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CHAPTER 1

Complex Numbers

1.1 The Real Number System

The number system as we know it today is a result of gradual development as indicated in the following list.

(1) **Natural numbers** 1, 2, 3, 4, . . . , also called *positive integers*, were first used in counting. If a and b are natural numbers, the *sum* $a + b$ and *product* $a \cdot b$, $(a)(b)$ or ab are also natural numbers. For this reason, the set of natural numbers is said to be *closed* under the operations of *addition* and *multiplication* or to satisfy the *closure property* with respect to these operations.

(2) **Negative integers and zero,** denoted by $-1,\ -2,\ -3, \ldots$ and 0, respectively, permit solutions of equations such as $x + b = a$ where a and b are any natural numbers. This leads to the operation of *subtraction*, or *inverse of addition*, and we write $x = a - b$.

The set of positive and negative integers and zero is called the set of *integers* and is closed under the operations of addition, multiplication, and subtraction.

(3) **Rational numbers** or *fractions* such as $\frac{3}{4},\ -\frac{8}{3}, \ldots$ permit solutions of equations such as $bx = a$ for all integers a and b where $b \neq 0$. This leads to the operation of *division* or *inverse of multiplication*, and we write $x = a/b$ or $a \div b$ (called the *quotient* of a and b) where a is the *numerator* and b is the *denominator*.

The set of integers is a part or *subset* of the rational numbers, since integers correspond to rational numbers a/b where $b = 1$.

The set of rational numbers is closed under the operations of addition, subtraction, multiplication, and division, so long as division by zero is excluded.

(4) **Irrational numbers** such as $\sqrt{2}$ and π are numbers that cannot be expressed as a/b where a and b are integers and $b \neq 0$.

The set of rational and irrational numbers is called the set of *real* numbers. It is assumed that the student is already familiar with the various operations on real numbers.

1.2 Graphical Representation of Real Numbers

Real numbers can be represented by points on a line called the *real axis*, as indicated in Fig. 1-1. The point corresponding to zero is called the *origin*.

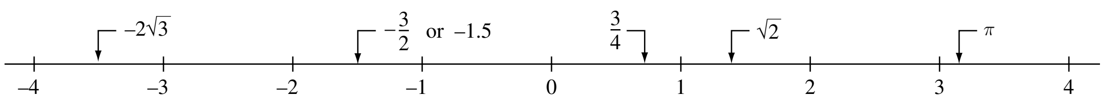

Fig. 1-1

Conversely, to each point on the line there is one and only one real number. If a point A corresponding to a real number a lies to the right of a point B corresponding to a real number b, we say that a is *greater than* b or b is *less than* a and write $a > b$ or $b < a$, respectively.

The set of all values of x such that $a < x < b$ is called an *open interval* on the real axis while $a \leq x \leq b$, which also includes the endpoints a and b, is called a *closed interval*. The symbol x, which can stand for any real number, is called a *real variable*.

The *absolute value* of a real number a, denoted by $|a|$, is equal to a if $a > 0$, to $-a$ if $a < 0$ and to 0 if $a = 0$. The distance between two points a and b on the real axis is $|a - b|$.

1.3 The Complex Number System

There is no real number x that satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and similar equations, the set of *complex numbers* is introduced.

We can consider a *complex number* as having the form $a + bi$ where a and b are real numbers and i, which is called the *imaginary unit*, has the property that $i^2 = -1$. If $z = a + bi$, then a is called the *real part* of z and b is called the *imaginary part* of z and are denoted by $\text{Re}\{z\}$ and $\text{Im}\{z\}$, respectively. The symbol z, which can stand for any complex number, is called a *complex variable*.

Two complex numbers $a + bi$ and $c + di$ are *equal* if and only if $a = c$ and $b = d$. We can consider real numbers as a subset of the set of complex numbers with $b = 0$. Accordingly the complex numbers $0 + 0i$ and $-3 + 0i$ represent the real numbers 0 and -3, respectively. If $a = 0$, the complex number $0 + bi$ or bi is called a *pure imaginary number*.

The *complex conjugate,* or briefly *conjugate,* of a complex number $a + bi$ is $a - bi$. The complex conjugate of a complex number z is often indicated by $\bar{z}$ or z^*.

1.4 Fundamental Operations with Complex Numbers

In performing operations with complex numbers, we can proceed as in the algebra of real numbers, replacing i^2 by -1 when it occurs.

(1) *Addition*

$$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$$

(2) *Subtraction*

$$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$$

(3) *Multiplication*

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

(4) *Division*

If $c \neq 0$ and $d \neq 0$, then

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2} \\ &= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i\end{aligned}$$

1.5 Absolute Value

The *absolute value* or *modulus* of a complex number $a + bi$ is defined as $|a + bi| = \sqrt{a^2 + b^2}$.

EXAMPLE 1.1: $|-4 + 2i| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$.

If $z_1, z_2, z_3, \ldots, z_m$ are complex numbers, the following properties hold.

(1) $|z_1 z_2| = |z_1||z_2|$ or $|z_1 z_2 \cdots z_m| = |z_1||z_2| \cdots |z_m|$

(2) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ if $z_2 \neq 0$

(3) $|z_1 + z_2| \leq |z_1| + |z_2|$ or $|z_1 + z_2 + \cdots + z_m| \leq |z_1| + |z_2| + \cdots + |z_m|$

(4) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

1.6 Axiomatic Foundation of the Complex Number System

From a strictly logical point of view, it is desirable to define a complex number as an ordered pair (a, b) of real numbers a and b subject to certain operational definitions, which turn out to be equivalent to those above. These definitions are as follows, where all letters represent real numbers.

A. Equality $(a, b) = (c, d)$ if and only if $a = c,\ b = d$
B. Sum $(a, b) + (c, d) = (a + c, b + d)$
C. Product $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
$m(a, b) = (ma, mb)$

From these we can show [Problem 1.14] that $(a, b) = a(1, 0) + b(0, 1)$ and we associate this with $a + bi$ where i is the symbol for $(0, 1)$ and has the property that $i^2 = (0, 1)(0, 1) = (-1, 0)$ [which can be considered equivalent to the real number -1] and $(1, 0)$ can be considered equivalent to the real number 1. The ordered pair $(0, 0)$ corresponds to the real number 0.

From the above, we can prove the following.

THEOREM 1.1: Suppose z_1, z_2, z_3 belong to the set S of complex numbers. Then

(1) $z_1 + z_2$ and $z_1 z_2$ belong to S — Closure law
(2) $z_1 + z_2 = z_2 + z_1$ — Commutative law of addition
(3) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ — Associative law of addition
(4) $z_1 z_2 = z_2 z_1$ — Commutative law of multiplication
(5) $z_1(z_2 z_3) = (z_1 z_2) z_3$ — Associative law of multiplication
(6) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ — Distributive law
(7) $z_1 + 0 = 0 + z_1 = z_1,\ 1 \cdot z_1 = z_1 \cdot 1 = z_1$, 0 is called the *identity with respect to addition,* 1 is called the *identity with respect to multiplication.*
(8) For any complex number z_1 there is a unique number z in S such that $z + z_1 = 0$; [z is called the *inverse of* z_1 *with respect to addition* and is denoted by $-z_1$].
(9) For any $z_1 \neq 0$ there is a unique number z in S such that $z_1 z = z z_1 = 1$; [z is called the *inverse of* z_1 *with respect to multiplication* and is denoted by z_1^{-1} or $1/z_1$].

In general, any set such as S, whose members satisfy the above, is called a *field*.

1.7 Graphical Representation of Complex Numbers

Suppose real scales are chosen on two mutually perpendicular axes $X'OX$ and $Y'OY$ [called the x and y *axes*, respectively] as in Fig. 1-2. We can locate any point in the plane determined by these lines by the ordered pair of real numbers (x, y) called *rectangular coordinates* of the point. Examples of the location of such points are indicated by P, Q, R, S, and T in Fig. 1-2.

Since a complex number $x + iy$ can be considered as an ordered pair of real numbers, we can represent such numbers by points in an xy plane called the *complex plane* or *Argand diagram.* The complex number represented by P, for example, could then be read as either (3, 4) or $3 + 4i$. To each complex number there corresponds one and only one point in the plane, and conversely to each point in the plane there corresponds one and only one complex number. Because of this we often refer to the complex number z as the *point* z. Sometimes, we refer to the x and y axes as the *real* and *imaginary* axes, respectively, and to the complex plane as the *z plane*. The distance between two points, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, in the complex plane is given by $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

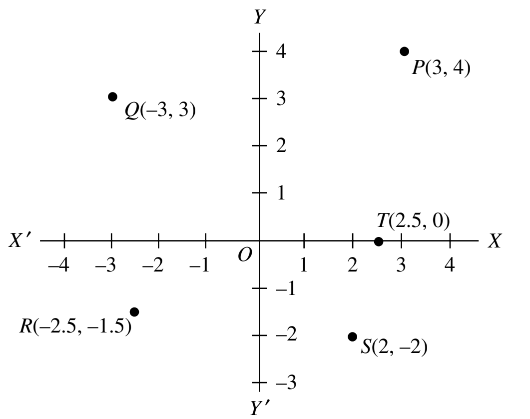

Fig. 1-2

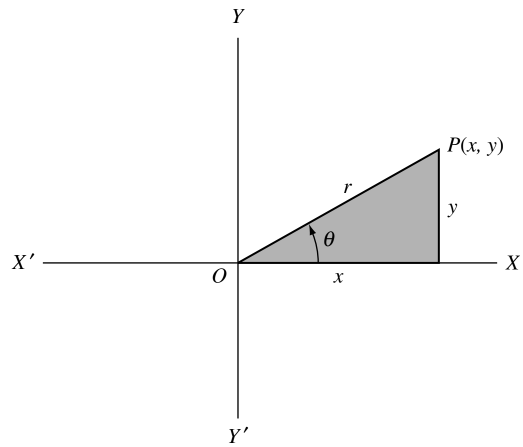

Fig. 1-3

1.8 Polar Form of Complex Numbers

Let P be a point in the complex plane corresponding to the complex number (x, y) or $x + iy$. Then we see from Fig. 1-3 that

$$x = r\cos\theta, \quad y = r\sin\theta$$

where $r = \sqrt{x^2 + y^2} = |x + iy|$ is called the *modulus* or *absolute value* of $z = x + iy$ [denoted by mod z or $|z|$]; and θ, called the *amplitude* or *argument* of $z = x + iy$ [denoted by arg z], is the angle that line OP makes with the positive x axis.

It follows that

$$z = x + iy = r(\cos\theta + i\sin\theta) \tag{1.1}$$

which is called the *polar form* of the complex number, and r and θ are called *polar coordinates*. It is sometimes convenient to write the abbreviation cis θ for $\cos\theta + i\sin\theta$.

For any complex number $z \neq 0$ there corresponds only one value of θ in $0 \leq \theta < 2\pi$. However, any other interval of length 2π, for example $-\pi < \theta \leq \pi$, can be used. Any particular choice, decided upon in advance, is called the *principal range*, and the value of θ is called its *principal value*.

