an analytic function satisfy *Laplace's equation* denoted by

$$\frac{\partial^2 \Psi}{\partial^2 x}+\frac{\partial^2 \Psi}{\partial^2 y}=0 \quad \text{or} \quad \nabla^2\Psi=0 \quad \text{where } \nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{3.4}$$

The operator ∇^2 is often called the *Laplacian*.

Functions such as $u(x, y)$ and $v(x, y)$ which satisfy Laplace's equation in a region $\mathcal{R}$ are called *harmonic functions* and are said to be *harmonic in* $\mathcal{R}$.

3.5 Geometric Interpretation of the Derivative

Let z_0 [Fig. 3-1] be a point P in the z plane and let w_0 [Fig. 3-2] be its image P' in the w plane under the transformation $w = f(z)$. Since we suppose that $f(z)$ is single-valued, the point z_0 maps into only one point w_0.

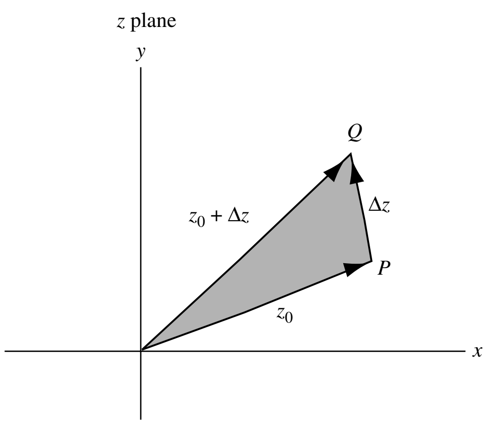

Fig. 3-1

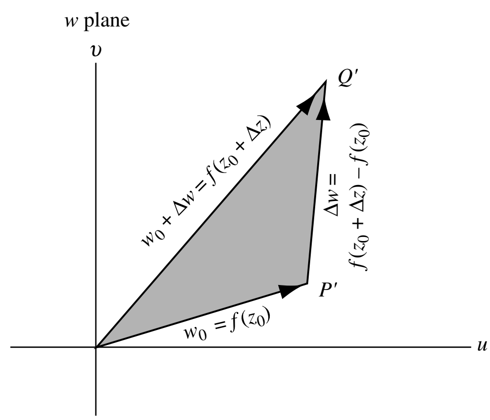

Fig. 3-2

If we give z_0 an increment Δz, we obtain the point Q of Fig. 3-1. This point has image Q' in the w plane. Thus, from Fig. 3-2, we see that $P'Q'$ represents the complex number $\Delta w = f(z_0 + \Delta z) - f(z_0)$. It follows that the derivative at z_0 (if it exists) is given by

$$\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{Q \to P} \frac{P'Q'}{PQ} \tag{3.5}$$

that is, the limit of the ratio $P'Q'$ to PQ as point Q approaches point P. The above interpretation clearly holds when z_0 is replaced by any point z.

3.6 Differentials

Let $\Delta z = dz$ be an increment given to z. Then

$$\Delta w = f(z + \Delta z) - f(z) \tag{3.6}$$

is called the increment in $w = f(z)$. If $f(z)$ is continuous and has a continuous first derivative in a region, then

$$\Delta w = f'(z)\Delta z + \epsilon\Delta z = f'(z)\,dz + \epsilon\,dz \tag{3.7}$$

where $\epsilon \to 0$ as $\Delta z \to 0$. The expression

$$dw = f'(z)\,dz \tag{3.8}$$

is called the *differential of w or f(z)*, or the *principal part* of Δw. Note that $\Delta w \neq dw$ in general. We call dz the *differential of z*.

Because of the definitions (3.1) and (3.8), we often write

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta x \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \tag{3.9}$$

It is emphasized that dz and dw are not the limits of Δz and Δw as $\Delta z \to 0$, since these limits are zero whereas dz and dw are not necessarily zero. Instead, given dz, we determine dw from (3.8), i.e., dw is a dependent variable determined from the independent variable dz for a given z.

It is useful to think of d/dz as being an *operator* that, when operating on $w = f(z)$, leads to $dw/dz = f'(z)$.

3.7 Rules for Differentiation

Suppose $f(z)$, $g(z)$, and $h(z)$ are analytic functions of z. Then the following differentiation rules (identical with those of elementary calculus) are valid.

1. $\dfrac{d}{dz}\{f(z) + g(z)\} = \dfrac{d}{dz}f(z) + \dfrac{d}{dz}g(z) = f'(z) + g'(z)$
2. $\dfrac{d}{dz}\{f(z) - g(z)\} = \dfrac{d}{dz}f(z) - \dfrac{d}{dz}g(z) = f'(z) - g'(z)$
3. $\dfrac{d}{dz}\{cf(z)\} = c\dfrac{d}{dz}f(z) = cf'(z)$ where c is any constant
4. $\dfrac{d}{dz}\{f(z)g(z)\} = f(z)\dfrac{d}{dz}g(z) + g(z)\dfrac{d}{dz}f(z) = f(z)g'(z) + g(z)f'(z)$
5. $\dfrac{d}{dz}\left\{\dfrac{f(z)}{g(z)}\right\} = \dfrac{g(z)(d/dz)f(z) - f(z)(d/dz)g(z)}{[g(z)]^2} = \dfrac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$ if $g(z) \neq 0$
6. If $w = f(\zeta)$ where $\zeta = g(z)$ then

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz} = f'(\zeta)\frac{d\zeta}{dz} = f'\{g(z)\}g'(z) \tag{3.10}$$

Similarly, if $w = f(\zeta)$ where $\zeta = g(\eta)$ and $\eta = h(z)$, then

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{d\eta}\cdot\frac{d\eta}{dz} \tag{3.11}$$

The results (3.10) and (3.11) are often called *chain rules* for differentiation of composite functions.

7. If $w = f(z)$ has a single-valued inverse f^{-1}, then $z = f^{-1}(w)$, and dw/dz and dz/dw are related by

$$\frac{dw}{dz} = \frac{1}{dz/dw} \tag{3.12}$$

8. If $z = f(t)$ and $w = g(t)$ where t is a parameter, then

$$\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{g'(t)}{f'(t)} \tag{3.13}$$

Similar rules can be formulated for differentials. For example,

$$d\{f(z) + g(z)\} = df(z) + dg(z) = f'(z)\,dz + g'(z)\,dz = \{f'(z) + g'(z)\}\,dz$$
$$d\{f(z)g(z)\} = f(z)\,dg(z) + g(z)\,df(z) = \{f(z)g'(z) + g(z)f'(z)\}\,dz$$

3.8 Derivatives of Elementary Functions

In the following, we assume that the functions are defined as in Chapter 2. In the cases where functions have branches, i.e., are multi-valued, the branch of the function on the right is chosen so as to correspond to the branch of the function on the left. Note that the results are identical with those of elementary calculus.

1. $\frac{d}{dz}(c) = 0$
2. $\frac{d}{dz}z^n = nz^{n-1}$
3. $\frac{d}{dz}e^z = e^z$
4. $\frac{d}{dz}a^z = a^z \ln a$
5. $\frac{d}{dz}\sin z = \cos z$
6. $\frac{d}{dz}\cos z = -\sin z$
7. $\frac{d}{dz}\tan z = \sec^2 z$
8. $\frac{d}{dz}\cot z = -\csc^2 z$
9. $\frac{d}{dz}\sec z = \sec z \tan z$
10. $\frac{d}{dz}\csc z = -\csc z \cot z$
11. $\frac{d}{dz}\log_e z = \frac{d}{dz}\ln z = \frac{1}{z}$
12. $\frac{d}{dz}\log_a z = \frac{\log_a e}{z}$
13. $\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1-z^2}}$
14. $\frac{d}{dz}\cos^{-1} z = \frac{-1}{\sqrt{1-z^2}}$
15. $\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$
16. $\frac{d}{dz}\cot^{-1} z = \frac{-1}{1+z^2}$
17. $\frac{d}{dz}\sec^{-1} z = \frac{1}{z\sqrt{z^2-1}}$
18. $\frac{d}{dz}\csc^{-1} z = \frac{-1}{z\sqrt{z^2-1}}$
19. $\frac{d}{dz}\sinh z = \cosh z$
20. $\frac{d}{dz}\cosh z = \sinh z$
21. $\frac{d}{dz}\tanh z = \operatorname{sech}^2 z$
22. $\frac{d}{dz}\coth z = -\operatorname{csch}^2 z$
23. $\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z$
24. $\frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z$
25. $\frac{d}{dz}\sinh^{-1} z = \frac{1}{\sqrt{1+z^2}}$
26. $\frac{d}{dz}\cosh^{-1} z = \frac{1}{\sqrt{z^2-1}}$
27. $\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}$
28. $\frac{d}{dz}\coth^{-1} z = \frac{1}{1-z^2}$
29. $\frac{d}{dz}\operatorname{sech}^{-1} z = \frac{-1}{z\sqrt{1-z^2}}$
30. $\frac{d}{dz}\operatorname{csch}^{-1} z = \frac{-1}{z\sqrt{z^2+1}}$

3.9 Higher Order Derivatives

If $w = f(z)$ is analytic in a region, its derivative is given by $f'(z)$, w', or dw/dz. If $f'(z)$ is also analytic in the region, its derivative is denoted by $f''(z)$, w'', or $(d/dz)(dw/dz) = d^2w/dz^2$. Similarly, the nth derivative of $f(z)$, if it exists, is denoted by $f^{(n)}(z)$, $w^{(n)}$, or d^nw/dz^n where n is called the *order* of the derivative. Thus derivatives of first, second, third, ... orders are given by $f'(z), f''(z), f'''(z), \ldots$. Computations of these higher order derivatives follow by repeated application of the above differentiation rules.

One of the most remarkable theorems valid for functions of a complex variable and not necessarily valid for functions of a real variable is the following:

THEOREM 3.1. Suppose $f(z)$ is analytic in a region $\mathcal{R}$. Then so also are $f'(z), f''(z), \ldots$ analytic in $\mathcal{R}$, i.e., all higher derivatives exist in $\mathcal{R}$.

This important theorem is proved in Chapter 5.

3.10 L'Hospital's Rule

Let $f(z)$ and $g(z)$ be analytic in a region containing the point z_0 and suppose that $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$. Then, *L'Hospital's rule* states that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{3.14}$$

In the case of $f'(z_0) = g'(z_0) = 0$, the rule may be extended. See Problems 3.21–3.24.

We sometimes say that the left side of (3.14) has the "indeterminate form" $0/0$, although such terminology is somewhat misleading since there is usually nothing indeterminate involved. Limits represented by so-called indeterminate forms ∞/∞, $0 \cdot \infty$, ∞°, 0°, 1^∞, and $\infty - \infty$ can often be evaluated by appropriate modifications of L'Hospital's rule.

3.11 Singular Points

A point at which $f(z)$ fails to be analytic is called a *singular point* or *singularity* of $f(z)$. Various types of singularities exist.

1. **Isolated Singularities.** The point $z = z_0$ is called an *isolated singularity* or *isolated singular point* of $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0 (i.e., there exists a deleted δ neighborhood of z_0 containing no singularity). If no such δ can be found, we call z_0 a *non-isolated singularity*.

   If z_0 is not a singular point and we can find $\delta > 0$ such that $|z - z_0| = \delta$ encloses no singular point, then we call z_0 an *ordinary point* of $f(z)$.
2. **Poles.** If z_0 is an isolated singularity and we can find a positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$, then $z = z_0$ is called a *pole of order n*. If $n = 1$, z_0 is called a *simple pole*.

   EXAMPLE 3.1
   (a) $f(z) = 1/(z-2)^3$ has a pole of order 3 at $z = 2$.
   (b) $f(z) = (3z-2)/(z-1)^2(z+1)(z-4)$ has a pole of order 2 at $z = 1$, and simple poles at $z = -1$ and $z = 4$.

   If $g(z) = (z - z_0)^n f(z)$, where $f(z_0) \neq 0$ and n is a positive integer, then $z = z_0$ is called a *zero of order n* of $g(z)$. If $n = 1$, z_0 is called a *simple zero*. In such a case, z_0 is a pole of order n of the function $1/g(z)$.

3. **Branch Points** of multiple-valued functions, already considered in Chapter 2, are non-isolated singular points since a multiple-valued function is not continuous and, therefore, not analytic in a deleted neighborhood of a branch point.