1.9 De Moivre's Theorem

Let $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then we can show that [see Problem 1.19]

$$z_1 z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\} \tag{1.2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\} \tag{1.3}$$

A generalization of (1.2) leads to

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n \{\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i \sin(\theta_1 + \theta_2 + \cdots + \theta_n)\} \tag{1.4}$$

and if $z_1 = z_2 = \cdots = z_n = z$ this becomes

$$z^n = \{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta) \tag{1.5}$$

which is often called *De Moivre's theorem.*

1.10 Roots of Complex Numbers

A number w is called an nth *root* of a complex number z if $w^n = z$, and we write $w = z^{1/n}$. From De Moivre's theorem we can show that if n is a positive integer,

$$\begin{aligned} z^{1/n} &= \{r(\cos\theta + i\sin\theta)\}^{1/n} \\ &= r^{1/n}\left\{\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right\} \quad k = 0, 1, 2, \ldots, n-1 \end{aligned} \tag{1.6}$$

from which it follows that there are n different values for $z^{1/n}$, i.e., n different nth roots of z, provided $z \neq 0$.

1.11 Euler's Formula

By assuming that the infinite series expansion $e^x = 1 + x + (x^2/2!) + (x^3/3!) + \cdots$ of elementary calculus holds when $x = i\theta$, we can arrive at the result

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.7}$$

which is called *Euler's formula*. It is more convenient, however, simply to take (1.7) as a definition of $e^{i\theta}$. In general, we define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) \tag{1.8}$$

In the special case where $y = 0$ this reduces to e^x.

Note that in terms of (1.7) De Moivre's theorem reduces to $(e^{i\theta})^n = e^{in\theta}$.

1.12 Polynomial Equations

Often in practice we require solutions of polynomial equations having the form

$$a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0 \tag{1.9}$$

where $a_0 \neq 0, a_1, \ldots, a_n$ are given complex numbers and n is a positive integer called the *degree* of the equation. Such solutions are also called *zeros* of the polynomial on the left of (1.9) or *roots of the equation*.

A very important theorem called the *fundamental theorem of algebra* [to be proved in Chapter 5] states that every polynomial equation of the form (1.9) has at least one root in the complex plane. From this we can show that it has in fact n complex roots, some or all of which may be identical.

If $z_1, z_2, \ldots, z_n$ are the n roots, then (1.9) can be written

$$a_0(z - z_1)(z - z_2)\cdots(z - z_n) = 0 \tag{1.10}$$

which is called the *factored form* of the polynomial equation.

1.13 The *n*th Roots of Unity

The solutions of the equation $z^n = 1$ where n is a positive integer are called the nth *roots of unity* and are given by

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{2k\pi i/n} \quad k = 0, 1, 2, \ldots, n-1 \tag{1.11}$$

If we let $\omega = \cos 2\pi/n + i\sin 2\pi/n = e^{2\pi i/n}$, the n roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. Geometrically, they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one with center at the origin. This circle has the equation $|z| = 1$ and is often called the *unit circle*.

1.14 Vector Interpretation of Complex Numbers

A complex number $z = x + iy$ can be considered as a vector OP whose *initial point* is the origin O and whose *terminal point* P is the point (x, y) as in Fig. 1-4. We sometimes call $OP = x + iy$ the *position vector* of P. Two vectors having the same *length* or *magnitude* and *direction* but different initial points, such as OP and AB in Fig. 1-4, are considered equal. Hence we write $OP = AB = x + iy$.

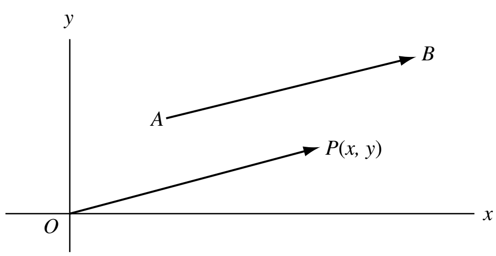

Fig. 1-4

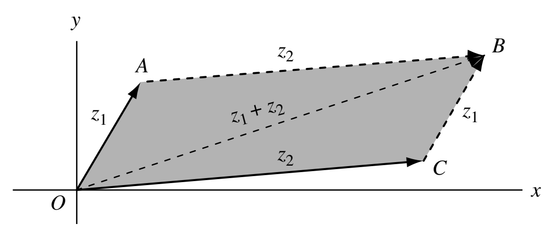

Fig. 1-5

Addition of complex numbers corresponds to the *parallelogram law* for addition of vectors [see Fig. 1-5]. Thus to add the complex numbers z_1 and z_2, we complete the parallelogram $OABC$ whose sides OA and OC correspond to z_1 and z_2. The diagonal OB of this parallelogram corresponds to $z_1 + z_2$. See Problem 1.5.

1.15 Stereographic Projection

Let $\mathcal{P}$ [Fig. 1-6] be the the complex plane and consider a sphere $\mathcal{S}$ tangent to $\mathcal{P}$ at $z = 0$. The diameter NS is perpendicular to $\mathcal{P}$ and we call points N and S the *north* and *south poles* of $\mathcal{S}$. Corresponding to any point A on $\mathcal{P}$ we can construct line NA intersecting $\mathcal{S}$ at point A'. Thus to each point of the complex plane $\mathcal{P}$ there corresponds one and only one point of the sphere $\mathcal{S}$, and we can represent any complex number by a point on the sphere. For completeness we say that the point N itself corresponds to the "point at infinity" of the plane. The set of all points of the complex plane including the point at infinity is called the *entire complex plane*, the *entire z plane*, or the *extended complex plane*.

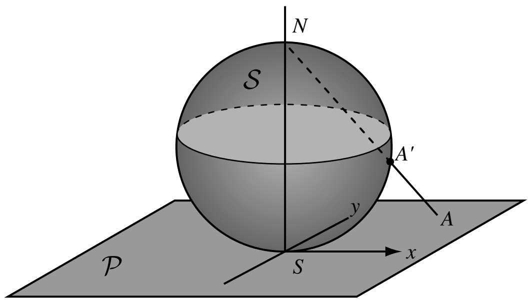

Fig. 1-6

The above method for mapping the plane on to the sphere is called *stereographic projection*. The sphere is sometimes called the *Riemann sphere*. When the diameter of the Riemann sphere is chosen to be unity, the equator corresponds to the unit circle of the complex plane.

1.16 Dot and Cross Product

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers [vectors]. The *dot product* [also called the *scalar product*] of z_1 and z_2 is defined as the real number

$$z_1 \cdot z_2 = x_1x_2 + y_1y_2 = |z_1||z_2|\cos\theta \tag{1.12}$$

where θ is the angle between z_1 and z_2 which lies between 0 and π.

The *cross product* of z_1 and z_2 is defined as the vector $z_1 \times z_2 = (0, 0, x_1y_2 - y_1x_2)$ perpendicular to the complex plane having magnitude

$$|z_1 \times z_2| = x_1y_2 - y_1x_2 = |z_1||z_2|\sin\theta \tag{1.13}$$

THEOREM 1.2: Let z_1 and z_2 be non-zero. Then:

(1) A necessary and sufficient condition that z_1 and z_2 be perpendicular is that $z_1 \cdot z_2 = 0$.
(2) A necessary and sufficient condition that z_1 and z_2 be parallel is that $|z_1 \times z_2| = 0$.
(3) The magnitude of the projection of z_1 on z_2 is $|z_1 \cdot z_2|/|z_2|$.
(4) The area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$.

1.17 Complex Conjugate Coordinates

A point in the complex plane can be located by rectangular coordinates (x, y) or polar coordinates (r, θ). Many other possibilities exist. One such possibility uses the fact that $x = \frac{1}{2}(z + \bar{z})$, $y = (1/2i)(z - \bar{z})$ where $z = x + iy$. The coordinates $(z, \bar{z})$ that locate a point are called *complex conjugate coordinates* or briefly *conjugate coordinates* of the point [see Problems 1.43 and 1.44].

1.18 Point Sets

Any collection of points in the complex plane is called a (*two-dimensional*) point set, and each point is called a *member* or *element* of the set. The following fundamental definitions are given here for reference.

(1) **Neighborhoods.** A *delta, or* δ, *neighborhood* of a point z_0 is the set of all points z such that $|z - z_0| < \delta$ where δ is any given positive number. A *deleted* δ *neighborhood* of z_0 is a neighborhood of z_0 in which the point z_0 is omitted, i.e., $0 < |z - z_0| < \delta$.

(2) **Limit Points.** A point z_0 is called a *limit point, cluster point,* or *point of accumulation* of a point set S if every deleted δ neighborhood of z_0 contains points of S.

Since δ can be any positive number, it follows that S must have infinitely many points. Note that z_0 may or may not belong to the set S.

(3) **Closed Sets.** A set S is said to be *closed* if every limit point of S belongs to S, i.e., if S contains all its limit points. For example, the set of all points z such that $|z| \leq 1$ is a closed set.

(4) **Bounded Sets.** A set S is called *bounded* if we can find a constant M such that $|z| < M$ for every point z in S. An *unbounded set* is one which is not bounded. A set which is both bounded and closed is called *compact*.

(5) **Interior, Exterior and Boundary Points.** A point z_0 is called an *interior point* of a set S if we can find a δ neighborhood of z_0 all of whose points belong to S. If every δ neighborhood of z_0 contains points belonging to S and also points not belonging to S, then z_0 is called a *boundary point*. If a point is not an interior or boundary point of a set S, it is an *exterior point* of S.

(6) **Open Sets.** An *open set* is a set which consists only of interior points. For example, the set of points z such that $|z| < 1$ is an open set.

(7) **Connected Sets.** An open set S is said to be *connected* if any two points of the set can be joined by a path consisting of straight line segments (i.e., a *polygonal path*) all points of which are in S.

(8) **Open Regions or Domains.** An open connected set is called an *open region* or *domain*.

(9) **Closure of a Set.** If to a set S we add all the limit points of S, the new set is called the *closure* of S and is a closed set.

(10) **Closed Regions.** The closure of an open region or domain is called a *closed region*.

(11) **Regions.** If to an open region or domain we add some, all or none of its limit points, we obtain a set called a *region*. If all the limit points are added, the region is *closed*; if none are added, the region is *open*. In this book whenever we use the word *region* without qualifying it, we shall mean *open region* or *domain*.

(12) **Union and Intersection of Sets.** A set consisting of all points belonging to set S_1 or set S_2 or to both sets S_1 and S_2 is called the *union* of S_1 and S_2 and is denoted by $S_1 \cup S_2$.

A set consisting of all points belonging to both sets S_1 and S_2 is called the *intersection* of S_1 and S_2 and is denoted by $S_1 \cap S_2$.

(13) **Complement of a Set.** A set consisting of all points which do not belong to S is called the *complement* of S and is denoted by $\tilde{S}$ or S^c.

(14) **Null Sets and Subsets.** It is convenient to consider a set consisting of no points at all. This set is called the *null set* and is denoted by $\varnothing$. If two sets S_1 and S_2 have no points in common (in which case they are called *disjoint* or *mutually exclusive sets*), we can indicate this by writing $S_1 \cap S_2 = \varnothing$.

Any set formed by choosing some, all or none of the points of a set S is called a *subset* of S. If we exclude the case where all points of S are chosen, the set is called a *proper subset* of S.

(15) **Countability of a Set.** Suppose a set is finite or its elements can be placed into a one to one correspondence with the natural numbers 1, 2, 3, ... Then the set is called *countable* or *denumerable*; otherwise it is *non-countable* or *non-denumerable*.

The following are two important theorems on point sets.

(1) **Weierstrass–Bolzano Theorem.** Every bounded infinite set has at least one limit point.

(2) **Heine–Borel Theorem.** Let S be a compact set each point of which is contained in one or more of the open sets $A_1, A_2, \ldots$ [which are then said to *cover* S]. Then there exists a finite number of the sets $A_1, A_2, \ldots$ which will cover S.

SOLVED PROBLEMS

Fundamental Operations with Complex Numbers

1.1. Perform each of the indicated operations.

Solution

(a) $(3+2i)+(-7-i)=3-7+2i-i=-4+i$
(b) $(-7-i)+(3+2i)=-7+3-i+2i=-4+i$

The results (a) and (b) illustrate the *commutative law of addition*.

(c) $(8-6i)-(2i-7)=8-6i-2i+7=15-8i$
(d) $(5+3i)+\{(-1+2i)+(7-5i)\}=(5+3i)+\{-1+2i+7-5i\}=(5+3i)+(6-3i)=11$
(e) $\{(5+3i)+(-1+2i)\}+(7-5i)=\{5+3i-1+2i\}+(7-5i)=(4+5i)+(7-5i)=11$

The results (d) and (e) illustrate the *associative law of addition*.

(f) $(2-3i)(4+2i)=2(4+2i)-3i(4+2i)=8+4i-12i-6i^2=8+4i-12i+6=14-8i$
(g) $(4+2i)(2-3i)=4(2-3i)+2i(2-3i)=8-12i+4i-6i^2=8-12i+4i+6=14-8i$

The results (f) and (g) illustrate the *commutative law of multiplication*.

(h) $$(2-i)\{(-3+2i)(5-4i)\}=(2-i)\{-15+12i+10i-8i^2\}$$
$$=(2-i)(-7+22i)=-14+44i+7i-22i^2=8+51i$$

(i) $$\{(2-i)(-3+2i)\}(5-4i)=\{-6+4i+3i-2i^2\}(5-4i)$$
$$=(-4+7i)(5-4i)=-20+16i+35i-28i^2=8+51i$$

The results (h) and (i) illustrate the *associative law of multiplication*.

(j) $(-1+2i)\{(7-5i)+(-3+4i)\}=(-1+2i)(4-i)=-4+i+8i-2i^2=-2+9i$

Another Method.

$$\begin{aligned}(-1+2i)\{(7-5i)+(-3+4i)\}&=(-1+2i)(7-5i)+(-1+2i)(-3+4i)\\&=\{-7+5i+14i-10i^2\}+\{3-4i-6i+8i^2\}\\&=(3+19i)+(-5-10i)=-2+9i\end{aligned}$$

The above illustrates the *distributive law*.

(k) $$\frac{3-2i}{-1+i}=\frac{3-2i}{-1+i}\cdot\frac{-1-i}{-1-i}=\frac{-3-3i+2i+2i^2}{1-i^2}=\frac{-5-i}{2}=-\frac{5}{2}-\frac{1}{2}i$$

Another Method. By definition, $(3-2i)/(-1+i)$ is that number $a+bi$, where a and b are real, such that $(-1+i)(a+bi)=-a-b+(a-b)i=3-2i$. Then $-a-b=3$, $a-b=-2$ and solving simultaneously, $a=-5/2$, $b=-1/2$ or $a+bi=-5/2-i/2$.

(l) $$\frac{5+5i}{3-4i}+\frac{20}{4+3i}=\frac{5+5i}{3-4i}\cdot\frac{3+4i}{3+4i}+\frac{20}{4+3i}\cdot\frac{4-3i}{4-3i}$$
$$=\frac{15+20i+15i+20i^2}{9-16i^2}+\frac{80-60i}{16-9i^2}=\frac{-5+35i}{25}+\frac{80-60i}{25}=3-i$$

(m) $$\frac{3i^{30}-i^{19}}{2i-1}=\frac{3(i^2)^{15}-(i^2)^9i}{2i-1}=\frac{3(-1)^{15}-(-1)^9i}{-1+2i}$$
$$=\frac{-3+i}{-1+2i}\cdot\frac{-1-2i}{-1-2i}=\frac{3+6i-i-2i^2}{1-4i^2}=\frac{5+5i}{5}=1+i$$

1.2. Suppose $z_1 = 2 + i$, $z_2 = 3 - 2i$ and $z_3 = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$. Evaluate each of the following.

Solution

(a) $$|3z_1 - 4z_2| = |3(2+i) - 4(3-2i)| = |6 + 3i - 12 + 8i|$$
$$= |-6 + 11i| = \sqrt{(-6)^2 + (11)^2} = \sqrt{157}$$

(b) $$z_1^3 - 3z_1^2 + 4z_1 - 8 = (2+i)^3 - 3(2+i)^2 + 4(2+i) - 8$$
$$= \{(2)^3 + 3(2)^2(i) + 3(2)(i)^2 + i^3\} - 3(4 + 4i + i^2) + 8 + 4i - 8$$
$$= 8 + 12i - 6 - i - 12 - 12i + 3 + 8 + 4i - 8 = -7 + 3i$$

(c) $$(\bar{z}_3)^4 = \left(\overline{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}\right)^4 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^4 = \left[\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2\right]^2$$
$$= \left[\frac{1}{4} + \frac{\sqrt{3}}{2}i + \frac{3}{4}i^2\right]^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{4} - \frac{\sqrt{3}}{2}i + \frac{3}{4}i^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

(d) $$\left|\frac{2z_2 + z_1 - 5 - i}{2z_1 - z_2 + 3 - i}\right|^2 = \left|\frac{2(3-2i) + (2+i) - 5 - i}{2(2+i) - (3-2i) + 3 - i}\right|^2$$
$$= \left|\frac{3 - 4i}{4 + 3i}\right|^2 = \frac{|3-4i|^2}{|4+3i|^2} = \frac{(\sqrt{(3)^2 + (-4)^2})^2}{(\sqrt{(4)^2 + (3)^2})^2} = 1$$

1.3. Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$.

Solution

The given equation can be written as $3x + 5y + i(2y - x) = 7 + 5i$. Then equating real and imaginary parts, $3x + 5y = 7$, $2y - x = 5$. Solving simultaneously, $x = -1$, $y = 2$.

1.4. Prove: (a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, (b) $|z_1 z_2| = |z_1||z_2|$.

Solution

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then

(a) $$\overline{z_1 + z_2} = \overline{x_1 + iy_1 + x_2 + iy_2} = \overline{x_1 + x_2 + i(y_1 + y_2)}$$
$$= x_1 + x_2 - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{x_1 + iy_1} + \overline{x_2 + iy_2} = \bar{z}_1 + \bar{z}_2$$

(b) $$|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = |x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)|$$
$$= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = |z_1||z_2|$$

Another Method.

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = z_1 z_2 \bar{z}_1 \bar{z}_2 = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2 \text{ or } |z_1 z_2| = |z_1||z_2|$$

where we have used the fact that the conjugate of a product of two complex numbers is equal to the product of their conjugates (see Problem 1.55).

Graphical Representation of Complex Numbers. Vectors

1.5. Perform the indicated operations both analytically and graphically:

(a) $(3 + 4i) + (5 + 2i)$, (b) $(6 - 2i) - (2 - 5i)$, (c) $(-3 + 5i) + (4 + 2i) + (5 - 3i) + (-4 - 6i)$.

Solution

(a) *Analytically*. $(3+4i)+(5+2i)=3+5+4i+2i=8+6i$

Graphically. Represent the two complex numbers by points P_1 and P_2, respectively, as in Fig. 1-7. Complete the parallelogram with OP_1 and OP_2 as adjacent sides. Point P represents the sum, $8+6i$, of the two given complex numbers. Note the similarity with the parallelogram law for addition of vectors OP_1 and OP_2 to obtain vector OP. For this reason it is often convenient to consider a complex number $a+bi$ as a vector having *components* a and b in the directions of the positive x and y axes, respectively.

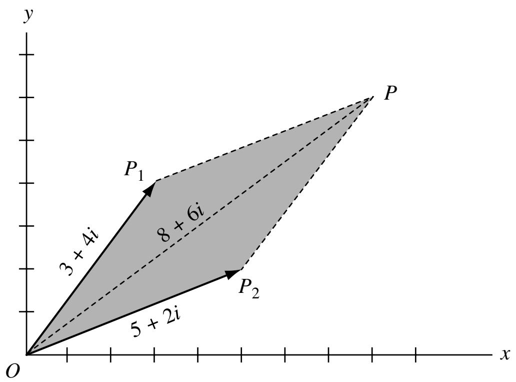

Fig. 1-7

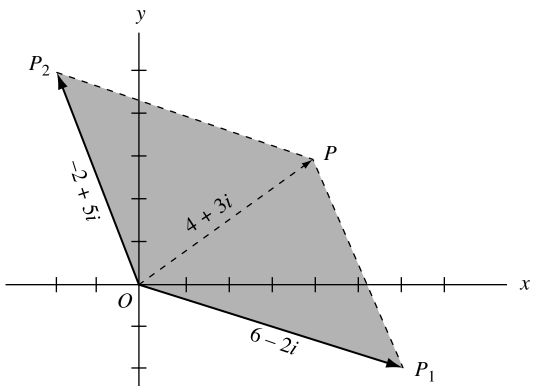

Fig. 1-8

(b) *Analytically*. $(6-2i)-(2-5i)=6-2-2i+5i=4+3i$
Graphically. $(6-2i)-(2-5i)=6-2i+(-2+5i)$. We now add $6-2i$ and $(-2+5i)$ as in part (a). The result is indicated by OP in Fig. 1-8.

(c) *Analytically*.

$$(-3+5i)+(4+2i)+(5-3i)+(-4-6i)=(-3+4+5-4)+(5i+2i-3i-6i)=2-2i$$

Graphically. Represent the numbers to be added by z_1, z_2, z_3, z_4, respectively. These are shown graphically in Fig. 1-9. To find the required sum proceed as shown in Fig. 1-10. At the terminal point of vector z_1 construct vector z_2. At the terminal point of z_2 construct vector z_3, and at the terminal point of z_3 construct vector z_4. The required sum, sometimes called the *resultant*, is obtained by constructing the vector OP from the initial point of z_1 to the terminal point of z_4, i.e., $OP=z_1+z_2+z_3+z_4=2-2i$.

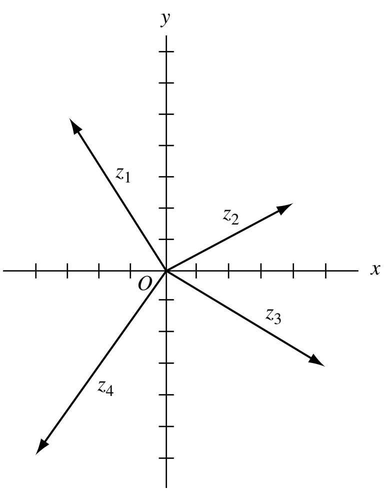

Fig. 1-9

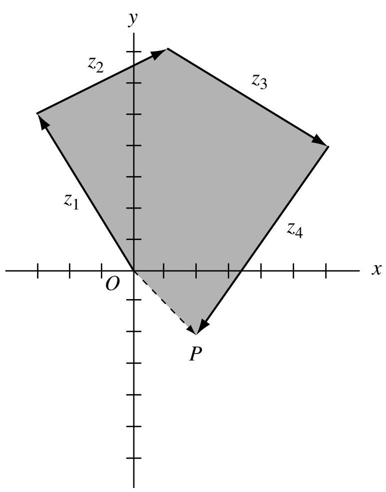

Fig. 1-10

1.6. Suppose z_1 and z_2 are two given complex numbers (vectors) as in Fig. 1-11. Construct graphically

(a) $3z_1 - 2z_2$, (b) $\frac{1}{2}z_2 + \frac{5}{3}z_1$

Solution

(a) In Fig. 1-12, $OA = 3z_1$ is a vector having length 3 times vecter z_1 and the same direction. $OB = -2z_2$ is a vector having length 2 times vector z_2 and the opposite direction. Then vector $OC = OA + OB = 3z_1 - 2z_2$.