EXAMPLE 3.2
(a) $f(z) = (z-3)^{1/2}$ has a branch point at $z = 3$.
(b) $f(z) = \ln(z^2 + z - 2)$ has branch points where $z^2 + z - 2 = 0$, i.e., at $z = 1$ and $z = -2$.

4. **Removable Singularities.** An isolated singular point z_0 is called a *removable singularity* of $f(z)$ if $\lim_{z\to z_0} f(z)$ exists. By defining $f(z_0) = \lim_{z\to z_0} f(z)$, it can then be shown that $f(z)$ is not only continuous at z_0 but is also analytic at z_0.

EXAMPLE 3.3 The singular point $z = 0$ is a removable singularity of $f(z) = \sin z/z$ since $\lim_{z\to 0}(\sin z/z) = 1$.

5. **Essential Singularities.** An isolated singularity that is not a pole or removable singularity is called an *essential singularity*.

EXAMPLE 3.4 $f(z) = e^{1/(z-2)}$ has an essential singularity at $z = 2$.

If a function has an isolated singularity, then the singularity is either removable, a pole, or an essential singularity. For this reason, a pole is sometimes called a *non-essential singularity*. Equivalently, $z = z_0$ is an essential singularity if we cannot find any positive integer n such that $\lim_{z\to z_0}(z - z_0)^n f(z) = A \neq 0$.

6. **Singularities at Infinity.** The type of singularity of $f(z)$ at $z = \infty$ [the point at infinity; see pages 7 and 47] is the same as that of $f(1/w)$ at $w = 0$.

EXAMPLE 3.5 The function $f(z) = z^3$ has a pole of order 3 at $z = \infty$, since $f(1/w) = 1/w^3$ has a pole of order 3 at $w = 0$.

For methods of classifying singularities using infinite series, see Chapter 6.

3.12 Orthogonal Families

Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic and $f'(z) \neq 0$. Then the one-parameter families of curves

$$u(x, y) = \alpha, \quad v(x, y) = \beta \tag{3.15}$$

where α and β are constants, are *orthogonal*, i.e., each member of one family [shown heavy in Fig. 3-3] is perpendicular to each member of the other family [shown dashed in Fig. 3-3] at the point of intersection. The corresponding image curves in the w plane consisting of lines parallel to the u and v axes also form orthogonal families [see Fig. 3-4].

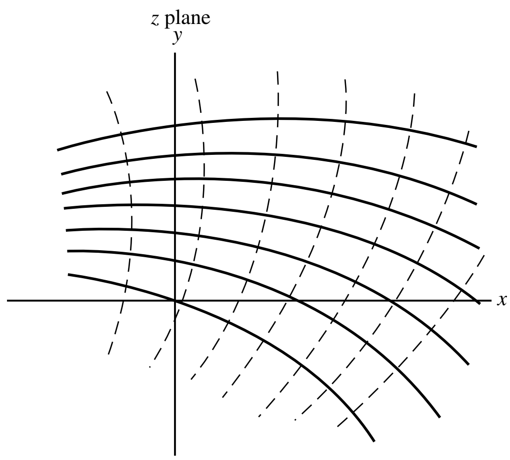

Fig. 3-3

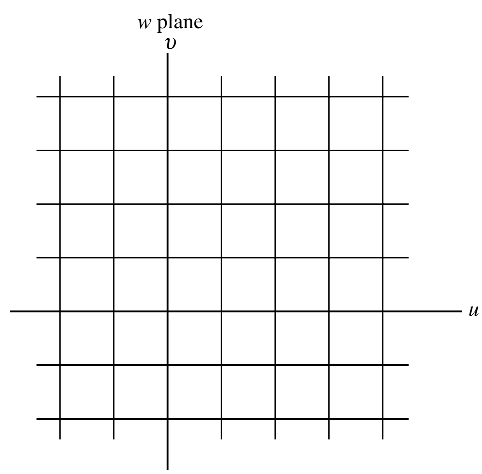

Fig. 3-4

In view of this, one might conjecture that if the mapping function $f(z)$ is analytic and $f'(z) \neq 0$, then the angle between any two intersecting curves C_1 and C_2 in the z plane would equal (both in magnitude and sense) the angle between corresponding intersecting image curves C_1' and C_2' in the w plane. This conjecture is in fact correct and leads to the subject of *conformal mapping*, which is of such great importance in both theory and application that two chapters (8 and 9) will be devoted to it.

3.13 Curves

Suppose $\phi(t)$ and $\psi(t)$ are real functions of the real variable t assumed continuous in $t_1 \le t \le t_2$. Then the parametric equations

$$z = x + iy = \phi(t) + i\psi(t) = z(t), \quad t_1 \le t \le t_2 \tag{3.16}$$

define a *continuous curve* or *arc* in the z plane joining points $a = z(t_1)$ and $b = z(t_2)$ [see Fig. 3-5].

If $t_1 \neq t_2$ while $z(t_1) = z(t_2)$, i.e., $a = b$, the endpoints coincide and the curve is said to be *closed*. A closed curve that does not intersect itself anywhere is called a *simple closed curve*. For example, the curve of Fig. 3-6 is a simple closed curve while that of Fig. 3-7 is not.

If $\phi(t)$ and $\psi(t)$ [and thus $z(t)$] have continuous derivatives in $t_1 \le t \le t_2$, the curve is often called a *smooth curve* or *arc*. A curve, which is composed of a finite number of smooth arcs, is called a *piecewise* or *sectionally smooth* curve or sometimes a *contour*. For example, the boundary of a square is a piecewise smooth curve or contour.

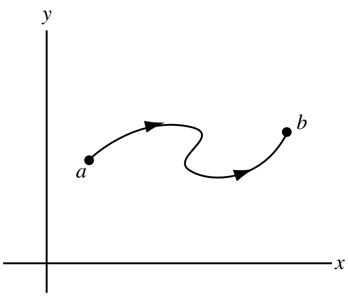

Fig. 3-5

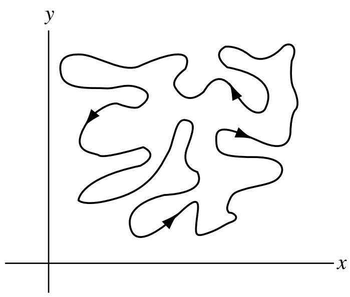

Fig. 3-6

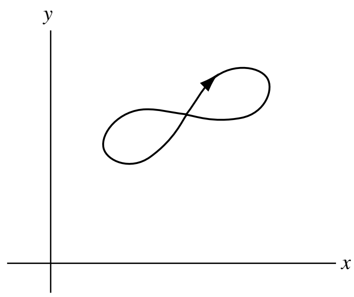

Fig. 3-7

Unless otherwise specified, whenever we refer to a curve or simple closed curve, we shall assume it to be piecewise smooth.

3.14 Applications to Geometry and Mechanics

We can consider $z(t)$ as a position vector whose terminal point describes a curve C in a definite *sense* or *direction* as t varies from t_1 to t_2. If $z(t)$ and $z(t + \Delta t)$ represent position vectors of points P and Q, respectively, then

$$\frac{\Delta z}{\Delta t} = \frac{z(t + \Delta t) - z(t)}{\Delta t}$$

is a vector in the direction of Δz [Fig. 3-8]. If $\lim_{\Delta t \to 0} \Delta z/\Delta t = dz/dt$ exists, the limit is a vector in the direction of the *tangent* to C at point P and is given by

$$\frac{dz}{dt} = \frac{dx}{dt} + i\frac{dy}{dt}$$

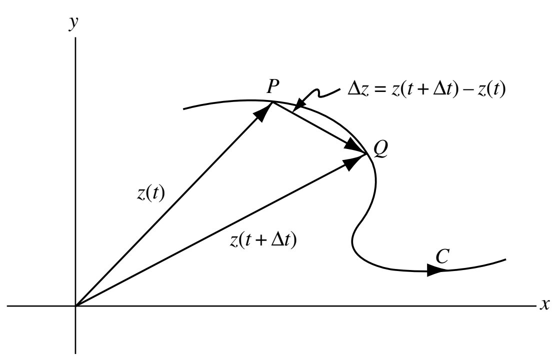

Fig. 3-8

If t is time, dz/dt represents the *velocity* with which the terminal point describes the curve. Similarly, d^2z/dt^2 represents its *acceleration* along the curve.

3.15 Complex Differential Operators

Let us define the operators ∇ (*del*) and $\overline{\nabla}$ (*del bar*) by

$$\nabla \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \bar{z}}, \quad \overline{\nabla} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} = 2\frac{\partial}{\partial z} \tag{3.17}$$

where the equivalence in terms of the conjugate coordinates z and $\bar{z}$ (page 7) follows from Problem 3.32.

3.16 Gradient, Divergence, Curl, and Laplacian

The operator ∇ enables us to define the following operations. In all cases, we consider $F(x, y)$ as a real continuously differentiable function of x and y (scalar), while $A(x, y) = P(x, y) + iQ(x, y)$ is a complex continuously differentiable function of x and y (vector).

In terms of conjugate coordinates,

$$F(x, y) = F\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = G(z, \bar{z}) \quad \text{and} \quad A(x, y) = B(z, \bar{z})$$

.

1. **Gradient.** We define the *gradient* of a real function F (scalar) by

$$\text{grad } F = \nabla F = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = 2\frac{\partial G}{\partial \bar{z}} \tag{3.18}$$

   Geometrically, if $\nabla F \neq 0$, then ∇F represents a vector normal to the curve $F(x, y) = c$ where c is a constant (see Problem 3.33).

   Similarly, the gradient of a complex function $A = P + iQ$ (vector) is defined by

$$\begin{aligned} \text{grad } A = \nabla A &= \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P + iQ) \\ &= \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) = 2\frac{\partial B}{\partial \bar{z}} \end{aligned} \tag{3.19}$$

   In particular, if B is an analytic function of z, then $\partial B/\partial \bar{z} = 0$ and so the gradient is zero, i.e., $\partial P/\partial x = \partial Q/\partial y$, $\partial P/\partial y = -(\partial Q/\partial x)$, which shows that the Cauchy–Riemann equations are satisfied in this case.

2. **Divergence.** By using the definition of dot product of two complex numbers (page 7) extended to the case of operators, we define the *divergence* of a complex function (vector) by

$$\begin{aligned} \text{div } A = \nabla \cdot A = \text{Re}\{\overline{\nabla} A\} &= \text{Re}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P + iQ)\right\} \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2\ \text{Re}\left\{\frac{\partial B}{\partial z}\right\} \end{aligned} \tag{3.20}$$

   Similarly, we can define the divergence of a real function. It should be noted that the divergence of a complex or real function (vector or scalar) is always a real function (scalar).

3. **Curl.** By using the definition of cross product of two complex numbers (page 7), we define the *curl* of a complex function as the vector

$$\nabla \times A = \left(0,\ 0,\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

orthogonal to the x-y plane having magnitude

$$|\text{curl } A| = |\nabla \times A| = |\text{Im}\{\overline{\nabla} A\}| = \left|\text{Im}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(P + iQ)\right\}\right|$$
$$= \left|\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right| = \left|2\ \text{Im}\left\{\frac{\partial B}{\partial z}\right\}\right| \tag{3.21}$$

4. **Laplacian.** The *Laplacian operator* is defined as the dot or scalar product of ∇ with itself, i.e.,

$$\nabla \cdot \nabla \equiv \nabla^2 \equiv \text{Re}\{\overline{\nabla}\nabla\} = \text{Re}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right\}$$
$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}} \tag{3.22}$$

Note that if A is analytic, $\nabla^2 A = 0$ so that $\nabla^2 P = 0$ and $\nabla^2 Q = 0$, i.e., P and Q are harmonic.

Some Identities Involving Gradient, Divergence, and Curl

Suppose A_1, A_2, and A are differentiable functions. Then the following identities hold

1. $\text{grad}(A_1 + A_2) = \text{grad } A_1 + \text{grad } A_2$
2. $\text{div}(A_1 + A_2) = \text{div } A_1 + \text{div } A_2$
3. $\text{curl}(A_1 + A_2) = \text{curl } A_1 + \text{curl } A_2$
4. $\text{grad}(A_1 A_2) = (A_1)(\text{grad } A_2) + (\text{grad } A_1)(A_2)$
5. $|\text{curl grad } A| = 0$ if A is real or, more generally, if $\text{Im}\{A\}$ is harmonic.
6. $\text{div grad } A = 0$ if A is imaginary or, more generally, if $\text{Re}\{A\}$ is harmonic.