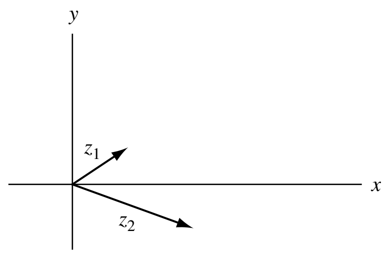

Fig. 1-11

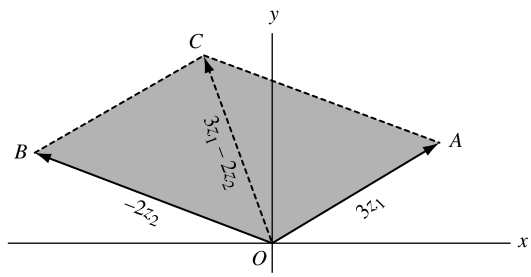

Fig. 1-12

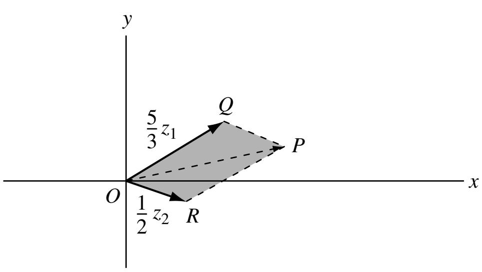

Fig. 1-13

(b) The required vector (complex number) is represented by OP in Fig. 1-13.

1.7. Prove (a) $|z_1 + z_2| \le |z_1| + |z_2|$, (b) $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$, (c) $|z_1 - z_2| \ge |z_1| - |z_2|$ and give a graphical interpretation.

Solution

(a) *Analytically*. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then we must show that

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \le \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

Squaring both sides, this will be true if

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \le x_1^2 + y_1^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} + x_2^2 + y_2^2$$

i.e., if
$$x_1x_2 + y_1y_2 \le \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

or if (squaring both sides again)

$$x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \le x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2$$

or
$$2x_1x_2y_1y_2 \le x_1^2y_2^2 + y_1^2x_2^2$$

But this is equivalent to $(x_1y_2 - x_2y_1)^2 \ge 0$, which is true. Reversing the steps, which are reversible, proves the result.

Graphically. The result follows graphically from the fact that $|z_1|$, $|z_2|$, $|z_1 + z_2|$ represent the lengths of the sides of a triangle (see Fig. 1-14) and that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side.

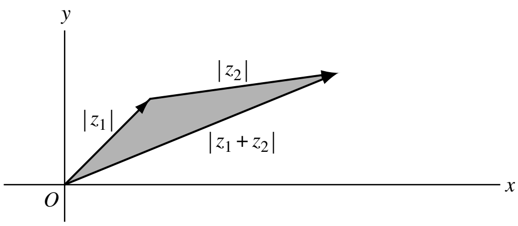

Fig. 1-14

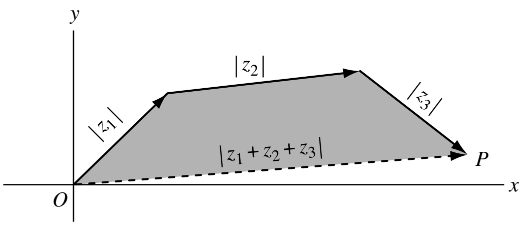

Fig. 1-15

(b) *Analytically*. By part (a),

$$|z_1 + z_2 + z_3| = |z_1 + (z_2 + z_3)| \le |z_1| + |z_2 + z_3| \le |z_1| + |z_2| + |z_3|$$

Graphically. The result is a consequence of the geometric fact that, in a plane, a straight line is the shortest distance between two points O and P (see Fig. 1-15).

(c) *Analytically*. By part (a), $|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$. Then $|z_1 - z_2| \ge |z_1| - |z_2|$. An equivalent result obtained on replacing z_2 by $-z_2$ is $|z_1 + z_2| \ge |z_1| - |z_2|$.
Graphically. The result is equivalent to the statement that a side of a triangle has length greater than or equal to the difference in lengths of the other two sides.

1.8. Let the position vectors of points $A(x_1, y_1)$ and $B(x_2, y_2)$ be represented by z_1 and z_2, respectively. (a) Represent the vector AB as a complex number. (b) Find the distance between points A and B.

Solution

(a) From Fig. 1-16, $OA + AB = OB$ or

$$AB = OB - OA = z_2 - z_1 = (x_2 + iy_2) - (x_1 + iy_1) = (x_2 - x_1) + i(y_2 - y_1)$$

(b) The distance between points A and B is given by

$$|AB| = |(x_2 - x_1) + i(y_2 - y_1)| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

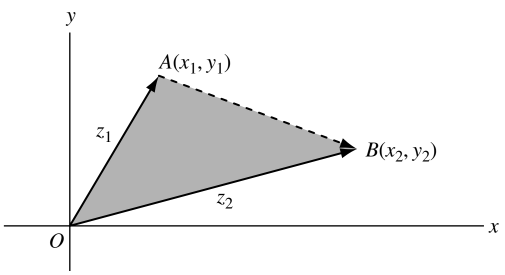

Fig. 1-16

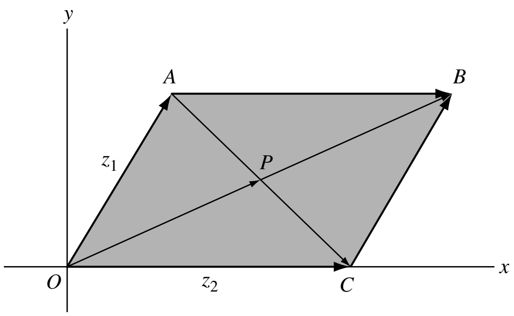

Fig. 1-17

1.9. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ represent two non-collinear or non-parallel vectors. If a and b are real numbers (scalars) such that $az_1 + bz_2 = 0$, prove that $a = 0$ and $b = 0$.

Solution

The given condition $az_1 + bz_2 = 0$ is equivalent to

$$a(x_1 + iy_1) + b(x_2 + iy_2) = 0 \text{ or } ax_1 + bx_2 + i(ay_1 + by_2) = 0.$$

Then $ax_1 + bx_2 = 0$ and $ay_1 + by_2 = 0$. These equations have the simultaneous solution $a = 0$, $b = 0$ if $y_1/x_1 \neq y_2/x_2$, i.e., if the vectors are non-collinear or non-parallel vectors.

1.10. Prove that the diagonals of a parallelogram bisect each other.

Solution

Let $OABC$ [Fig. 1-17] be the given parallelogram with diagonals intersecting at P.
Since $z_1 + AC = z_2$, $AC = z_2 - z_1$. Then $AP = m(z_2 - z_1)$ where $0 \le m \le 1$.
Since $OB = z_1 + z_2$, $OP = n(z_1 + z_2)$ where $0 \le n \le 1$.
But $OA + AP = OP$, i.e., $z_1 + m(z_2 - z_1) = n(z_1 + z_2)$ or $(1 - m - n)z_1 + (m - n)z_2 = 0$. Hence, by Problem 1.9, $1 - m - n = 0$, $m - n = 0$ or $m = \frac{1}{2}$, $n = \frac{1}{2}$ and so P is the midpoint of both diagonals.

1.11. Find an equation for the straight line that passes through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Solution

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be the position vectors of A and B, respectively. Let $z = x + iy$ be the position vector of any point P on the line joining A and B.

From Fig. 1-18,

$$OA + AP = OP \quad \text{or} \quad z_1 + AP = z, \quad \text{i.e., } AP = z - z_1$$
$$OA + AB = OB \quad \text{or} \quad z_1 + AB = z_2, \quad \text{i.e., } AB = z_2 - z_1$$

Since AP and AB are collinear, $AP = tAB$ or $z - z_1 = t(z_2 - z_1)$ where t is real, and the required equation is

$$z = z_1 + t(z_2 - z_1) \quad \text{or} \quad z = (1 - t)z_1 + tz_2$$

Using $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z = x + iy$, this can be written

$$x - x_1 = t(x_2 - x_1), \quad y - y_1 = t(y_2 - y_1) \quad \text{or} \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

The first two are called *parametric equations* of the line and t is the parameter; the second is called the equation of the line in *standard form.*

Another Method. Since AP and PB are collinear, we have for real numbers m and n:

$$mAP = nPB \quad \text{or} \quad m(z - z_1) = n(z_2 - z)$$

Solving,

$$z = \frac{mz_1 + nz_2}{m + n} \quad \text{or} \quad x = \frac{mx_1 + nx_2}{m + n}, \quad y = \frac{my_1 + ny_2}{m + n}$$

which is called the *symmetric form.*

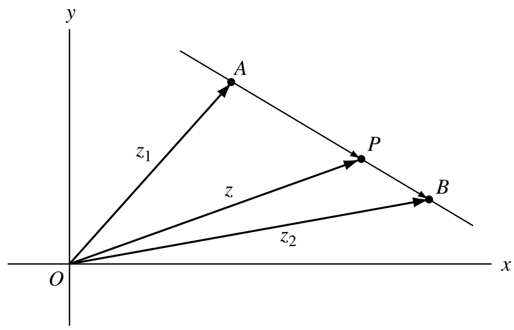

Fig. 1-18

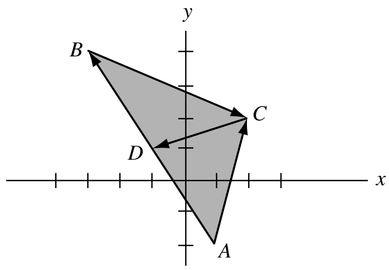

Fig. 1-19

1.12. Let $A(1, -2)$, $B(-3, 4)$, $C(2, 2)$ be the three vertices of triangle ABC. Find the length of the median from C to the side AB.

Solution

The position vectors of A, B, and C are given by $z_1 = 1 - 2i$, $z_2 = -3 + 4i$ and $z_3 = 2 + 2i$, respectively. Then, from Fig. 1-19,

$$AC = z_3 - z_1 = 2 + 2i - (1 - 2i) = 1 + 4i$$
$$BC = z_3 - z_2 = 2 + 2i - (-3 + 4i) = 5 - 2i$$
$$AB = z_2 - z_1 = -3 + 4i - (1 - 2i) = -4 + 6i$$
$$AD = \tfrac{1}{2}AB = \tfrac{1}{2}(-4 + 6i) = -2 + 3i \quad \text{since } D \text{ is the midpoint of } AB.$$
$$AC + CD = AD \quad \text{or} \quad CD = AD - AC = -2 + 3i - (1 + 4i) = -3 - i.$$

Then the length of median CD is $|CD| = |-3 - i| = \sqrt{10}$.

1.13. Find an equation for (a) a circle of radius 4 with center at (−2, 1), (b) an ellipse with major axis of length 10 and foci at (−3, 0) and (3, 0).

Solution

(a) The center can be represented by the complex number $-2 + i$. If z is any point on the circle [Fig. 1-20], the distance from z to $-2 + i$ is

$$|z - (-2 + i)| = 4$$

Then $|z + 2 - i| = 4$ is the required equation. In rectangular form, this is given by

$$|(x + 2) + i(y - 1)| = 4, \quad \text{i.e., } (x + 2)^2 + (y - 1)^2 = 16$$

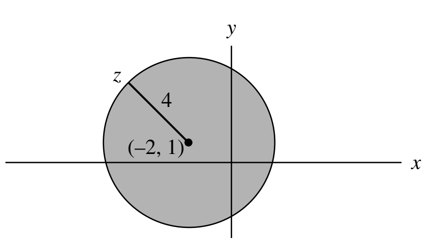

Fig. 1-20

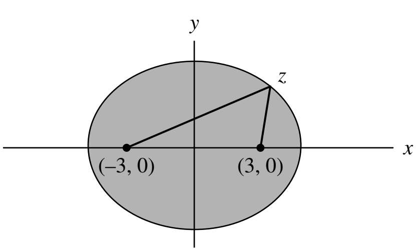

Fig. 1-21

(b) The sum of the distances from any point z on the ellipse [Fig. 1-21] to the foci must equal 10. Hence, the required equation is

$$|z+3| + |z-3| = 10$$

In rectangular form, this reduces to $x^2/25 + y^2/16 = 1$ (see Problem 1.74).

Axiomatic Foundations of Complex Numbers

1.14. Use the definition of a complex number as an ordered pair of real numbers and the definitions on page 3 to prove that $(a, b) = a(1, 0) + b(0, 1)$ where $(0, 1)(0, 1) = (-1, 0)$.

Solution

From the definitions of sum and product on page 3, we have

$$(a, b) = (a, 0) + (0, b) = a(1, 0) + b(0, 1)$$

where

$$(0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1,\ 0 \cdot 1 + 1 \cdot 0) = (-1, 0)$$

By identifying (1, 0) with 1 and (0, 1) with i, we see that $(a, b) = a + bi$.

1.15. Suppose $z_1 = (a_1, b_1)$, $z_2 = (a_2, b_2)$, and $z_3 = (a_3, b_3)$. Prove the distributive law:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Solution

We have

$$\begin{aligned} z_1(z_2+z_3) &= (a_1, b_1)\{(a_2, b_2) + (a_3, b_3)\} = (a_1, b_1)(a_2 + a_3, b_2 + b_3) \\ &= \{a_1(a_2+a_3) - b_1(b_2+b_3),\ a_1(b_2+b_3) + b_1(a_2+a_3)\} \\ &= (a_1a_2 - b_1b_2 + a_1a_3 - b_1b_3,\ a_1b_2 + b_1a_2 + a_1b_3 + b_1a_3) \\ &= (a_1a_2 - b_1b_2,\ a_1b_2 + b_1a_2) + (a_1a_3 - b_1b_3,\ a_1b_3 + b_1a_3) \\ &= (a_1, b_1)(a_2, b_2) + (a_1, b_1)(a_3, b_3) = z_1z_2 + z_1z_3 \end{aligned}$$

Polar Form of Complex Numbers

1.16. Express each of the following complex numbers in polar form.

(a) $2 + 2\sqrt{3}i$, (b) $-5 + 5i$, (c) $-\sqrt{6} - \sqrt{2}i$, (d) $-3i$

Solution

(a) $2 + 2\sqrt{3}i$ [See Fig. 1-22.]

Modulus or absolute value, $r = |2 + 2\sqrt{3}i| = \sqrt{4+12} = 4$.
Amplitude or argument, $\theta = \sin^{-1} 2\sqrt{3}/4 = \sin^{-1}\sqrt{3}/2 = 60^\circ = \pi/3$ (radians).
Then

$$2 + 2\sqrt{3}i = r(\cos\theta + i\sin\theta) = 4(\cos 60^\circ + i\sin 60^\circ) = 4(\cos\pi/3 + i\sin\pi/3)$$

The result can also be written as 4 cis $\pi/3$ or, using Euler's formula, as $4e^{\pi i/3}$.

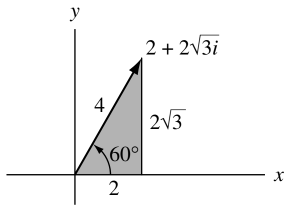

Fig. 1-22

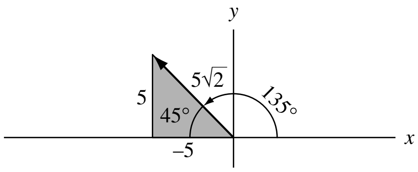

Fig. 1-23

(b) $-5+5i$ [See Fig. 1-23.]

$$r = |-5+5i| = \sqrt{25+25} = 5\sqrt{2}$$
$$\theta = 180^\circ - 45^\circ = 135^\circ = 3\pi/4 \text{ (radians)}$$

Then

$$-5+5i = 5\sqrt{2}(\cos 135^\circ + i \sin 135^\circ) = 5\sqrt{2} \text{ cis } 3\pi/4 = 5\sqrt{2}e^{3\pi i/4}$$

(c) $-\sqrt{6}-\sqrt{2}i$ [See Fig. 1-24.]

$$r = |-\sqrt{6}-\sqrt{2}i| = \sqrt{6+2} = 2\sqrt{2}$$
$$\theta = 180^\circ + 30^\circ = 210^\circ = 7\pi/6 \text{ (radians)}$$

Then

$$-\sqrt{6}-\sqrt{2}i = 2\sqrt{2}(\cos 210^\circ + i \sin 210^\circ) = 2\sqrt{2} \text{ cis } 7\pi/6 = 2\sqrt{2}e^{7\pi i/e}$$

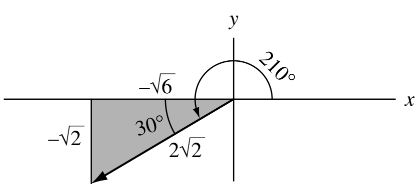

Fig. 1-24

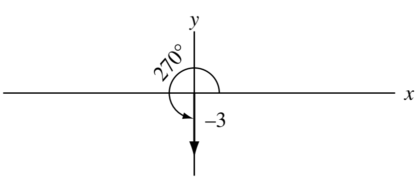

Fig. 1-25

(d) $-3i$ [See Fig. 1-25.]

$$r = |-3i| = |0-3i| = \sqrt{0+9} = 3$$
$$\theta = 270^\circ = 3\pi/2 \text{ (radians)}$$

Then

$$-3i = 3(\cos 3\pi/2 + i \sin 3\pi/2) = 3 \text{ cis } 3\pi/2 = 3e^{3\pi i/2}$$

1.17. Graph each of the following: (a) $6(\cos 240^\circ + i \sin 240^\circ)$, (b) $4e^{3\pi i/5}$, (c) $2e^{-\pi i/4}$.

Solution

(a) $6(\cos 240^\circ + i \sin 240^\circ) = 6 \text{ cis } 240^\circ = 6 \text{ cis } 4\pi/3 = 6e^{4\pi i/3}$ can be represented graphically by OP in Fig. 1-26.

If we start with vector OA, whose magnitude is 6 and whose direction is that of the positive x axis, we can obtain OP by rotating OA counterclockwise through an angle of 240°. In general, $re^{i\theta}$ is equivalent to a vector obtained by rotating a vector of magnitude r and direction that of the positive x axis, counterclockwise through an angle θ.

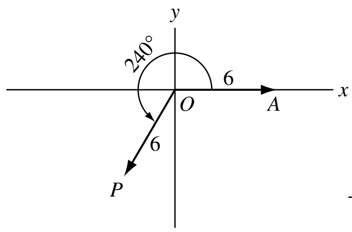

Fig. 1-26

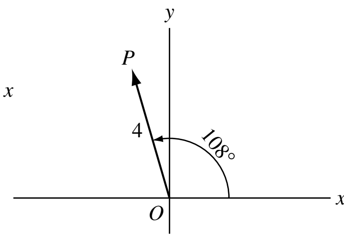

Fig. 1-27

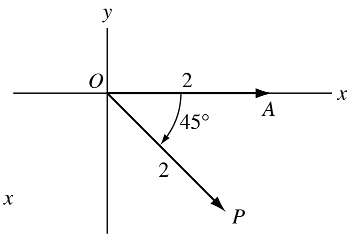

Fig. 1-28

(b) $$4e^{3\pi i/5} = 4(\cos 3\pi/5 + i\sin 3\pi/5) = 4(\cos 108^\circ + i\sin 108^\circ)$$

is represented by OP in Fig. 1-27.

(c) $$2e^{-\pi i/4} = 2\{\cos(-\pi/4) + i\sin(-\pi/4)\} = 2\{\cos(-45^\circ) + i\sin(-45^\circ)\}$$

This complex number can be represented by vector OP in Fig. 1-28. This vector can be obtained by starting with vector OA, whose magnitude is 2 and whose direction is that of the positive x axis, and rotating it counterclockwise through an angle of -45° (which is the same as rotating it *clockwise* through an angle of 45°).

1.18. A man travels 12 miles northeast, 20 miles 30° west of north, and then 18 miles 60° south of west. Determine (a) analytically and (b) graphically how far and in what direction he is from his starting point.

Solution

(a) *Analytically.* Let O be the starting point (see Fig. 1-29). Then the successive displacements are represented by vectors OA, AB, and BC. The result of all three displacements is represented by the vector

$$OC = OA + AB + BC$$

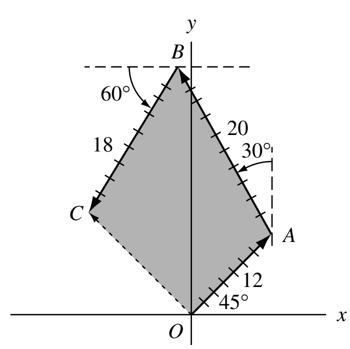

Fig. 1-29

Now

$$OA = 12(\cos 45^\circ + i\sin 45^\circ) = 12e^{\pi i/4}$$

$$AB = 20\{\cos(90^\circ + 30^\circ) + i\sin(90^\circ + 30^\circ)\} = 20e^{2\pi i/3}$$

$$BC = 18\{\cos(180^\circ + 60^\circ) + i\sin(180^\circ + 60^\circ)\} = 18e^{4\pi i/3}$$

Then

$$\begin{aligned} OC &= 12e^{\pi i/4} + 20e^{2\pi i/3} + 18e^{4\pi i/3} \\ &= \{12\cos 45^\circ + 20\cos 120^\circ + 18\cos 240^\circ\} + i\{12\sin 45^\circ + 20\sin 120^\circ + 18\sin 240^\circ\} \\ &= \{(12)(\sqrt{2}/2) + (20)(-1/2) + (18)(-1/2)\} + i\{(12)(\sqrt{2}/2) + (20)(\sqrt{3}/2) + (18)(-\sqrt{3}/2) \\ &= (6\sqrt{2} - 19) + (6\sqrt{2} + \sqrt{3})i \end{aligned}$$

If $r(\cos\theta + i\sin\theta) = 6\sqrt{2} - 19 + (6\sqrt{2} + \sqrt{3})i$, then $r = \sqrt{(6\sqrt{2} - 19)^2 + (6\sqrt{2} + \sqrt{3})^2} = 14.7$ approximately, and $\theta = \cos^{-1}(6\sqrt{2} - 19)/r = \cos^{-1}(-.717) = 135^\circ 49'$ approximately.

Thus, the man is 14.7 miles from his starting point in a direction $135^\circ 49' - 90^\circ = 45^\circ 49'$ west of north.

(b) *Graphically.* Using a convenient unit of length such as PQ in Fig. 1-29, which represents 2 miles, and a protractor to measure angles, construct vectors OA, AB, and BC. Then, by determining the number of units in OC and the angle that OC makes with the y axis, we obtain the approximate results of (a).

De Moivre's Theorem

1.19. Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Prove:

(a) $z_1 z_2 = r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$, (b) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$.