SOLVED PROBLEMS

Derivatives

3.1. Using the definition, find the derivative of $w = f(z) = z^3 - 2z$ at the point where (a) $z = z_0$, (b) $z = -1$.

Solution

(a) By definition, the derivative at $z = z_0$ is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta_z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^3 - 2(z_0 + \Delta z) - \{z_0^3 - 2z_0\}}{\Delta z}$$
$$= \lim_{\Delta z \to 0} \frac{z_0^3 + 3z_0^2 \Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2z_0 - 2\Delta z - z_0^3 + 2z_0}{\Delta z}$$
$$= \lim_{\Delta z \to 0} 3z_0^2 + 3z_0 \Delta z + (\Delta z)^2 - 2 = 3z_0^2 - 2$$

In general, $f'(z) = 3z^2 - 2$ for all z.

(b) From (a), or directly, we find that if $z_0 = -1$, then $f'(-1) = 3(-1)^2 - 2 = 1$.

3.2. Show that $(d/dz)\bar{z}$ does not exist anywhere, i.e., $f(z) = \bar{z}$ is non-analytic anywhere.

Solution

By definition,

$$\frac{d}{dz}f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

if this limit exists independent of the manner in which $\Delta z = \Delta x + i\Delta y$ approaches zero.

Then

$$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x+iy+\Delta x+i\Delta y} - \overline{x+iy}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\Delta y - (x - iy)}{\Delta x + i\Delta y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, the required limit is

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

If $\Delta x = 0$, the required limit is

$$\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

Then, since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist, i.e., $f(z) = \bar{z}$ is *non-analytic* anywhere.

3.3. Given $w = f(z) = (1+z)/(1-z)$, find (a) dw/dz and (b) determine where $f(z)$ is non-analytic.

Solution

(a) ***Method 1.*** Using the definition

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\dfrac{1+(z+\Delta z)}{1-(z+\Delta z)} - \dfrac{1+z}{1-z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2}{(1-z-\Delta z)(1-z)} = \frac{2}{(1-z)^2}$$

independent of the manner in which $\Delta z \to 0$, provided $z \neq 1$.

Method 2. Using differentiation rules. By the quotient rule [see Problem 3.10(c)], we have if $z \neq 1$,

$$\frac{d}{dz}\left(\frac{1+z}{1-z}\right) = \frac{(1-z)\dfrac{d}{dz}(1+z) - (1+z)\dfrac{d}{dz}(1-z)}{(1-z)^2} = \frac{(1-z)(1) - (1+z)(-1)}{(1-z)^2} = \frac{2}{(1-z)^2}$$

(b) The function $f(z)$ is analytic for all finite values of z except $z = 1$ where the derivative does not exist and the function is non-analytic. The point $z = 1$ is a *singular point* of $f(z)$.

3.4. (a) If $f(z)$ is analytic at z_0, prove that it must be continuous at z_0.

(b) Give an example to show that the converse of (a) is not necessarily true.

Solution

(a) Since

$$f(z_0 + h) - f(z_0) = \frac{f(z_0 + h) - f(z_0)}{h} \cdot h$$

where $h = \Delta z \neq 0$, we have

$$\lim_{h \to 0} f(z_0 + h) - f(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \cdot \lim_{h \to 0} h = f'(z_0) \cdot 0 = 0$$

because $f'(z_0)$ exists by hypothesis. Thus

$$\lim_{h \to 0} f(z_0 + h) - f(z_0) = 0 \qquad \text{or} \qquad \lim_{h \to 0} f(z_0 + h) = f(z_0)$$

showing that $f(z)$ is continuous at z_0.

(b) The function $f(z) = \bar{z}$ is continuous at z_0. However, by Problem 3.2, $f(z)$ is not analytic anywhere. This shows that a function, which is continuous, need not have a derivative, i.e., need not be analytic.

Cauchy–Riemann Equations

3.5. Prove that a (a) necessary and (b) sufficient condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region $\mathcal{R}$ is that the Cauchy–Riemann equations $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -(\partial v/\partial x)$ are satisfied in $\mathcal{R}$ where it is supposed that these partial derivatives are continuous in $\mathcal{R}$.

Solution

(a) *Necessity.* In order for $f(z)$ to be analytic, the limit

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\Delta y} \qquad (1)$$

must exist independent of the manner in which Δz (or Δx and Δy) approaches zero. We consider two possible approaches.

Case 1. $\Delta y = 0$, $\Delta x \to 0$. In this case, (1) becomes

$$\lim_{\Delta x \to 0} \left\{ \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \left[\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right] \right\} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

provided the partial derivatives exist.

Case 2. $\Delta x = 0$, $\Delta y \to 0$. In this case, (1) becomes

$$\lim_{\Delta y \to 0} \left\{ \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} \right\} = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now $f(z)$ cannot possibly be analytic unless these two limits are identical. Thus, a necessary condition that $f(z)$ be analytic is

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{or} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

(b) *Sufficiency*. Since $\partial u/\partial x$ and $\partial u/\partial y$ are supposed to be continuous, we have

$$\begin{aligned} \Delta u &= u(x + \Delta x, y + \Delta y) - u(x, y) \\ &= \{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)\} + \{u(x, y + \Delta y) - u(x, y)\} \\ &= \left(\frac{\partial u}{\partial x} + \epsilon_1 \right) \Delta x + \left(\frac{\partial u}{\partial y} + \eta_1 \right) \Delta y = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \eta_1 \Delta y \end{aligned}$$

where $\epsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Similarly, since $\partial v/\partial x$ and $\partial v/\partial y$ are supposed to be continuous, we have

$$\Delta v = \left(\frac{\partial v}{\partial x} + \epsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_2 \Delta x + \eta_2 \Delta y$$

where $\epsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Then

$$\Delta w = \Delta u + i\Delta v = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \epsilon \Delta x + \eta \Delta y \tag{2}$$

where $\epsilon = \epsilon_1 + i\epsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

By the Cauchy–Riemann equations, (2) can be written

$$\begin{aligned} \Delta w &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \epsilon \Delta x + \eta \Delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \epsilon \Delta x + \eta \Delta y \end{aligned}$$

Then, on dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

so that the derivative exists and is unique, i.e., $f(z)$ is analytic in $\mathcal{R}$.

3.6. Given $f(z) = u + iv$ is analytic in a region $\mathcal{R}$. Prove that u and v are harmonic in $\mathcal{R}$ if they have continuous second partial derivatives in $\mathcal{R}$.

Solution

If $f(z)$ is analytic in $\mathcal{R}$, then the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{2}$$

are satisfied in $\mathcal{R}$. Assuming u and v have continuous second partial derivatives, we can differentiate both sides of (1) with respect to x and (2) with respect to y to obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{3}$$

and

$$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \tag{4}$$

from which

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

i.e., u is harmonic.

Similarly, by differentiating both sides of (1) with respect to y and (2) with respect to x, we find

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

and v is harmonic.

It will be shown later (Chapter 5) that if $f(z)$ is analytic in $\mathcal{R}$, all its derivatives exist and are continuous in $\mathcal{R}$. Hence, the above assumptions will not be necessary.

3.7. (a) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.

(b) Find v such that $f(z) = u + iv$ is analytic.

Solution

(a)
$$\frac{\partial u}{\partial x} = (e^{-x})(\sin y) + (-e^{-x})(x \sin y - y \cos y) = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y) = -2e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \cos y \quad (1)$$
$$\frac{\partial u}{\partial y} = e^{-x}(x \cos y + y \sin y - \cos y) = xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y) = -xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y \quad (2)$$

Adding (1) and (2) yields $(\partial^2 u/\partial x^2) + (\partial^2 u/\partial y^2) = 0$ and u is harmonic.

(b) From the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y \quad (3)$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x} \cos y - xe^{-x} \cos y - ye^{-z} \sin y \quad (4)$$

Integrate (3) with respect to y, keeping x constant. Then

$$\begin{aligned} v &= -e^{-x} \cos y + xe^{-x} \cos y + e^{-x}(y \sin y + \cos y) + F(x) \\ &= ye^{-x} \sin y + xe^{-x} \cos y + F(x) \quad (5) \end{aligned}$$

where $F(x)$ is an arbitrary real function of x.

Substitute (5) into (4) and obtain

$$-ye^{-x} \sin y - xe^{-x} \cos y + e^{-x} \cos y + F'(x) = -ye^{-x} \sin y - xe^{-x} \cos y - ye^{-x} \sin y$$

or $F'(x) = 0$ and $F(x) = c$, a constant. Then, from (5),

$$v = e^{-x}(y \sin y + x \cos y) + c$$

For another method, see Problem 3.40.

3.8. Find $f(z)$ in Problem 3.7.

Solution

Method 1

We have $\quad f(z) = f(x + iy) = u(x, y) + iv(x, y).$

Putting $y = 0$ $\quad f(x) = u(x, 0) + iv(x, 0)$.

Replacing x by z, $\quad f(z) = u(z, 0) + iv(z, 0)$.

Then, from Problem 3.7, $u(z, 0) = 0$, $v(z, 0) = ze^{-z}$ and so $f(z) = u(z, 0) + iv(z, 0) = ize^{-z}$, apart from an arbitrary additive constant.

Method 2

Apart from an arbitrary additive constant, we have from the results of Problem 3.7,

$$\begin{aligned} f(z) &= u + iv = e^{-x}(x \sin y - y \cos y) + ie^{-x}(y \sin y + x \cos y) \\ &= e^{-x}\left\{x\left(\frac{e^{iy} - e^{-iy}}{2i}\right) - y\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\} + ie^{-x}\left\{y\left(\frac{e^{iy} - e^{-iy}}{2i}\right) + x\left(\frac{e^{iy} + e^{-iy}}{2}\right)\right\} \\ &= i(x + iy)e^{-(x+iy)} = ize^{-z} \end{aligned}$$

Method 3

We have $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$. Then, substituting into $u(x, y) + iv(x, y)$, we find after much tedious labor that $\bar{z}$ disappears and we are left with the result ize^{-z}.

In general, method 1 is preferable over methods 2 and 3 when both u and v are known. If only u (or v) is known, another procedure is given in Problem 3.101.

Differentials

3.9. Given $w = f(z) = z^3 - 2z^2$. Find: (a) Δw, (b) dw, (c) $\Delta w - dw$.

Solution

(a) $$\begin{aligned} \Delta w = f(z + \Delta z) - f(z) &= \{(z + \Delta z)^3 - 2(z + \Delta z)^2\} - \{z^3 - 2z^2\} \\ &= z^3 + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - 2z^2 - 4z\Delta z - 2(\Delta z)^2 - z^3 + 2z^2 \\ &= (3z^2 - 4z)\Delta z + (3z - 2)(\Delta z)^2 + (\Delta z)^3 \end{aligned}$$

(b) dw = principal part of $\Delta w = (3z^2 - 4z)\Delta z = (3z^2 - 4z)\, dz$, since, by definition, $\Delta z = dz$. Note that $f'(z) = 3z^2 - 4z$ and $dw = (3z^2 - 4z)\, dz$, i.e., $dw/dz = 3z^2 - 4z$.

(c) From (a) and (b), $\Delta w - dw = (3z - 2)(\Delta z)^2 + (\Delta z)^3 = \epsilon\Delta z$ where $\epsilon = (3z - 2)\Delta z + (\Delta z)^2$. Note that $\epsilon \to 0$ as $\Delta z \to 0$, i.e., $(\Delta w - dw)/\Delta z \to 0$ as $\Delta z \to 0$. It follows that $\Delta w - dw$ is an infinitesimal of higher order than Δz.