Solution

(a)
$$\begin{aligned} z_1 z_2 &= \{r_1(\cos\theta_1 + i\sin\theta_1)\}\{r_2(\cos\theta_2 + i\sin\theta_2)\} \\ &= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} \end{aligned}$$

(b)
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1}{r_2}\left\{\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right\} \\ &= \frac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\} \end{aligned}$$

In terms of Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, the results state that if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ and $z_1/z_2 = r_1 e^{i\theta_1}/r_2 e^{i\theta_2} = (r_1/r_2)e^{i(\theta_1-\theta_2)}$.

1.20. Prove De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where n is any positive integer.

Solution

We use the *principle of mathematical induction.* Assume that the result is true for the particular positive integer k, i.e., assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$. Then, multiplying both sides by $\cos\theta + i\sin\theta$, we find

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) = \cos(k+1)\theta + i\sin(k+1)\theta$$

by Problem 1.19. Thus, *if* the result is true for $n = k$, then it is also true for $n = k+1$. But, since the result is clearly true for $n = 1$, it must also be true for $n = 1+1 = 2$ and $n = 2+1 = 3$, etc., and so must be true for all positive integers.

The result is equivalent to the statement $(e^{i\theta})^n = e^{ni\theta}$.

1.21. Prove the identities: (a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$;
(b) $(\sin 5\theta)/(\sin\theta) = 16\cos^4\theta - 12\cos^2\theta + 1$, if $\theta \neq 0, \pm\pi, \pm 2\pi, \ldots$.

Solution

We use the *binomial formula*

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

where the coefficients

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

also denoted by $C(n,r)$ or ${}_nC_r$, are called the *binomial coefficients.* The number $n!$ or *factorial* n, is defined as the product $n(n-1)\cdots 3\cdot 2\cdot 1$ and we define $0! = 1$.

From Problem 1.20, with $n = 5$, and the binomial formula,

$$\begin{aligned}\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta + \binom{5}{1}(\cos^4\theta)(i\sin\theta) + \binom{5}{2}(\cos^3\theta)(i\sin\theta)^2 \\ &\quad + \binom{5}{3}(\cos^2\theta)(i\sin\theta)^3 + \binom{5}{4}(\cos\theta)(i\sin\theta)^4 + (i\sin\theta)^5 \\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\ &\quad - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta \\ &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &\quad + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)\end{aligned}$$

Hence

(a) $$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

(b) $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

or

$$\begin{aligned}\frac{\sin 5\theta}{\sin\theta} &= 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\ &= 16\cos^4\theta - 12\cos^2\theta + 1\end{aligned}$$

provided $\sin\theta \neq 0$, i.e., $\theta \neq 0, \pm\pi, \pm 2\pi, \ldots$.

1.22. Show that (a) $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, (b) $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$.

Solution

We have

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2)$$

(a) Adding (1) and (2),

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad \text{or} \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

(b) Subtracting (2) from (1),

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \quad \text{or} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

1.23. Prove the identities (a) $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$, (b) $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

Solution

(a) $$\sin^3\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 = \frac{(e^{i\theta}-e^{-i\theta})^3}{8i^3} = -\frac{1}{8i}\{(e^{i\theta})^3 - 3(e^{i\theta})^2(e^{-i\theta}) + 3(e^{i\theta})(e^{-i\theta})^2 - (e^{-i\theta})^3\}$$
$$= -\frac{1}{8i}(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}) = \frac{3}{4}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right) - \frac{1}{4}\left(\frac{e^{3i\theta}-e^{-3i\theta}}{2i}\right)$$
$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

(b) $$\cos^4\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^4 = \frac{(e^{i\theta}+e^{-i\theta})^4}{16}$$
$$= \frac{1}{16}\{(e^{i\theta})^4 + 4(e^{i\theta})^3(e^{-i\theta}) + 6(e^{i\theta})^2(e^{-i\theta})^2 + 4(e^{i\theta})(e^{-i\theta})^3 + (e^{-i\theta})^4\}$$
$$= \frac{1}{16}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}) = \frac{1}{8}\left(\frac{e^{4i\theta}+e^{-4i\theta}}{2}\right) + \frac{1}{2}\left(\frac{e^{2i\theta}+e^{-2i\theta}}{2}\right) + \frac{3}{8}$$
$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

1.24. Given a complex number (vector) z, interpret geometrically $ze^{i\alpha}$ where α is real.

Solution

Let $z = re^{i\theta}$ be represented graphically by vector OA in Fig. 1-30. Then

$$ze^{i\alpha} = re^{i\theta}\cdot e^{i\alpha} = re^{i(\theta+\alpha)}$$

is the vector represented by OB.

Hence multiplication of a vector z by $e^{i\alpha}$ amounts to rotating z counterclockwise through angle α. We can consider $e^{i\alpha}$ as an *operator* that acts on z to produce this rotation.

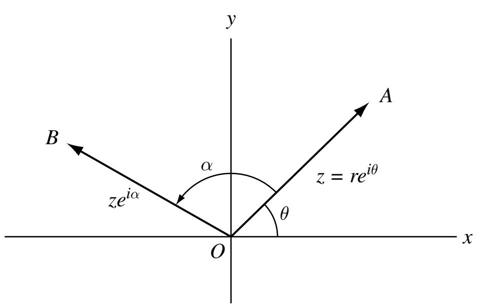

Fig. 1-30

1.25. Prove: $e^{i\theta} = e^{i(\theta+2k\pi)}$, $k = 0, \pm 1, \pm 2, \ldots$.

Solution

$$e^{i(\theta+2k\pi)} = \cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi) = \cos\theta + i\sin\theta = e^{i\theta}$$

1.26. Evaluate each of the following.

(a) $[3(\cos 40^\circ + i\sin 40^\circ)][4(\cos 80^\circ + i\sin 80^\circ)]$, (b) $\dfrac{(2\text{ cis }15^\circ)^7}{(4\text{ cis }45^\circ)^3}$, (c) $\left(\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$

Solution

(a) $$[3(\cos 40^\circ + i\sin 40^\circ)][4(\cos 80^\circ + i\sin 80^\circ)] = 3\cdot 4[\cos(40^\circ+80^\circ) + i\sin(40^\circ+80^\circ)]$$
$$= 12(\cos 120^\circ + i\sin 120^\circ)$$
$$= 12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -6 + 6\sqrt{3}i$$

(b) $$\frac{(2 \text{ cis } 15^\circ)^7}{(4 \text{ cis } 45^\circ)^3} = \frac{128 \text{ cis } 105^\circ}{64 \text{ cis } 135^\circ} = 2 \text{ cis}(105^\circ - 135^\circ)$$

$$= 2[\cos(-30^\circ) + i\sin(-30^\circ)] = 2[\cos 30^\circ - i\sin 30^\circ] = \sqrt{3} - i$$

(c) $$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left\{\frac{2 \text{ cis}(60^\circ)}{2 \text{ cis}(-60^\circ)}\right\}^{10} = (\text{cis } 120^\circ)^{10} = \text{cis } 1200^\circ = \text{cis } 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Another Method.

$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left(\frac{2e^{\pi i/3}}{2e^{-\pi i/3}}\right)^{10} = (e^{2\pi i/3})^{10} = e^{20\pi i/3}$$

$$= e^{6\pi i}e^{2\pi i/3} = (1)[\cos(2\pi/3) + i\sin(2\pi/3)] = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

1.27. Prove that (a) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$, (b) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$, stating appropriate conditions of validity.

Solution

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then $\arg z_1 = \theta_1$, $\arg z_2 = \theta_2$.

(a) Since $z_1 z_2 = r_1 r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$, $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$.

(b) Since $z_1/z_2 = (r_1/r_2)\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$, $\arg(z_1/z_2) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$.

Since there are many possible values for $\theta_1 = \arg z_1$ and $\theta_2 = \arg z_2$, we can only say that the two sides in the above equalities are equal for *some* values of $\arg z_1$ and $\arg z_2$. They may not hold even if principal values are used.

Roots of Complex Numbers

1.28. (a) Find all values of z for which $z^5 = -32$, and (b) locate these values in the complex plane.

Solution

(a) In polar form, $-32 = 32\{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\}$, $k = 0, \pm1, \pm2, \ldots$.
Let $z = r(\cos\theta + i\sin\theta)$. Then, by De Moivre's theorem,

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta) = 32\{\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\}$$

and so $r^5 = 32$, $5\theta = \pi + 2k\pi$, from which $r = 2$, $\theta = (\pi + 2k\pi)/5$. Hence

$$z = 2\left\{\cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)\right\}$$

If $k = 0$, $z = z_1 = 2(\cos\pi/5 + i\sin\pi/5)$.
If $k = 1$, $z = z_2 = 2(\cos 3\pi/5 + i\sin 3\pi/5)$.
If $k = 2$, $z = z_3 = 2(\cos 5\pi/5 + i\sin 5\pi/5) = -2$.
If $k = 3$, $z = z_4 = 2(\cos 7\pi/5 + i\sin 7\pi/5)$.
If $k = 4$, $z = z_5 = 2(\cos 9\pi/5 + i\sin 9\pi/5)$.

By considering $k = 5, 6, \ldots$ as well as negative values, $-1, -2, \ldots$, repetitions of the above five values of z are obtained. Hence, these are the only *solutions* or *roots* of the given equation. These five roots are called the *fifth roots of* -32 and are collectively denoted by $(-32)^{1/5}$. In general, $a^{1/n}$ represents the nth roots of a and there are n such roots.

(b) The values of z are indicated in Fig. 1-31. Note that they are equally spaced along the circumference of a circle with center at the origin and radius 2. Another way of saying this is that the roots are represented by the vertices of a regular polygon.

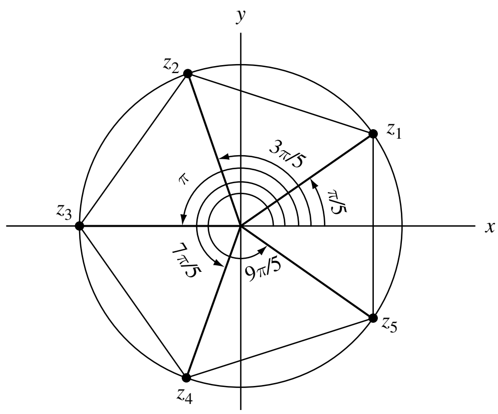

Fig. 1-31

1.29. Find each of the indicated roots and locate them graphically.

(a) $(-1+i)^{1/3}$, (b) $(-2\sqrt{3}-2i)^{1/4}$

Solution

(a) $(-1+i)^{1/3}$

$$-1+i = \sqrt{2}\{\cos(3\pi/4 + 2k\pi) + i\sin(3\pi/4 + 2k\pi)\}$$

$$(-1+i)^{1/3} = 2^{1/6}\left\{\cos\left(\frac{3\pi/4 + 2k\pi}{3}\right) + i\sin\left(\frac{3\pi/4 + 2k\pi}{3}\right)\right\}$$

If $k = 0$, $z_1 = 2^{1/6}(\cos\pi/4 + i\sin\pi/4)$.
If $k = 1$, $z_2 = 2^{1/6}(\cos 11\pi/12 + i\sin 11\pi/12)$.
If $k = 2$, $z_3 = 2^{1/6}(\cos 19\pi/12 + i\sin 19\pi/12)$.

These are represented graphically in Fig. 1-32.

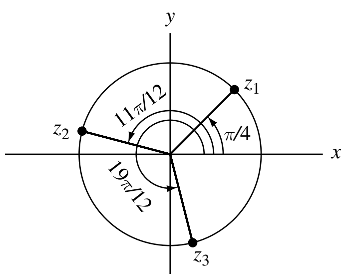

Fig. 1-32

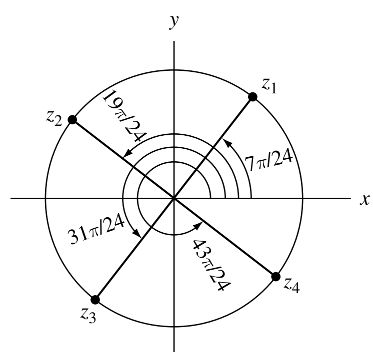

Fig. 1-33

(b) $(-2\sqrt{3}-2i)^{1/4}$

$$-2\sqrt{3}-2i = 4\{\cos(7\pi/6 + 2k\pi) + i\sin(7\pi/6 + 2k\pi)\}$$

$$(-2\sqrt{3}-2i)^{1/4} = 4^{1/4}\left\{\cos\left(\frac{7\pi/6 + 2k\pi}{4}\right) + i\sin\left(\frac{7\pi/6 + 2k\pi}{4}\right)\right\}$$

If $k = 0$, $z_1 = \sqrt{2}(\cos 7\pi/24 + i\sin 7\pi/24)$.
If $k = 1$, $z_2 = \sqrt{2}(\cos 19\pi/24 + i\sin 19\pi/24)$.
If $k = 2$, $z_3 = \sqrt{2}(\cos 31\pi/24 + i\sin 31\pi/24)$.
If $k = 3$, $z_4 = \sqrt{2}(\cos 43\pi/24 + i\sin 43\pi/24)$.

These are represented graphically in Fig. 1-33.

1.30. Find the square roots of $-15 - 8i$.

Solution

Method 1.

$$-15 - 8i = 17\{\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\}$$

where $\cos\theta = -15/17$, $\sin\theta = -8/17$. Then the square roots of $-15 - 8i$ are

$$\sqrt{17}(\cos\theta/2 + i\sin\theta/2) \tag{1}$$

and

$$\sqrt{17}\{\cos(\theta/2 + \pi) + i\sin(\theta/2 + \pi)\} = -\sqrt{17}(\cos\theta/2 + i\sin\theta/2) \tag{2}$$

Now

$$\cos\theta/2 = \pm\sqrt{(1 + \cos\theta)/2} = \pm\sqrt{(1 - 15/17)/2} = \pm 1/\sqrt{17}$$
$$\sin\theta/2 = \pm\sqrt{(1 - \cos\theta)/2} = \pm\sqrt{(1 + 15/17)/2} = \pm 4/\sqrt{17}$$

Since θ is an angle in the third quadrant, $\theta/2$ is an angle in the second quadrant. Hence, $\cos\theta/2 = -1/\sqrt{17}$, $\sin\theta/2 = 4/\sqrt{17}$, and so from (1) and (2) the required square roots are $-1 + 4i$ and $1 - 4i$. As a check, note that $(-1 + 4i)^2 = (1 - 4i)^2 = -15 - 8i$.

Method 2.
Let $p + iq$, where p and q are real, represent the required square roots. Then

$$(p + iq)^2 = p^2 - q^2 + 2pqi = -15 - 8i$$

or

$$p^2 - q^2 = -15 \tag{3}$$
$$pq = -4 \tag{4}$$

Substituting $q = -4/p$ from (4) into (3), it becomes $p^2 - 16/p^2 = -15$ or $p^4 + 15p^2 - 16 = 0$, i.e., $(p^2 + 16)(p^2 - 1) = 0$ or $p^2 = -16$, $p^2 = 1$. Since p is real, $p = \pm 1$. From (4), if $p = 1$, $q = -4$; if $p = -1$, $q = 4$. Thus the roots are $-1 + 4i$ and $1 - 4i$.

Polynomial Equations

1.31. Solve the *quadratic equation* $az^2 + bz + c = 0$, $a \neq 0$.

Solution

Transposing c and dividing by $a \neq 0$,

$$z^2 + \frac{b}{a}z = -\frac{c}{a}$$

Adding $(b/2a)^2$ [completing the square],

$$z^2 + \frac{b}{a}z + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2. \quad \text{Then} \quad \left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square roots,

$$z + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}. \quad \text{Hence} \quad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.32. Solve the equation $z^2 + (2i - 3)z + 5 - i = 0$.

Solution

From Problem 1.31, $a = 1$, $b = 2i - 3$, $c = 5 - i$ and so the solutions are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(2i-3) \pm \sqrt{(2i-3)^2 - 4(1)(5-i)}}{2(1)} = \frac{3 - 2i \pm \sqrt{-15-8i}}{2}$$
$$= \frac{3 - 2i \pm (1-4i)}{2} = 2 - 3i \quad \text{or} \quad 1 + i$$

using the fact that the square roots of $-15 - 8i$ are $\pm(1 - 4i)$ [see Problem 1.30]. These are found to satisfy the given equation.

1.33. Suppose the real rational number p/q (where p and q have no common factor except ± 1, i.e., p/q is in lowest terms) satisfies the polynomial equation $a_0z^n + a_1z^{n-1} + \cdots + a_n = 0$ where $a_0, a_1, \ldots, a_n$ are integers. Show that p and q must be factors of a_n and a_0, respectively.

Solution

Substituting $z = p/q$ in the given equation and multiplying by q^n yields

$$a_0p^n + a_1p^{n-1}q + \cdots + a_{n-1}pq^{n-1} + a_nq^n = 0 \tag{1}$$

Dividing by p and transposing the last term,

$$a_0p^{n-1} + a_1p^{n-2}q + \cdots + a_{n-1}q^{n-1} = -\frac{a_nq^n}{p} \tag{2}$$

Since the left side of (2) is an integer, so also is the right side. But since p has no factor in common with q, it cannot divide q^n and so must divide a_n.

Similarly, on dividing (1) by q and transposing the first term, we find that q must divide a_0.

1.34. Solve $6z^4 - 25z^3 + 32z^2 + 3z - 10 = 0$.

Solution

The integer factors of 6 and -10 are, respectively, $\pm 1, \pm 2, \pm 3, \pm 6$ and $\pm 1, \pm 2, \pm 5, \pm 10$. Hence, by Problem 1.33, the possible rational solutions are $\pm 1, \pm 1/2, \pm 1/3, \pm 1/6, \pm 2, \pm 2/3, \pm 5, \pm 5/2, \pm 5/3, \pm 5/6, \pm 10, \pm 10/3$.

By trial, we find that $z = -1/2$ and $z = 2/3$ are solutions, and so the polynomial

$$(2z+1)(3z-2) = 6z^2 - z - 2 \text{ is a factor of } 6z^4 - 25z^3 + 32z^2 + 3z - 10$$

the other factor being $z^2 - 4z + 5$ as found by long division. Hence

$$6z^4 - 25z^3 + 32z^2 + 3z - 10 = (6z^2 - z - 2)(z^2 - 4z + 5) = 0$$

The solutions of $z^2 - 4z + 5 = 0$ are [see Problem 1.31]

$$z = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Then the solutions are $-1/2, 2/3, 2+i, 2-i$.

1.35. Prove that the sum and product of all the roots of $a_0z^n + a_1z^{n-1} + \cdots + a_n = 0$ where $a_0 \neq 0$, are $-a_1/a_0$ and $(-1)^n a_n/a_0$, respectively.

Solution

If $z_1, z_2, \ldots, z_n$ are the n roots, the equation can be written in factored form as

$$a_0(z - z_1)(z - z_2)\cdots(z - z_n) = 0$$

Direct multiplication shows that

$$a_0\{z^n - (z_1 + z_2 + \cdots + z_n)z^{n-1} + \cdots + (-1)^n z_1 z_2 \cdots z_n\} = 0$$

It follows that $-a_0(z_1 + z_2 + \cdots + z_n) = a_1$ and $a_0(-1)^n z_1 z_2 \cdots z_n = a_n$, from which

$$z_1 + z_2 + \cdots + z_n = -a_1/a_0, \quad z_1 z_2 \cdots z_n = (-1)^n a_n/a_0$$

as required.

1.36. Suppose $p + qi$ is a root of $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$ where $a_0 \neq 0, a_1, \ldots, a_n$, p and q are real. Prove that $p - qi$ is also a root.

Solution

Let $p + qi = re^{i\theta}$ in polar form. Since this satisfies the equation,

$$a_0 r^n e^{in\theta} + a_1 r^{n-1} e^{i(n-1)\theta} + \cdots + a_{n-1} r e^{i\theta} + a_n = 0$$

Taking the conjugate of both sides

$$a_0 r^n e^{-in\theta} + a_1 r^{n-1} e^{-i(n-1)\theta} + \cdots + a_{n-1} r e^{-i\theta} + a_n = 0$$

we see that $re^{-i\theta} = p - qi$ is also a root. The result does not hold if $a_0, \ldots, a_n$ are not all real (see Problem 1.32).

The theorem is often expressed in the statement: The zeros of a polynomial with real coefficients occur in conjugate pairs.

The *n*th Roots of Unity

1.37. Find all the 5th roots of unity.

Solution

$$z^5 = 1 = \cos 2k\pi + i \sin 2k\pi = e^{2k\pi i}$$

where $k = 0, \pm 1, \pm 2, \ldots$. Then

$$z = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} = e^{2k\pi i/5}$$

where it is sufficient to use $k = 0, 1, 2, 3, 4$ since all other values of k lead to repetition.

Thus the roots are $1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$. If we call $e^{2\pi i/5} = \omega$, these can be denoted by $1, \omega, \omega^2, \omega^3, \omega^4$.

1.38. Suppose $n = 2, 3, 4, \ldots$. Prove that

(a)
$$\cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \cos \frac{6\pi}{n} + \cdots + \cos \frac{2(n-1)\pi}{n} = -1$$

(b)
$$\sin \frac{2\pi}{n} + \sin \frac{4\pi}{n} + \sin \frac{6\pi}{n} + \cdots + \sin \frac{2(n-1)\pi}{n} = 0$$

Solution

Consider the equation $z^n - 1 = 0$ whose solutions are the nth roots of unity,

$$1, e^{2\pi i/n}, e^{4\pi i/n}, e^{6\pi i/n}, \ldots, e^{2(n-1)\pi i/n}$$

By Problem 1.35, the sum of these roots is zero. Then

$$1 + e^{2\pi i/n} + e^{4\pi i/n} + e^{6\pi i/n} + \cdots + e^{2(n-1)\pi i/n} = 0$$

i.e.,

$$\left\{1 + \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n}\right\} + i\left\{\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \cdots + \sin\frac{2(n-1)\pi}{n}\right\} = 0$$

from which the required results follow.

Dot and Cross Product

1.39. Suppose $z_1 = 3 - 4i$ and $z_2 = -4 + 3i$. Find: (a) $z_1 \cdot z_2$, (b) $|z_1 \times z_2|$.

Solution

(a) $z_1 \cdot z_2 = \text{Re}\{\bar{z}_1 z_2\} = \text{Re}\{(3 + 4i)(-4 + 3i)\} = \text{Re}\{-24 - 7i\} = -24$

Another Method. $z_1 \cdot z_2 = (3)(-4) + (-4)(3) = -24$

(b) $|z_1 \times z_2| = |\text{Im}\{\bar{z}_1 z_2\}| = |\text{Im}\{(3 + 4i)(-4 + 3i)\}| = |\text{Im}\{-24 - 7i\}| = |-7| = 7$

Another Method. $|z_1 \times z_2| = |(3)(3) - (-4)(-4)| = |-7| = 7$

1.40. Find the acute angle between the vectors in Problem 1.39.

Solution

From Problem 1.39(a), we have

$$\cos\theta = \frac{z_1 \cdot z_2}{|z_1||z_2|} = \frac{-24}{|3 - 4i||-4 + 3i|} = \frac{-24}{25} = -.96$$

Then the acute angle is $\cos^{-1} .96 = 16°16'$ approximately.