Differentiation Rules. Derivatives of Elementary Functions

3.10. Prove the following assuming that $f(z)$ and $g(z)$ are analytic in a region $\mathcal{R}$.

(a) $\dfrac{d}{dz}\{f(z) + g(z)\} = \dfrac{d}{dz} f(z) + \dfrac{d}{dz} g(z)$

(b) $\dfrac{d}{dz}\{f(z)g(z)\} = f(z)\dfrac{d}{dz} g(z) + g(z)\dfrac{d}{dz} f(z)$

(c) $\dfrac{d}{dz}\left\{\dfrac{f(z)}{g(z)}\right\} = \dfrac{g(z)\dfrac{d}{dz} f(z) - f(z)\dfrac{d}{dz} g(z)}{[g(z)]^2}$ if $g(z) \neq 0$

Solution

(a) $$\frac{d}{dz}\{f(z)+g(z)\} = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)+g(z+\Delta z)-\{f(z)+g(z)\}}{\Delta z}$$
$$= \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} + \lim_{\Delta z\to 0}\frac{g(z+\Delta z)-g(z)}{\Delta z}$$
$$= \frac{d}{dz}f(z) + \frac{d}{dz}g(z)$$

(b) $$\frac{d}{dz}\{f(z)g(z)\} = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z)}{\Delta z}$$
$$= \lim_{\Delta z\to 0}\frac{f(z+\Delta z)\{g(z+\Delta z)-g(z)\}+g(z)\{f(z+\Delta z)-f(z)\}}{\Delta z}$$
$$= \lim_{\Delta z\to 0} f(z+\Delta z)\left\{\frac{g(z+\Delta z)-g(z)}{\Delta z}\right\} + \lim_{\Delta z\to 0} g(z)\left\{\frac{f(z+\Delta z)-f(z)}{\Delta z}\right\}$$
$$= f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$$

Note that we have used the fact that $\lim_{\Delta z\to 0} f(z+\Delta z) = f(z)$ which follows since $f(z)$ is analytic and thus continuous (see Problem 3.4).

Another Method

Let $U = f(z)$, $V = g(z)$. Then $\Delta U = f(z+\Delta z) - f(z)$ and $\Delta V = g(z+\Delta z) - g(z)$, i.e., $f(z+\Delta z) = U + \Delta U$, $g(z+\Delta z) = V + \Delta V$. Thus

$$\frac{d}{dz}UV = \lim_{\Delta z\to 0}\frac{(U+\Delta U)(V+\Delta V)-UV}{\Delta z} = \lim_{\Delta z\to 0}\frac{U\Delta V + V\Delta U + \Delta U\Delta V}{\Delta z}$$
$$= \lim_{\Delta z\to 0}\left(U\frac{\Delta V}{\Delta z} + V\frac{\Delta U}{\Delta z} + \frac{\Delta U}{\Delta z}\Delta V\right) = U\frac{dV}{dz} + V\frac{dU}{dz}$$

where it is noted that $\Delta V \to 0$ as $\Delta z \to 0$, since V is supposed analytic and thus continuous.

A similar procedure can be used to prove (a).

(c) We use the second method in (b). Then

$$\frac{d}{dz}\left(\frac{U}{V}\right) = \lim_{\Delta z\to 0}\frac{1}{\Delta z}\left\{\frac{U+\Delta U}{V+\Delta V} - \frac{U}{V}\right\} = \lim_{\Delta z\to 0}\frac{V\Delta U - U\Delta V}{\Delta z(V+\Delta V)V}$$
$$= \lim_{\Delta z\to 0}\frac{1}{(V+\Delta V)V}\left\{V\frac{\Delta U}{\Delta z} - U\frac{\Delta V}{\Delta z}\right\} = \frac{V(dU/dz) - U(dV/dz)}{V^2}$$

The first method of (b) can also be used.

3.11. Prove that (a) $(d/dz)e^z = e^z$, (b) $(d/dz)e^{az} = ae^{az}$ where a is any constant.

Solution

(a) By definition, $w = e^z = e^{x+iy} = e^x(\cos y + i\sin y) = u + iv$ or $u = e^x\cos y$, $v = e^x\sin y$.

Since $\partial u/\partial x = e^x\cos y = \partial v/\partial y$ and $\partial v/\partial x = e^x\sin y = -(\partial u/\partial y)$, the Cauchy–Riemann equations are satisfied. Then, by Problem 3.5, the required derivative exists and is equal to

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = e^x\cos y + ie^x\sin y = e^z$$

(b) Let $w = e^{\zeta}$ where $\zeta = az$. Then, by part (a) and Problem 3.39,

$$\frac{d}{dz}e^{az} = \frac{d}{dz}e^{\zeta} = \frac{d}{d\zeta}e^{\zeta} \cdot \frac{d\zeta}{dz} = e^{\zeta} \cdot a = ae^{az}$$

We can also proceed as in part (a).

3.12. Prove that: (a) $\frac{d}{dz}\sin z = \cos z$, (b) $\frac{d}{dz}\cos z = -\sin z$, (c) $\frac{d}{dz}\tan z = \sec^2 z$.

Solution

(a) We have $w = \sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$. Then

$$u = \sin x \cosh y, \quad v = \cos x \sinh y$$

Now $\partial u/\partial x = \cos x \cosh y = \partial v/\partial y$ and $\partial v/\partial x = -\sin x \sinh y = -(\partial u/\partial y)$ so that the Cauchy–Riemann equations are satisfied. Hence, by Problem 3.5, the required derivative is equal to

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \cos x \cosh y - i \sin x \sinh y = \cos(x + iy) = \cos z$$

Another Method

Since $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, we have, using Problem 3.11(b),

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{1}{2i}\frac{d}{dz}e^{iz} - \frac{1}{2i}\frac{d}{dz}e^{-iz} = \frac{1}{2}e^{iz} + \frac{1}{2}e^{-iz} = \cos z$$

(b)
$$\frac{d}{dz}\cos z = \frac{d}{dz}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = \frac{1}{2}\frac{d}{dz}e^{iz} + \frac{1}{2}\frac{d}{dz}e^{-iz}$$
$$= \frac{i}{2}e^{iz} - \frac{i}{2}e^{-iz} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

The first method of part (a) can also be used.

(c) By the quotient rule of Problem 3.10(c), we have

$$\frac{d}{dz}\tan z = \frac{d}{dz}\left(\frac{\sin z}{\cos z}\right) = \frac{\cos z \frac{d}{dz}\sin z - \sin z \frac{d}{dz}\cos z}{\cos^2 z}$$
$$= \frac{(\cos z)(\cos z) - (\sin z)(-\sin z)}{\cos^2 z} = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z$$

3.13. Prove that $\frac{d}{dz}z^{1/2} = \dfrac{1}{2z^{1/2}}$, realizing that $z^{1/2}$ is a multiple-valued function.

Solution

A function must be single-valued in order to have a derivative. Thus, since $z^{1/2}$ is multiple-valued (in this case two-valued), we must restrict ourselves to one branch of this function at a time.

Case 1

Let us first consider that branch of $w = z^{1/2}$ for which $w = 1$ where $z = 1$. In this case, $w^2 = z$ so that

$$\frac{dz}{dw} = 2w \quad \text{and so} \quad \frac{dw}{dz} = \frac{1}{2w} \quad \text{or} \quad \frac{d}{dz}z^{1/2} = \frac{1}{2z^{1/2}}$$

Case 2

Next, we consider that branch of $w = z^{1/2}$ for which $w = -1$ where $z = 1$. In this case too, we have $w^2 = z$ so that

$$\frac{dz}{dw} = 2w \quad \text{and} \quad \frac{dw}{dz} = \frac{1}{2w} \quad \text{or} \quad \frac{d}{dz}z^{1/2} = \frac{1}{2z^{1/2}}$$

In both cases, we have $(d/dz)z^{1/2} = 1/(2z^{1/2})$. Note that the derivative does not exist at the branch point $z = 0$. In general, a function does not have a derivative, i.e., is not analytic, at a branch point. Thus branch points are singular points.

3.14. Prove that $\dfrac{d}{dz}\ln z = \dfrac{1}{z}$.

Solution

Let $w = \ln z$. Then $z = e^w$ and $dz/dw = e^w = z$. Hence

$$\frac{d}{dz}\ln z = \frac{dw}{dz} = \frac{1}{dz/dw} = \frac{1}{z}$$

Note that the result is valid regardless of the particular branch of $\ln z$. Also observe that the derivative does not exist at the branch point $z = 0$, illustrating further the remark at the end of Problem 3.13.

3.15. Prove that $\dfrac{d}{dz}\ln f(z) = \dfrac{f'(z)}{f(z)}$.

Solution

Let $w = \ln \zeta$ where $\zeta = f(z)$. Then

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz} = \frac{1}{\zeta}\cdot\frac{d\zeta}{dz} = \frac{f'(z)}{f(z)}$$

3.16. Prove that: (a) $\dfrac{d}{dz}\sin^{-1} z = \dfrac{1}{\sqrt{1-z^2}}$, (b) $\dfrac{d}{dz}\tanh^{-1} z = \dfrac{1}{1-z^2}$.

Solution

(a) If we consider the principal branch of $\sin^{-1} z$, we have by Problem 2.22 and by Problem 3.15

$$\begin{aligned}\frac{d}{dz}\sin^{-1} z &= \frac{d}{dz}\left\{\frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right)\right\} = \frac{1}{i}\frac{d}{dz}\left(iz + \sqrt{1-z^2}\right)\Big/\left(iz + \sqrt{1-z^2}\right) \\ &= \frac{1}{i}\left\{i + \frac{1}{2}(1-z^2)^{-1/2}(-2z)\right\}\Big/\left(iz + \sqrt{1-z^2}\right) \\ &= \left(1 + \frac{iz}{\sqrt{1-z^2}}\right)\Big/\left(iz + \sqrt{1-z^2}\right) = \frac{1}{\sqrt{1-z^2}}\end{aligned}$$

The result is also true if we consider other branches.

(b) We have, on considering the principal branch,

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) = \frac{1}{2}\ln(1+z) - \frac{1}{2}\ln(1-z)$$

Then

$$\frac{d}{dz}\tanh^{-1} z = \frac{1}{2}\frac{d}{dz}\ln(1+z) - \frac{1}{2}\frac{d}{dz}\ln(1-z) = \frac{1}{2}\left(\frac{1}{1+z}\right) + \frac{1}{2}\left(\frac{1}{1-z}\right) = \frac{1}{1-z^2}$$

Note that in both parts (a) and (b), the derivatives do not exist at the branch points $z = \pm 1$.

3.17. Using rules of differentiation, find the derivatives of each of the following:

(a) $\cos^2(2z+3i)$, (b) $z\tan^{-1}(\ln z)$, (c) $\{\tanh^{-1}(iz+2)\}^{-1}$, (d) $(z-3i)^{4z+2}$.

Solution

(a) Let $\eta = 2z + 3i$, $\zeta = \cos\eta$, $w = \zeta^2$ from which $w = \cos^2(2z+3i)$. Then, using the chain rule, we have

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{d\eta}\cdot\frac{d\eta}{dz} = (2\zeta)(-\sin\eta)(2) = (2\cos\eta)(-\sin\eta)(2) = -4\cos(2z+3i)\sin(2z+3i)$$

Another Method

$$\begin{aligned}\frac{d}{dz}\{\cos(2z+3i)\}^2 &= 2\{\cos(2z+3i)\}\left\{\frac{d}{dz}\cos(2z+3i)\right\}\\ &= 2\{\cos(2z+3i)\}\{-\sin(2z+3i)\}\left\{\frac{d}{dz}(2z+3i)\right\}\\ &= -4\cos(2z+3i)\sin(2z+3i)\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dz}\{(z)[\tan^{-1}(\ln z)]\} &= z\frac{d}{dz}[\tan^{-1}(\ln z)] + [\tan^{-1}(\ln z)]\frac{d}{dz}(z)\\ &= z\left\{\frac{1}{1+(\ln z)^2}\right\}\frac{d}{dz}(\ln z) + \tan^{-1}(\ln z) = \frac{1}{1+(\ln z)^2} + \tan^{-1}(\ln z)\end{aligned}$$

(c)
$$\begin{aligned}\frac{d}{dz}\{\tanh^{-1}(iz+2)\}^{-1} &= -1\{\tanh^{-1}(iz+2)\}^{-2}\frac{d}{dz}\{\tanh^{-1}(iz+2)\}\\ &= -\{\tanh^{-1}(iz+2)\}^{-2}\left\{\frac{1}{1-(iz+2)^2}\right\}\frac{d}{dz}(iz+2)\\ &= \frac{-i\{\tanh^{-1}(iz+2)\}^{-2}}{1-(iz+2)^2}\end{aligned}$$

(d)
$$\begin{aligned}\frac{d}{dz}\left\{(z-3i)^{4z+2}\right\} &= \frac{d}{dz}\left\{e^{(4z+2)\ln(z-3i)}\right\} = e^{(4z+2)\ln(z-3i)}\frac{d}{dz}\{(4z+2)\ln(z-3i)\}\\ &= e^{(4z+2)\ln(z-3i)}\left\{(4z+2)\frac{d}{dz}[\ln(z-3i)] + \ln(z-3i)\frac{d}{dz}(4z+2)\right\}\\ &= e^{(4z+2)\ln(z-3i)}\left\{\frac{4z+2}{z-3i} + 4\ln(z-3i)\right\}\\ &= (z-3i)^{4z+1}(4z+2) + 4(z-3i)^{4z+2}\ln(z-3i)\end{aligned}$$

3.18. Suppose $w^3 - 3z^2w + 4\ln z = 0$. Find dw/dz.

Solution

Differentiating with respect to z, considering w as an implicit function of z, we have

$$\frac{d}{dz}(w^3) - 3\frac{d}{dz}(z^2w) + 4\frac{d}{dz}(\ln z) = 0 \quad \text{or} \quad 3w^2\frac{dw}{dz} - 3z^2\frac{dw}{dz} - 6zw + \frac{4}{z} = 0$$

Then, solving for dw/dz, we obtain $\dfrac{dw}{dz} = \dfrac{6zw - 4/z}{3w^2 - 3z^2}$.

3.19. Given $w = \sin^{-1}(t-3)$ and $z = \cos(\ln t)$. Find dw/dz.

Solution

$$\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{1/\sqrt{1-(t-3)^2}}{-\sin(\ln t)(1/t)} = -\frac{t}{\sin(\ln t)\sqrt{1-(t-3)^2}}$$

3.20. In Problem 3.18, find d^2w/dz^2.

Solution

$$\frac{d^2w}{dz^2} = \frac{d}{dz}\left(\frac{dw}{dz}\right) = \frac{d}{dz}\left(\frac{6zw - 4/z}{3w^2 - 3z^2}\right)$$
$$= \frac{(3w^2-3z^2)(6z\,dw/dz + 6w + 4/z^2) - (6zw - 4/z)(6w\,dw/dz - 6z)}{(3w^2-3z^2)^2}$$

The required result follows on substituting the value of dw/dz from Problem 3.18 and simplifying.

L'Hospital's Rule

3.21. Suppose $f(z)$ is analytic in a region $\mathcal{R}$ including the point z_0. Prove that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z - z_0)$$

where $\eta \to 0$ as $z \to z_0$.

Solution

Let $\dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = \eta$ so that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) = \eta(z - z_0)$$

Then, since $f(z)$ is analytic at z_0, we have as required

$$\lim_{z\to z_0} \eta = \lim_{z\to z_0}\left\{\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right\} = f'(z_0) - f'(z_0) = 0$$

3.22. Suppose $f(z)$ and $g(z)$ are analytic at z_0, and $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$. Prove that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

Solution

By Problem 3.21 we have, using the fact that $f(z_0) = g(z_0) = 0$,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta_1(z - z_0) = f'(z_0)(z - z_0) + \eta_1(z - z_0)$$
$$g(z) = g(z_0) + g'(z_0)(z - z_0) + \eta_2(z - z_0) = g'(z_0)(z - z_0) + \eta_2(z - z_0)$$

where $\lim_{z\to z_0}\eta_1 = \lim_{z\to z_0}\eta_2 = 0$. Then, as required,

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \lim_{z\to z_0}\frac{\{f'(z_0) + \eta_1\}(z - z_0)}{\{g'(z_0) + \eta_2\}(z - z_0)} = \frac{f'(z_0)}{g'(z_0)}$$

Another Method

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\bigg/\frac{g(z)-g(z_0)}{z-z_0}$$
$$=\left(\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\right)\bigg/\left(\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}\right)=\frac{f'(z_0)}{g'(z_0)}.$$

3.23. Evaluate (a) $\lim_{z\to i}\dfrac{z^{10}+1}{z^6+1}$, (b) $\lim_{z\to 0}\dfrac{1-\cos z}{z^2}$, (c) $\lim_{z\to 0}\dfrac{1-\cos z}{\sin z^2}$.

Solution

(a) Let $f(z)=z^{10}+1$ and $g(z)=z^6+1$. Then $f(i)=g(i)=0$. Also, $f(z)$ and $g(z)$ are analytic at $z=i$. Hence, by L'Hospital's rule,

$$\lim_{z\to i}\frac{z^{10}+1}{z^6+1}=\lim_{z\to i}\frac{10z^9}{6z^5}=\lim_{z\to i}\frac{5}{3}z^4=\frac{5}{3}$$

(b) Let $f(z)=1-\cos z$ and $g(z)=z^2$. Then $f(0)=g(0)=0$. Also, $f(z)$ and $g(z)$ are analytic at $z=0$. Hence, by L'Hospital's rule,

$$\lim_{z\to 0}\frac{1-\cos z}{z^2}=\lim_{z\to 0}\frac{\sin z}{2z}$$

Since $f_1(z)=\sin z$ and $g_1(z)=2z$ are analytic and equal to zero when $z=0$, we can apply L'Hospital's rule again to obtain the required limit,

$$\lim_{z\to 0}\frac{\sin z}{2z}=\lim_{z\to 0}\frac{\cos z}{2}=\frac{1}{2}$$

(c) ***Method 1.*** By repeated application of L'Hospital's rule, we have

$$\lim_{z\to 0}\frac{1-\cos z}{\sin z^2}=\lim_{z\to 0}\frac{\sin z}{2z\cos z^2}=\lim_{z\to 0}\frac{\cos z}{2\cos z^2-4z^2\sin z^2}=\frac{1}{2}$$

Method 2. Since $\lim_{z\to 0}\dfrac{\sin z}{z}=1$, we have by one application of L'Hospital's rule,

$$\lim_{z\to 0}\frac{1-\cos z}{\sin z^2}=\lim_{z\to 0}\frac{\sin z}{2z\cos z^2}=\lim_{z\to 0}\left(\frac{\sin z}{z}\right)\left(\frac{1}{2\cos z^2}\right)$$
$$=\lim_{z\to 0}\left(\frac{\sin z}{z}\right)\lim_{z\to 0}\left(\frac{1}{2\cos z^2}\right)=(1)\left(\frac{1}{2}\right)=\frac{1}{2}$$

Method 3. Since $\lim_{z\to 0}\dfrac{\sin z^2}{z^2}=1$ or, equivalently, $\lim_{z\to 0}\dfrac{z^2}{\sin z^2}=1$, we can write

$$\lim_{z\to 0}\frac{1-\cos z}{\sin z^2}=\lim_{z\to 0}\left(\frac{1-\cos z}{z^2}\right)\left(\frac{z^2}{\sin z^2}\right)=\lim_{z\to 0}\frac{1-\cos z}{z^2}=\frac{1}{2}$$

using part (b).

3.24. Evaluate $\lim_{z\to 0}(\cos z)^{1/z^2}$.

Solution

Let $w=(\cos z)^{1/z^2}$. Then $\ln w=(\ln\cos z)/z^2$ where we consider the principal branch of the logarithm. By L'Hospital's rule,

$$\lim_{z\to 0}\ln w=\lim_{z\to 0}\frac{\ln\cos z}{z^2}=\lim_{z\to 0}\frac{(-\sin z)/\cos z}{2z}$$
$$=\lim_{z\to 0}\left(\frac{\sin z}{z}\right)\left(-\frac{1}{2\cos z}\right)=(1)\left(-\frac{1}{2}\right)=-\frac{1}{2}$$

But since the logarithm is a continuous function, we have

$$\lim_{z\to 0} \ln w = \ln\left(\lim_{z\to 0} w\right) = -\frac{1}{2}$$

or $\lim_{z\to 0} w = e^{-1/2}$, which is the required value.

Note that since $\lim_{z\to 0} \cos z = 1$ and $\lim_{z\to 0} 1/z^2 = \infty$, the required limit has the "indeterminate form" 1^∞.

Singular Points

3.25. For each of the following functions, locate and name the singularities in the finite z plane and determine whether they are isolated singularities or not.

(a) $f(z) = \dfrac{z}{(z^2+4)^2}$, (b) $f(z) = \sec(1/z)$, (c) $f(z) = \dfrac{\ln(z-2)}{(z^2+2z+2)^4}$, (d) $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$

Solution

(a) $f(z) = \dfrac{z}{(z^2+4)^2} = \dfrac{z}{\{(z+2i)(z-2i)\}^2} = \dfrac{z}{(z+2i)^2(z-2i)^2}$.

Since

$$\lim_{z\to 2i} (z-2i)^2 f(z) = \lim_{z\to 2i} \frac{z}{(z+2i)^2} = \frac{1}{8i} \neq 0$$

$z = 2i$ is a pole of order 2. Similarly, $z = -2i$ is a pole of order 2.

Since we can find δ such that no singularity other than $z = 2i$ lies inside the circle $|z-2i| = \delta$ (e.g., choose $\delta = 1$), it follows that $z = 2i$ is an isolated singularity. Similarly, $z = -2i$ is an isolated singularity.

(b) Since $\sec(1/z) = 1/\cos(1/z)$, the singularities occur where $\cos(1/z) = 0$, i.e., $1/z = (2n+1)\pi/2$ or $z = 2/(2n+1)\pi$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Also, since $f(z)$ is not defined at $z = 0$, it follows that $z = 0$ is also a singularity.

Now, by L'Hospital's rule,

$$\begin{aligned}\lim_{z\to 2/(2n+1)\pi}\left\{z - \frac{2}{(2n+1)\pi}\right\}f(z) &= \lim_{z\to 2/(2n+1)\pi} \frac{z - 2/(2n+1)\pi}{\cos(1/z)} \\ &= \lim_{z\to 2/(2n+1)\pi} \frac{1}{-\sin(1/z)\{-1/z^2\}} \\ &= \frac{\{2/(2n+1)\pi\}^2}{\sin(2n+1)\pi/2} = \frac{4(-1)^n}{(2n+1)^2\pi^2} \neq 0\end{aligned}$$

Thus the singularities $z = 2/(2n+1)/\pi$, $n = 0, \pm 1, \pm 2, \ldots$ are *poles of order one*, i.e., *simple poles*. Note that these poles are located on the real axis at $z = \pm 2/\pi$, $\pm 2/3\pi$, $\pm 2/5\pi, \ldots$ and that there are infinitely many in a finite interval which includes 0 (see Fig. 3-9).

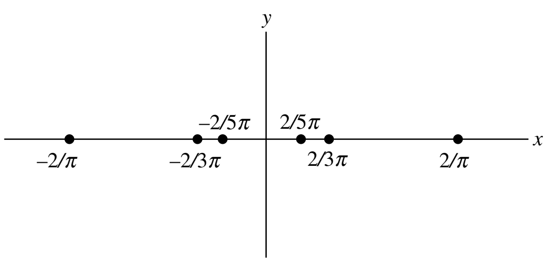

Fig. 3-9

Since we can surround each of these by a circle of radius δ, which contains no other singularity, it follows that they are isolated singularities. It should be noted that the δ required is smaller the closer the singularity is to the origin.

Since we cannot find any positive integer n such that $\lim_{z\to 0} (z-0)^n f(z) = A \neq 0$, it follows that $z = 0$ is an *essential singularity*. Also, since every circle of radius δ with center at $z = 0$ contains singular points other than $z = 0$, no matter how small we take δ, we see that $z = 0$ is a *non-isolated singularity*.

(c) The point $z = 2$ is a *branch point* and is a *non-isolated singularity*. Also, since $z^2 + 2z + 2 = 0$ where $z = -1 \pm i$, it follows that $z^2 + 2z + 2 = (z + 1 + i)(z + 1 - i)$ and that $z = -1 \pm i$ are *poles of order 4* which are *isolated singularities*.

(d) At first sight, it appears as if $z = 0$ is a branch point. To test this, let $z = re^{i\theta} = re^{i(\theta+2\pi)}$ where $0 \le \theta < 2\pi$.

If $z = re^{i\theta}$, we have

$$f(z) = \frac{\sin(\sqrt{r}e^{i\theta/2})}{\sqrt{r}e^{i\theta/2}}$$

If $z = re^{i(\theta+2\pi)}$, we have

$$f(z) = \frac{\sin(\sqrt{r}e^{i\theta/2}e^{\pi i})}{\sqrt{r}e^{i\theta/2}e^{\pi i}} = \frac{\sin(-\sqrt{r}e^{i\theta/2})}{-\sqrt{r}e^{i\theta/2}} = \frac{\sin(\sqrt{r}e^{i\theta/2})}{\sqrt{r}e^{i\theta/2}}$$

Thus, there is actually only one branch to the function, and so $z = 0$ cannot be a branch point. Since $\lim_{z\to 0} \sin\sqrt{z}/\sqrt{z} = 1$, it follows in fact that $z = 0$ is a *removable singularity*.

3.26. (a) Locate and name all the singularities of $f(z) = \dfrac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}$.

(b) Determine where $f(z)$ is analytic.

Solution

(a) The singularities in the finite z plane are located at $z = 1$ and $z = -2/3$; $z = 1$ is a *pole of order 3* and $z = -2/3$ is a *pole of order 2*.

To determine whether there is a singularity at $z = \infty$ (the point at infinity), let $z = 1/w$. Then

$$f(1/w) = \frac{(1/w)^8 + (1/w)^4 + 2}{(1/w - 1)^3(3/w + 2)^2} = \frac{1 + w^4 + 2w^8}{w^3(1-w)^3(3+2w)^2}$$

Thus, since $w = 0$ is a pole of order 3 for the function $f(1/w)$, it follows that $z = \infty$ is a pole of order 3 for the function $f(z)$.

Then the given function has three singularities: a pole of order 3 at $z = 1$, a pole of order 2 at $z = -2/3$, and a pole of order 3 at $z = \infty$.

(b) From (a) it follows that $f(z)$ is analytic everywhere in the finite z plane except at the points $z = 1$ and $-2/3$.

Orthogonal Families

3.27. Let $u(x, y) = \alpha$ and $v(x, y) = \beta$, where u and v are the real and imaginary parts of an analytic function $f(z)$ and α and β are any constants, represent two families of curves. Prove that if $f'(z) \neq 0$, then the families are orthogonal (i.e., each member of one family is perpendicular to each member of the other family at their point of intersection).

Solution

Consider any two members of the respective families, say $u(x, y) = \alpha_1$ and $v(x, y) = \beta_1$ where α_1 and β_1 are particular constants [Fig. 3-10].

Differentiating $u(x, y) = \alpha_1$ with respect to x yields

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

Then the slope of $u(x, y) = \alpha_1$ is

$$\frac{dy}{dx} = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}$$

Similarly, the slope of $v(x, y) = \beta_1$ is

$$\frac{dy}{dx} = -\frac{\partial v}{\partial x} \Big/ \frac{\partial v}{\partial y}$$

Now

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \neq 0 \quad \Rightarrow \quad \text{either } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \neq 0 \text{ or } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \neq 0$$

From these equations and inequalities, it follows that either the product of the slopes is -1 (when none of the partials is zero) or one slope is 0 and the other infinity, i.e., one tangent line is horizontal and the other is vertical, when

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$$

Thus, the curves are orthogonal if $f'(z) \neq 0$.

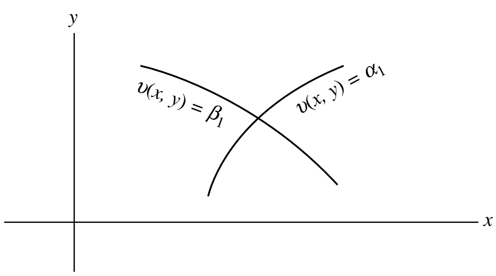

Fig. 3-10

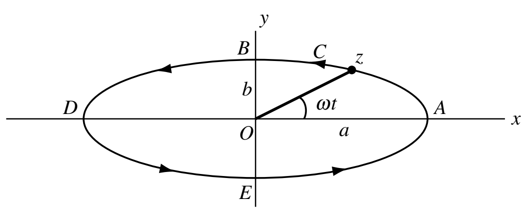

Fig. 3-11

3.28. Find the orthogonal trajectories of the family of curves in the xy plane which are defined by $e^{-x}(x \sin y - y \cos y) = \alpha$ where α is a real constant.

Solution

By Problems 3.7 and 3.27, it follows that $e^{-x}(y \sin y + x \cos y) = \beta$, where β is a real constant, is the required equation of the orthogonal trajectories.

Applications to Geometry and Mechanics

3.29. An ellipse C has the equation $z = a \cos \omega t + bi \sin \omega t$ where a, b, ω are positive constants, $a > b$, and t is a real variable. (a) Graph the ellipse and show that as t increases from $t = 0$ the ellipse is traversed in a counterclockwise direction. (b) Find a unit tangent vector to C at any point.

Solution

(a) As t increases from 0 to $\pi/2\omega$, $\pi/2\omega$ to π/ω, π/ω to $3\pi/2\omega$, and $3\pi/2\omega$ to $2\pi/\omega$, point z on C moves from A to B, B to D, D to E, and E to A, respectively (i.e., it moves in a counterclockwise direction as shown in Fig. 3-11).

(b) A tangent vector to C at any point t is

$$\frac{dz}{dt} = -a\omega \sin \omega t + b\omega i \cos \omega t$$

Then a unit tangent vector to C at any point t is

$$\frac{dz/dt}{|dz/dt|} = \frac{-a\omega \sin \omega t + b\omega i \cos \omega t}{|-a\omega \sin \omega t + b\omega i \cos \omega t|} = \frac{-a \sin \omega t + bi \cos \omega t}{\sqrt{a^2 \sin^2 \omega t + b^2 \cos^2 \omega t}}$$

3.30. In Problem 3.29, suppose that z is the position vector of a particle moving on C and that t is the time.

(a) Determine the velocity and speed of the particle at any time.

(b) Determine the acceleration both in magnitude and direction at any time.

(c) Prove that $d^2z/dt^2 = -\omega^2 z$ and give a physical interpretation.

(d) Determine where the velocity and acceleration have the greatest and least magnitudes.

Solution

(a) Velocity $= dz/dt = -a\omega \sin \omega t + b\omega i \cos \omega t$.

$$\text{Speed} = \text{magnitude of velocity} = |dz/dt| = \omega\sqrt{a^2 \sin^2 \omega t + b^2 \cos^2 \omega t}$$

(b) Acceleration $= d^2z/dt^2 = -a\omega^2 \cos \omega t - b\omega^2 i \sin \omega t$.

$$\text{Magnitude of acceleration} = |d^2z/dt^2| = \omega^2\sqrt{a^2 \cos^2 \omega t + b^2 \sin^2 \omega t}$$

(c) From (b) we see that

$$d^2z/dt^2 = -a\omega^2 \cos \omega t - b\omega^2 i \sin \omega t = -\omega^2(a \cos \omega t + bi \sin \omega t) = -\omega^2 z$$

Physically, this states that the acceleration at any time is always directed toward point O and has magnitude proportional to the instantaneous distance from O. As the particle moves, its projection on the x and y axes describes what is sometimes called *simple harmonic motion of period* $2\pi/\omega$. The acceleration is sometimes known as the *centripetal acceleration*.

(d) From (a) and (b), we have

$$\text{Magnitude of velocity} = \omega\sqrt{a^2 \sin^2 \omega t + b^2(1 - \sin^2 \omega t)} = \omega\sqrt{(a^2 - b^2)\sin^2 \omega t + b^2}$$

$$\text{Magnitude of acceleration} = \omega^2\sqrt{a^2 \cos^2 \omega t + b^2(1 - \cos^2 \omega t)} = \omega^2\sqrt{(a^2 - b^2)\cos^2 \omega t + b^2}$$

Then, the velocity has the greatest magnitude [given by ωa] where $\sin \omega t = \pm 1$, i.e., at points B and E [Fig. 3-11], and the least magnitude [given by ωb] where $\sin \omega t = 0$, i.e., at points A and D.

Similarly, the acceleration has the greatest magnitude [given by $\omega^2 a$] where $\cos \omega t = \pm 1$, i.e., at points A and D, and the least magnitude [given by $\omega^2 b$] where $\cos \omega t = 0$, i.e., at points B and E.

Theoretically, the planets of our solar system move in elliptical paths with the Sun at one focus. In practice, there is some deviation from an exact elliptical path.

Gradient, Divergence, Curl, and Laplacian

3.31. Prove the equivalence of the operators:

(a) $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}$, (b) $\dfrac{\partial}{\partial y} = i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right)$ where $z = x + iy$, $\bar{z} = x - iy$.

Solution

If F is any continuously differentiable function, then

(a) $\dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x} + \dfrac{\partial F}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial x} = \dfrac{\partial F}{\partial z} + \dfrac{\partial F}{\partial \bar{z}}$ showing the equivalence $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}$.

(b) $\dfrac{\partial F}{\partial y} = \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial y} + \dfrac{\partial F}{\partial \bar{z}}\dfrac{\partial \bar{z}}{\partial y} = \dfrac{\partial F}{\partial z}(i) + \dfrac{\partial F}{\partial \bar{z}}(-i) = i\left(\dfrac{\partial F}{\partial z} - \dfrac{\partial F}{\partial \bar{z}}\right)$ showing the equivalence $\dfrac{\partial}{\partial y} = i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right)$.

3.32. Show that (a) $\nabla \equiv \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial \bar{z}}$, (b) $\overline{\nabla} \equiv \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial z}$.

Solution

From the equivalences established in Problem 3.31, we have

(a) $\nabla \equiv \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}} + i^2\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right) = 2\dfrac{\partial}{\partial \bar{z}}$

(b) $\overline{\nabla} \equiv \dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}} - i^2\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right) = 2\dfrac{\partial}{\partial z}$

3.33. Suppose $F(x, y) = c$ [where c is a constant and F is continuously differentiable] is a curve in the xy plane. Show that grad $F = \nabla F = (\partial F/\partial x) + i(\partial F/\partial y)$, is a vector normal to the curve.

Solution

We have $dF = (\partial F/\partial x)dx + (\partial F/\partial y)dy = 0$. In terms of dot product [see page X], this can be written

$$\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right) \cdot (dx + i\,dy) = 0$$

But $dx + i\,dy$ is a vector tangent to C. Hence $\nabla F = (\partial F/\partial x) + i(\partial F/\partial y)$ must be perpendicular to C.

3.34. Show that $\dfrac{\partial P}{\partial x} - \dfrac{\partial Q}{\partial y} + i\left(\dfrac{\partial Q}{\partial x} + \dfrac{\partial P}{\partial y}\right) = 2\dfrac{\partial B}{\partial \bar{z}}$ where $B(z, \bar{z}) = P(x, y) + iQ(x, y)$.

Solution

From Problem 3.32, $\nabla B = 2(\partial B/\partial \bar{z})$. Hence

$$\nabla B = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P + iQ) = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) = 2\frac{\partial B}{\partial \bar{z}}$$

3.35. Let C be the curve in the xy plane defined by $3x^2y - 2y^3 = 5x^4y^2 - 6x^2$. Find a unit vector normal to C at $(1, -1)$.

Solution

Let $F(x, y) = 3x^2y - 2y^3 - 5x^4y^2 + 6x^2 = 0$. By Problem 3.33, a vector normal to C at $(1, -1)$ is

$$\nabla F = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = (6xy - 20x^3y^2 + 12x) + i(3x^2 - 6y^2 - 10x^4y) = -14 + 7i$$

Then a unit vector normal to C at $(1, -1)$ is $\dfrac{-14 + 7i}{|-14 + 7i|} = \dfrac{-2 + i}{\sqrt{5}}$. Another such unit vector is $\dfrac{2 - i}{\sqrt{5}}$.

3.36. Suppose $A(x, y) = 2xy - ix^2y^3$. Find (a) grad A, (b) div A, (c) |curl A|, (d) Laplacian of A.

Solution

(a) $\text{grad } A = \nabla A = \left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)(2xy - ix^2y^3) = \dfrac{\partial}{\partial x}(2xy - ix^2y^3) + i\dfrac{\partial}{\partial y}(2xy - ix^2y^3)$

$= 2y - 2ixy^3 + i(2x - 3ix^2y^3) = 2y + 3x^2y^2 + i(2x - 2xy^3)$

(b) $\text{div } A = \nabla \cdot A = \text{Re}\{\bar{\nabla} A\} = \text{Re}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(2xy - ix^2y^3)\right\}$

$$= \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2y^3) = 2y - 3x^2y^2$$

(c) $|\text{curl } A| = |\nabla \times A| = |\text{Im}\{\bar{\nabla} A\}| = \left|\text{Im}\left\{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(2xy - ix^2y^3)\right\}\right|$

$$= \left|\frac{\partial}{\partial x}(-x^2y^3) - \frac{\partial}{\partial y}(2xy)\right| = \left|-2xy^3 - 2x\right|$$

(d) $\text{Laplacian } A = \nabla^2 A = \text{Re}\{\bar{\nabla}\nabla A\} = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = \frac{\partial^2}{\partial x^2}(2xy - ix^2y^3) + \frac{\partial^2}{\partial y^2}(2xy - ix^2y^3)$

$$= \frac{\partial}{\partial x}(2y - 2ixy^3) + \frac{\partial}{\partial y}(2x - 3ix^2y^2) = -2iy^3 - 6ix^2y$$

Miscellaneous Problems

3.37. Prove that in polar form the Cauchy–Riemann equations can be written

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Solution

We have $x = r\cos\theta$, $y = r\sin\theta$ or $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2 + y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2 + y^2}\right) = \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2 + y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{x}{x^2 + y^2}\right) = \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta \quad (2)$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta \quad (3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta \quad (4)$$

From the Cauchy–Riemann equation $\partial u/\partial x = \partial v/\partial y$ we have, using (1) and (4),

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta - \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \quad (5)$$

From the Cauchy–Riemann equation $\partial u/\partial y = -(\partial v/\partial x)$ we have, using (2) and (3),

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0 \quad (6)$$

Multiplying (5) by $\cos\theta$, (6) by $\sin\theta$ and adding yields

$$\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta} = 0 \quad \text{or} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

.

Multiplying (5) by $-\sin\theta$, (6) by $\cos\theta$ and adding yields

$$\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} = 0 \quad \text{or} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

.

3.38. Prove that the real and imaginary parts of an analytic function of a complex variable when expressed in polar form satisfy the equation [Laplace's equation in polar form]

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = 0$$

Solution

From Problem 3.37,

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \tag{1}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \tag{2}$$

To eliminate v differentiate (1) partially with respect to r and (2) with respect to θ. Then

$$\frac{\partial^2 v}{\partial r\,\partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right) = \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} \tag{3}$$

$$\frac{\partial^2 v}{\partial \theta\,\partial r} = \frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial r}\right) = \frac{\partial}{\partial \theta}\left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \tag{4}$$

But

$$\frac{\partial^2 v}{\partial r\,\partial \theta} = \frac{\partial^2 v}{\partial \theta\,\partial r}$$

assuming the second partial derivatives are continuous. Hence, from (3) and (4),

$$r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Similarly, by elimination of u, we find

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

so that the required result is proved.

3.39. Suppose $w = f(\zeta)$ where $\zeta = g(z)$. Assuming f and g are analytic in a region $\mathcal{R}$, prove that

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz}$$

Solution

Let z be given an increment $\Delta z \neq 0$ so that $z + \Delta z$ is in $\mathcal{R}$. Then, as a consequence, ζ and w take on increments $\Delta\zeta$ and Δw, respectively, where

$$\Delta w = f(\zeta + \Delta\zeta) - f(\zeta), \quad \Delta\zeta = g(z + \Delta z) - g(z) \tag{1}$$

Note that as $\Delta z \to 0$, we have $\Delta w \to 0$ and $\Delta\zeta \to 0$.

If $\Delta\zeta \neq 0$, let us write $\epsilon = (\Delta w/\Delta\zeta) - (dw/d\zeta)$ so that $\epsilon \to 0$ as $\Delta\zeta \to 0$ and

$$\Delta w = \frac{dw}{d\zeta}\Delta\zeta + \epsilon\Delta\zeta \tag{2}$$

If $\Delta\zeta = 0$ for values of Δz, then (1) shows that $\Delta w = 0$ for these values of Δz. For such cases, we define $\epsilon = 0$.

It follows that in both cases, $\Delta\zeta \neq 0$ or $\Delta\zeta = 0$, (2) holds. Then dividing (2) by $\Delta z \neq 0$ and taking the limit as $\Delta z \to 0$, we have

$$\begin{aligned}\frac{dw}{dz} = \lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z} &= \lim_{\Delta z\to 0}\left(\frac{dw}{d\zeta}\frac{\Delta\zeta}{\Delta z} + \epsilon\frac{\Delta w}{\Delta z}\right)\\ &= \frac{dw}{d\zeta}\cdot\lim_{\Delta z\to 0}\frac{\Delta\zeta}{\Delta z} + \lim_{\Delta z\to 0}\epsilon\cdot\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}\\ &= \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz} + 0\cdot\frac{d\zeta}{dz} = \frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz}\end{aligned}$$

3.40. (a) Suppose $u_1(x, y) = \partial u/\partial x$ and $u_2(x, y) = \partial u/\partial y$. Prove that $f'(z) = u_1(z, 0) - iu_2(z, 0)$.

(b) Show how the result in (a) can be used to solve Problems 3.7 and 3.8.

Solution

(a) From Problem 3.5, we have $f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y} = u_1(x, y) - iu_2(x, y)$.

Putting $y = 0$, this becomes $f'(x) = u_1(x, 0) - iu_2(x, 0)$.

Then, replacing x by z, we have as required $f'(z) = u_1(z, 0) - iu_2(z, 0)$.

(b) Since we are given $u = e^{-x}(x\sin y - y\cos y)$, we have

$$\begin{aligned}u_1(x, y) &= \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y\\ u_2(x, y) &= \frac{\partial u}{\partial y} = xe^{-z}\cos y + ye^{-z}\sin y - e^{-x}\cos y\end{aligned}$$

so that from part (a),

$$f'(z) = u_1(z, 0) - iu_2(z, 0) = 0 - i(ze^{-z} - e^{-z}) = -i(ze^{-z} - e^{-z})$$

Integrating with respect to z we have, apart from a constant, $f(z) = ize^{-z}$. By separating this into real and imaginary parts, $v = e^{-x}(y\sin y + x\cos y)$ apart from a constant.

3.41. Suppose A is real or, more generally, suppose $\text{Im}\,A$ is harmonic. Prove that $|\text{curl grad}\,A| = 0$.

Solution

If $A = P + Qi$, we have

$$\text{grad}\,A = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(P + iQ) = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)$$

Then

$$\begin{aligned}|\text{curl grad}\,A| &= \left|\text{Im}\left[\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left\{\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + i\left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)\right\}\right]\right|\\ &= \left|\text{Im}\left[\frac{\partial^2 P}{\partial x^2} - \frac{\partial^2 Q}{\partial x\,\partial y} + i\left(\frac{\partial^2 P}{\partial x\,\partial y} + \frac{\partial^2 Q}{\partial x^2}\right) - i\left(\frac{\partial^2 P}{\partial y\,\partial x} - \frac{\partial^2 Q}{\partial y^2}\right) + \left(\frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 Q}{\partial y\,\partial x}\right)\right]\right|\\ &= \left|\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2}\right|\end{aligned}$$

Hence if $Q = 0$, i.e., A is real, or if Q is harmonic, $|\text{curl grad}\,A| = 0$.

3.42. Solve the partial differential equation $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = x^2 - y^2$.

Solution

Let $z = x + iy$, $\bar{z} = x - iy$ so that $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$. Then $x^2 - y^2 = \frac{1}{2}(z^2 + \bar{z}^2)$ and

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \nabla^2 U = 4\frac{\partial^2 U}{\partial z\,\partial \bar{z}}$$

Thus, the given partial differential equation becomes $4(\partial^2 U/\partial z\,\partial\bar{z}) = \frac{1}{2}(z^2 + \bar{z}^2)$ or

$$\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial \bar{z}}\right) = \frac{1}{8}(z^2 + \bar{z}^2) \tag{1}$$

Integrating (1) with respect to z (treating $\bar{z}$ as constant),

$$\frac{\partial U}{\partial \bar{z}} = \frac{z^3}{24} + \frac{z\bar{z}^2}{8} + F_1(\bar{z}) \tag{2}$$

where $F_1(\bar{z})$ is an arbitrary function of $\bar{z}$. Integrating (2) with respect to $\bar{z}$,

$$U = \frac{z^3\bar{z}}{24} + \frac{z\bar{z}^3}{24} + F(\bar{z}) + G(z) \tag{3}$$

where $F(\bar{z})$ is the function obtained by integrating $F_1(\bar{z})$, and $G(z)$ is an arbitrary function of z. Replacing z and $\bar{z}$ by $x + iy$ and $x - iy$, respectively, we obtain

$$U = \frac{1}{12}(x^4 - y^4) + F(x - iy) + G(x + iy)$$

SUPPLEMENTARY PROBLEMS

Derivatives

3.43. Using the definition, find the derivative of each function at the indicated points.

(a) $f(z) = 3z^2 + 4iz - 5 + i$; $z = 2$, (b) $f(z) = \dfrac{2z - i}{z + 2i}$; $z = -i$, (c) $f(z) = 3z^{-2}$; $z = 1 + i$.

3.44. Prove that $\dfrac{d}{dz}(z^2\bar{z})$ does not exist anywhere.

3.45. Determine whether $|z|^2$ has a derivative anywhere.

3.46. For each of the following functions determine the singular points, i.e., points at which the function is not analytic. Determine the derivatives at all other points. (a) $\dfrac{z}{z + i}$, (b) $\dfrac{3z - 2}{z^2 + 2z + 5}$.

Cauchy–Riemann Equations

3.47. Verify that the real and imaginary parts of the following functions satisfy the Cauchy–Riemann equations and thus deduce the analyticity of each function:

(a) $f(z) = z^2 + 5iz + 3 - i$, (b) $f(z) = ze^{-z}$, (c) $f(z) = \sin 2z$.

3.48. Show that the function $x^2 + iy^3$ is not analytic anywhere. Reconcile this with the fact that the Cauchy–Riemann equations are satisfied at $x = 0$, $y = 0$.

3.49. Prove that if $w = f(z) = u + iv$ is analytic in a region $\mathcal{R}$, then $dw/dz = \partial w/\partial x = -i(\partial w/\partial y)$.

3.50. (a) Prove that the function $u = 2x(1 - y)$ is harmonic. (b) Find a function v such that $f(z) = u + iv$ is analytic [i.e., find the conjugate function of u]. (c) Express $f(z)$ in terms of z.

3.51. Answer Problem 3.50 for the function $u = x^2 - y^2 - 2xy - 2x + 3y$.

3.52. Verify that the Cauchy–Riemann equations are satisfied for the functions (a) e^{z^2}, (b) $\cos 2z$, (c) $\sinh 4z$.

3.53. Determine which of the following functions u are harmonic. For each harmonic function, find the conjugate harmonic function v and express $u + iv$ as an analytic function of z.

(a) $3x^2y + 2x^2 - y^3 - 2y^2$, (b) $2xy + 3xy^2 - 2y^3$, (c) $xe^z \cos y - ye^z \sin y$, (d) $e^{-2xy} \sin(x^2 - y^2)$.

3.54. (a) Prove that $\psi = \ln[(x-1)^2 + (y-2)^2]$ is harmonic in every region which does not include the point (1, 2). (b) Find a function ϕ such that $\phi + i\psi$ is analytic. (c) Express $\phi + i\psi$ as a function of z.

3.55. Suppose $\text{Im}\{f'(z)\} = 6x(2y - 1)$ and $f(0) = 3 - 2i$, $f(1) = 6 - 5i$. Find $f(1 + i)$.

Differentials

3.56. Let $w = iz^2 - 4z + 3i$. Find (a) Δw, (b) dw, (c) $\Delta w - dw$ at the point $z = 2i$.

3.57. Suppose $w = (2z + 1)^3$, $z = -i$, $\Delta z = 1 + i$. Find (a) Δw and (b) dw.

3.58. Suppose $w = 3iz^2 + 2z + 1 - 3i$. Find (a) Δw, (b) dw, (c) $\Delta w/\Delta z$, (d) dw/dz where $z = i$.

3.59. (a) Suppose $w = \sin z$. Show that $\dfrac{\Delta w}{\Delta z} = \cos z\left(\dfrac{\sin \Delta z}{\Delta z}\right) - 2\sin z\left\{\dfrac{\sin^2(\Delta z/2)}{\Delta z}\right\}$.

(b) Assuming $\lim_{\Delta z \to 0} \dfrac{\sin \Delta z}{\Delta z} = 1$, prove that $\dfrac{dw}{dz} = \cos z$.

(c) Show that $dw = (\cos z)\, dz$.

3.60. (a) Let $w = \ln z$. Show that if $\Delta z/z = \zeta$, then $\Delta w/\Delta z = (1/z)\ln\{(1 + \zeta)^{1/\zeta}\}$.

(b) Assuming $\lim_{\zeta \to 0}(1 + \zeta)^{1/\zeta} = e$ prove that $dw/dz = 1/z$.

(c) Show that $d(\ln z) = dz/z$.

3.61. Giving restrictions on $f(z)$ and $g(z)$, prove that

(a) $d\{f(z)g(z)\} = \{f(z)g'(z) + g(z)f'(z)\}dz$
(b) $d\{f(z)/g(z)\} = \{g(z)f'(z) - f(z)g'(z)\}dz/\{g(z)\}^2$

Differentiation Rules. Derivatives of Elementary Functions

3.62. Suppose $f(z)$ and $g(z)$ are analytic in a region $\mathcal{R}$. Then prove that

(a) $d/dz\{2if(z) - (1 + i)g(z)\} = 2if'(z) - (1 + i)g'(z)$, (b) $d/dz\{f(z)\}^2 = 2f(z)f'(z)$,
(c) $d/dz\{f(z)\}^{-1} = -\{f(z)\}^{-2}f'(z)$.

3.63. Using differentiation rules, find the derivatives of each of the following functions:
(a) $(1 + 4i)z^2 - 3z - 2$, (b) $(2z + 3i)(z - i)$, (c) $(2z - i)/(z + 2i)$, (d) $(2iz + 1)^2$, (e) $(iz - 1)^{-3}$.

3.64. Find the derivatives of each of the following at the indicated points:

(a) $(z + 2i)(i - z)/(2z - 1)$, $z = i$, (b) $\{z + (z^2 + 1)^2\}^2$, $z = 1 + i$.

3.65. Prove that (a) $\frac{d}{dz}\sec z = \sec z \tan z$, (b) $\frac{d}{dz}\cot z = -\csc^2 z$.

3.66. Prove that (a) $\frac{d}{dz}(z^2+1)^{1/2} = \frac{z}{(z^2+1)^{1/2}}$, (b) $\frac{d}{dz}\ln(z^2+2z+2) = \frac{2z+2}{z^2+2z+2}$ indicating restrictions if any.

3.67. Find the derivatives of each of the following, indicating restrictions if any.

(a) $3\sin^2(z/2)$, (b) $\tan^3(z^2-3z+4i)$, (c) $\ln(\sec z + \tan z)$, (d) $\csc\{(z^2+1)^{1/2}\}$, (e) $(z^2-1)\cos(z+2i)$.

3.68. Prove that (a) $\frac{d}{dz}(1+z^2)^{3/2} = 3z(1+z^2)^{1/2}$, (b) $\frac{d}{dz}(z+2\sqrt{z})^{1/3} = \frac{1}{3}z^{-1/2}(z+2\sqrt{z})^{-2/3}(\sqrt{z}+1)$.

3.69. Prove that (a) $\frac{d}{dz}(\tan^{-1} z) = \frac{1}{z^2+1}$, (b) $\frac{d}{dz}(\sec^{-1} z) = \frac{1}{z\sqrt{z^2-1}}$.

3.70. Prove that (a) $\frac{d}{dz}\sinh^{-1} z = \frac{1}{\sqrt{1+z^2}}$, (b) $\frac{d}{dz}\operatorname{csch}^{-1} z = \frac{-1}{z\sqrt{z^2+1}}$.

3.71. Find the derivatives of each of the following:

(a) $\{\sin^{-1}(2z-1)\}^2$, (c) $\cos^{-1}(\sin z - \cos z)$, (e) $\coth^{-1}(z\csc 2z)$

(b) $\ln\{\cot^{-1} z^2\}$, (d) $\tan^{-1}(z+3i)^{-1/2}$, (f) $\ln(z - \frac{3}{2} + \sqrt{z^2-3z+2i})$

3.72. Suppose $w = \cos^{-1}(z-1)$, $z = \sinh(3\zeta + 2i)$ and $\zeta = \sqrt{t}$. Find dw/dt.

3.73. Let $w = t\sec(t-3i)$ and $z = \sin^{-1}(2t-1)$. Find dw/dz.

3.74. Suppose $w^2 - 2w + \sin 2z = 0$. Find (a) dw/dz, (b) d^2w/dz^2.

3.75. Given $w = \cos\zeta$, $z = \tan(\zeta + \pi i)$. Find d^2w/dz^2 at $\zeta = 0$.

3.76. Find (a) $d/dz\{z^{\ln z}\}$, (b) $d/dz\left\{[\sin(iz-2)]^{\tan^{-1}(z+3i)}\right\}$.

3.77. Find the second derivatives of each of the following:

(a) $3\sin^2(2z-1+i)$, (b) $\ln\tan z^2$, (c) $\sinh(z+1)^2$, (d) $\cos^{-1}(\ln z)$, (e) $\operatorname{sech}^{-1}\sqrt{1+z}$.

L'Hospital's Rule

3.78. Evaluate (a) $\lim_{z\to 2i} \frac{z^2+4}{2z^2+(3-4i)z-6i}$, (b) $\lim_{z\to e^{\pi i/3}} (z-e^{\pi i/3})\left(\frac{z}{z^3+1}\right)$, (c) $\lim_{z\to i} \frac{z^2-2iz-1}{z^4+2z^2+1}$.

3.79. Evaluate (a) $\lim_{z\to 0} \frac{z-\sin z}{z^3}$, (b) $\lim_{z\to m\pi i} (z-m\pi i)\left(\frac{e^z}{\sin z}\right)$.

3.80. Find $\lim_{z\to i} \frac{\tan^{-1}(z^2+1)^2}{\sin^2(z^2+1)}$ where the branch of the inverse tangent is chosen such that $\tan^{-1} 0 = 0$.

3.81. Evaluate $\lim_{z\to 0}\left(\frac{\sin z}{z}\right)^{1/z^2}$.

Singular Points

3.82. For each of the following functions locate and name the singularities in the finite z plane.

(a) $\frac{z^2-3z}{z^2+2z+2}$, (b) $\frac{\ln(z+3i)}{z^2}$, (c) $\sin^{-1}(1/z)$, (d) $\sqrt{z(z^2+1)}$, (e) $\frac{\cos z}{(z+i)^3}$

3.83. Show that $f(z) = (z+3i)^5/(z^2-2z+5)^2$ has double poles at $z = 1 \pm 2i$ and a simple pole at infinity.

3.84. Show that e^{x^2} has an essential singularity at infinity.

3.85. Locate and name all the singularities of each of the following functions.

(a) $(z+3)/(z^2-1)$, (b) $\csc(1/z^2)$, (c) $(z^2+1)/z^{3/2}$.

Orthogonal Families

3.86. Find the orthogonal trajectories of the following families of curves:

(a) $x^3y - xy^3 = \alpha$, (b) $e^{-x}\cos y + xy = \alpha$.

3.87. Find the orthogonal trajectories of the family of curves $r^2 \cos 2\theta = \alpha$.

3.88. By separating $f(z) = z + 1/z$ into real and imaginary parts, show that the families $(r^2+1)\cos\theta = \alpha r$ and $(r^2-1)\sin\theta = \beta r$ are orthogonal trajectories and verify this by another method.

3.89. Let n be any real constant. Prove that $r^n = \alpha \sec n\theta$ and $r^n = \beta \csc n\theta$ are orthogonal trajectories.

Applications to Geometry and Mechanics

3.90. A particle moves along a curve $z = e^{-t}(2\sin t + i\cos t)$.

(a) Find a unit tangent vector to the curve at the point where $t = \pi/4$.

(b) Determine the magnitudes of velocity and acceleration of the particle at $t = 0$ and $\pi/2$.

3.91. A particle moves along the curve $z = ae^{i\omega t}$. (a) Show that its speed is always constant and equal to ωa.
(b) Show that the magnitude of its acceleration is always constant and equal to $\omega^2 a$.
(c) Show that the acceleration is always directed toward $z = 0$.
(d) Explain the relationship of this problem to the problem of a stone being twirled at the end of a string in a horizontal plane.

3.92. The position at time t of a particle moving in the z plane is given by $z = 3te^{-4it}$. Find the magnitudes of (a) the velocity, (b) the acceleration of the particle at $t = 0$ and $t = \pi$.

3.93. A particle P moves along the line $x + y = 2$ in the z plane with a uniform speed of $3\sqrt{2}$ ft/sec from the point $z = -5 + 7i$ to $z = 10 - 8i$. If $w = 2z^2 - 3$ and P' is the image of P in the w plane, find the magnitudes of (a) the velocity and (b) the acceleration of P' after 3 seconds.

Gradient, Divergence, Curl, and Laplacian

3.94. Let $F = x^2y - xy^2$. Find (a) ∇F, (b) $\nabla^2 F$.

3.95. Let $B = 3z^2 + 4\bar{z}$. Find (a) grad B, (b) div B, (c) $|\text{curl } B|$, (d) Laplacian B.

3.96. Let C be the curve in the xy plane defined by $x^2 - xy + y^2 = 7$. Find a unit vector normal to C at (a) the point $(-1, 2)$, (b) any point.

3.97. Find an equation for the line normal to the curve $x^2y = 2xy + 6$ at the point (3, 2).

3.98. Show that $\nabla^2|f(z)|^2 = 4|f'(z)|^2$. Illustrate by choosing $f(z) = z^2 + iz$.

3.99. Prove $\nabla^2\{FG\} = F\nabla^2 G + G\nabla^2 F + 2\nabla F \cdot \nabla G$

3.100. Prove div grad $A = 0$ if A is imaginary or, more generally, if $\text{Re}\{A\}$ is harmonic.

Miscellaneous Problems

3.101. Let $f(z) = u(x, y) + iv(x, y)$. Prove that:

(a) $f(z) = 2u(z/2, -iz/2) + \text{constant}$, (b) $f(z) = 2iv(z/2, -iz/2) + \text{constant}$.

3.102. Use Problem 3.101 to find $f(z)$ if (a) $u(x, y) = x^4 - 6x^2y^2 + y^4$, (b) $v(x, y) = \sinh x \cos y$.

3.103. Suppose V is the instantaneous speed of a particle moving along any plane curve C. Prove that the normal component of the acceleration at any point of C is given by V^2/R where R is the radius of curvature at the point.

3.104. Find an analytic function $f(z)$ such that $\text{Re}\{f'(z)\} = 3x^2 - 4y - 3y^2$ and $f(1+i) = 0$.

3.105. Show that the family of curves

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

with $-a^2 < \lambda < -b^2$ is orthogonal to the family with $\lambda > -b^2 > -a^2$.

3.106. Prove that the equation $F(x, y) =$ constant can be expressed as $u(x, y) =$ constant where u is harmonic if and only if the following is a function of F:

$$\frac{\partial^2 F/\partial x^2 + \partial^2 F/\partial y^2}{(\partial F/\partial x)^2 + (\partial F/\partial y)^2}$$

3.107. Illustrate the result in Problem 3.106 by considering $(y+2)/(x-1) =$ constant.

3.108. Let $f'(z) = 0$ in a region $\mathcal{R}$. Prove that $f(z)$ must be a constant in $\mathcal{R}$.

3.109. Suppose $w = f(z)$ is analytic and expressed in polar coordinates (r, θ). Prove that

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r}$$

3.110. Suppose u and v are conjugate harmonic functions. Prove that

$$dv = \frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx$$

3.111. Given u and v are harmonic in a region $\mathcal{R}$. Prove that the following is analytic in $\mathcal{R}$:

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

3.112. Prove that $f(z) = |z|^4$ is differentiable but not analytic at $z = 0$.

3.113. Given $f(z)$ is analytic in a region $\mathcal{R}$ and $f(z)f'(z) \neq 0$ in $\mathcal{R}$, prove that $\psi = \ln|f(z)|$ is harmonic in $\mathcal{R}$.

3.114. Express the Cauchy–Riemann equations in terms of the curvilinear coordinates (ξ, η) where $x = e^{\xi}\cosh\eta$, $y = e^{\xi}\sinh\eta$.

3.115. Show that a solution of the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_0\cos\omega t$$

where L, R, C, E_0 and ω are constants, is given by

$$Q = \text{Re}\left\{\frac{E_0 e^{i\omega t}}{i\omega[R + i(\omega L - 1/\omega C)]}\right\}$$

The equation arises in the *theory of alternating currents* of electricity.

[*Hint*. Rewrite the right hand side as $E_0 e^{i\omega t}$ and then assume a solution of the form $Ae^{i\omega t}$ where A is to be determined.]

3.116. Show that $\nabla^2\{f(z)\}^n = n^2|f(z)|^{n-2}|f'(z)|^2$, stating restrictions on $f(z)$.

3.117. Solve the partial differential equation $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = \dfrac{8}{x^2+y^2}$.