1.41. Prove that the area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$.

Solution

$$\begin{aligned}\text{Area of parallelogram [Fig.1-34]} &= \text{(base)(height)}\\ &= (|z_2|)(|z_1|\sin\theta) = |z_1||z_2|\sin\theta = |z_1 \times z_2|\end{aligned}$$

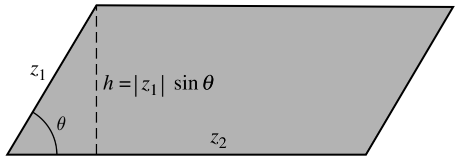

Fig. 1-34

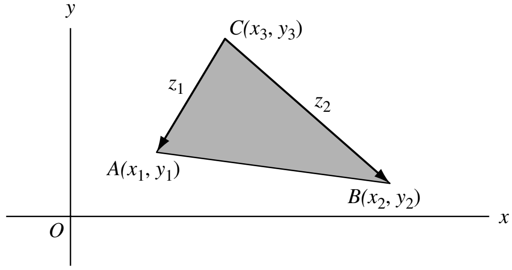

Fig. 1-35

1.42. Find the area of a triangle with vertices at $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$.

Solution

The vectors from C to A and B [Fig. 1-35] are, respectively, given by

$$z_1 = (x_1 - x_3) + i(y_1 - y_3) \quad \text{and} \quad z_2 = (x_2 - x_3) + i(y_2 - y_3)$$

Since the area of a triangle with sides z_1 and z_2 is half the area of the corresponding parallelogram, we have by Problem 1.41:

$$\begin{aligned}\text{Area of triangle} &= \tfrac{1}{2}|z_1 \times z_2| = \tfrac{1}{2}|\text{Im}\{[(x_1 - x_3) - i(y_1 - y_3)][(x_2 - x_3) + i(y_2 - y_3)]\}| \\ &= \tfrac{1}{2}|(x_1 - x_3)(y_2 - y_3) - (y_1 - y_3)(x_2 - x_3)| \\ &= \tfrac{1}{2}|x_1y_2 - y_1x_2 + x_2y_3 - y_2x_3 + x_3y_1 - y_3x_1| \\ &= \tfrac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|\end{aligned}$$

in determinant form.

Complex Conjugate Coordinates

1.43. Express each equation in terms of conjugate coordinates: (a) $2x + y = 5$, (b) $x^2 + y^2 = 36$.

Solution

(a) Since $z = x + iy$, $\bar{z} = x - iy$, $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$. Then, $2x + y = 5$ becomes

$$2\left(\frac{z + \bar{z}}{2}\right) + \left(\frac{z - \bar{z}}{2i}\right) = 5 \quad \text{or} \quad (2i + 1)z + (2i - 1)\bar{z} = 10i$$

The equation represents a straight line in the z plane.

(b) ***Method 1.*** The equation is $(x + iy)(x - iy) = 36$ or $z\bar{z} = 36$.

Method 2. Substitute $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$ in $x^2 + y^2 = 36$ to obtain $z\bar{z} = 36$.

The equation represents a circle in the z plane of radius 6 with center at the origin.

1.44. Prove that the equation of any circle or line in the z plane can be written as $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ where α and γ are real constants while β may be a complex constant.

Solution

The general equation of a circle in the xy plane can be written

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

which in conjugate coordinates becomes

$$Az\bar{z} + B\left(\frac{z + \bar{z}}{2}\right) + C\left(\frac{z - \bar{z}}{2i}\right) + D = 0 \quad \text{or} \quad Az\bar{z} + \left(\frac{B}{2} + \frac{C}{2i}\right)z + \left(\frac{B}{2} - \frac{C}{2i}\right)\bar{z} + D = 0$$

Calling $A = \alpha$, $(B/2) + (C/2i) = \beta$ and $D = \gamma$, the required result follows.

In the special case $A = \alpha = 0$, the circle degenerates into a line.

Point Sets

1.45. Given the point set S:$\{i, \frac{1}{2}i, \frac{1}{3}i, \frac{1}{4}i, \ldots\}$ or briefly $\{i/n\}$. (a) Is S bounded? (b) What are its limit points, if any? (c) Is S closed? (d) What are its interior and boundary points? (e) Is S open? (f) Is S connected? (g) Is S an open region or domain? (h) What is the closure of S? (i) What is the complement of S? (j) Is S countable? (k) Is S compact? (l) Is the closure of S compact?

Solution

(a) S is bounded since for every point z in S, $|z| < 2$ (for example), i.e., all points of S lie inside a circle of radius 2 with center at the origin.

(b) Since every deleted neighborhood of $z = 0$ contains points of S, a limit point is $z = 0$. It is the only limit point.

Note that since S is bounded and infinite, the Weierstrass–Bolzano theorem predicts *at least one* limit point.

(c) S is not closed since the limit point $z = 0$ does not belong to S.

(d) Every δ neighborhood of any point i/n (i.e., every circle of radius δ with center at i/n) contains points that belong to S and points that do not belong to S. Thus every point of S, as well as the point $z = 0$, is a boundary point. S has *no* interior points.

(e) S does not consist of any interior points. Hence, it cannot be open. Thus, S is neither open nor closed.

(f) If we join any two points of S by a polygonal path, there are points on this path that do not belong to S. Thus S is not connected.

(g) Since S is not an open connected set, it is not an open region or domain.

(h) The closure of S consists of the set S together with the limit point zero, i.e., $\{0, i, \frac{1}{2}i, \frac{1}{3}i, \ldots\}$.

(i) The complement of S is the set of all points not belonging to S, i.e., all points $z \neq i, i/2, i/3, \ldots$.

(j) There is a one to one correspondence between the elements of S and the natural numbers 1, 2, 3, ... as indicated below:

$$\begin{array}{ccccc} i & \frac{1}{2}i & \frac{1}{3}i & \frac{1}{4} & i\ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & \ldots \end{array}$$

Hence, S is countable.

(k) S is bounded but not closed. Hence, it is not compact.

(l) The closure of S is bounded and closed and so is compact.

1.46. Given the point sets $A = \{3, -i, 4, 2+i, 5\}$, $B = \{-i, 0, -1, 2+i\}$, $C = \{-\sqrt{2}i, \frac{1}{2}, 3\}$. Find (a) $A \cup B$, (b) $A \cap B$, (c) $A \cap C$, (d) $A \cap (B \cup C)$, (e) $(A \cap B) \cup (A \cap C)$, (f) $A \cap (B \cap C)$.

Solution

(a) $A \cup B$ consists of points belonging either to A or B or both and is given by $\{3, -i, 4, 2+i, 5, 0, -1\}$.

(b) $A \cap B$ consists of points belonging to both A and B and is given by $\{-i, 2+i\}$.

(c) $A \cap C = \{3\}$, consisting of only the member 3.

(d) $B \cup C = \{-i, 0, -1, 2+i, -\sqrt{2}i, \frac{1}{2}, 3\}$.

Hence $A \cap (B \cup C) = \{3, -i, 2+i\}$, consisting of points belonging to both A and $B \cup C$.

(e) $A \cap B = \{-i, 2+i\}$, $A \cap C = \{3\}$ from parts (b) and (c). Hence $(A \cap B) \cup (A \cap C) = \{-i, 2+i, 3\}$.

From this and the result of (d), we see that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, which illustrates the fact that A, B, C satisfy the *distributive law*. We can show that sets exhibit many of the properties valid in the algebra of numbers. This is of great importance in theory and application.

(f) $B \cap C = \emptyset$, the *null set*, since there are no points common to both B and C. Hence, $A \cap (B \cap C) = \emptyset$ also.

Miscellaneous Problems

1.47. A number is called an *algebraic number* if it is a solution of a polynomial equation

$$a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0 \text{ where } a_0, a_1, \ldots, a_n \text{ are integers.}$$

Prove that (a) $\sqrt{3}+\sqrt{2}$ and (b) $\sqrt[3]{4}-2i$ are algebraic numbers.

Solution

(a) Let $z = \sqrt{3}+\sqrt{2}$ or $z - \sqrt{2} = \sqrt{3}$. Squaring, $z^2 - 2\sqrt{2}z + 2 = 3$ or $z^2 - 1 = 2\sqrt{2}z$. Squaring again, $z^4 - 2z^2 + 1 = 8z^2$ or $z^4 - 10z^2 + 1 = 0$, a polynomial equation with integer coefficients having $\sqrt{3}+\sqrt{2}$ as a root. Hence, $\sqrt{3}+\sqrt{2}$ is an algebraic number.

(b) Let $z = \sqrt[3]{4}-2i$ or $z + 2i = \sqrt[3]{4}$. Cubing, $z^3 + 3z^2(2i) + 3z(2i)^2 + (2i)^3 = 4$ or $z^3 - 12z - 4 = i(8 - 6z^2)$. Squaring, $z^6 + 12z^4 - 8z^3 + 48z^2 + 96z + 80 = 0$, a polynomial equation with integer coefficients having $\sqrt[3]{4}-2i$ as a root. Hence, $\sqrt[3]{4}-2i$ is an algebraic number.

Numbers that are not algebraic, i.e., do not satisfy any polynomial equation with integer coefficients, are called *transcendental numbers*. It has been proved that the numbers π and e are transcendental. However, it is still not yet known whether numbers such as $e\pi$ or $e+\pi$, for example, are transcendental or not.

1.48. Represent graphically the set of values of z for which (a) $\left|\dfrac{z-3}{z+3}\right| = 2$, (b) $\left|\dfrac{z-3}{z+3}\right| < 2$.

Solution

(a) The given equation is equivalent to $|z-3| = 2|z+3|$ or, if $z = x+iy$, $|x+iy-3| = 2|x+iy+3|$, i.e.,

$$\sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$$

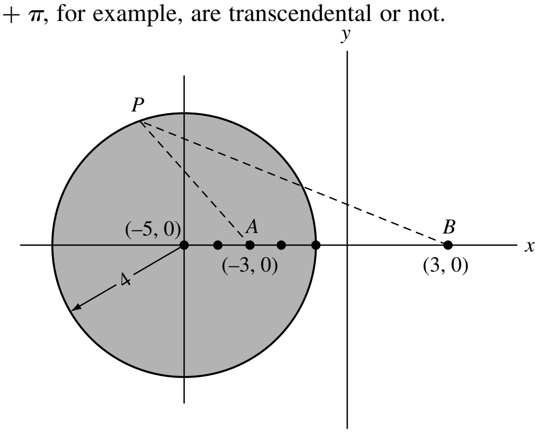

Fig. 1-36

Squaring and simplifying, this becomes

$$x^2 + y^2 + 10x + 9 = 0$$

or

$$(x+5)^2 + y^2 = 16$$

i.e., $|z+5| = 4$, a circle of radius 4 with center at $(-5, 0)$ as shown in Fig. 1-36.

Geometrically, any point P on this circle is such that the distance from P to point $B(3, 0)$ is twice the distance from P to point $A(-3, 0)$.

Another Method.

$$\left|\frac{z-3}{z+3}\right| = 2 \text{ is equivalent to} \left(\frac{z-3}{z+3}\right)\left(\frac{\bar{z}-3}{\bar{z}+3}\right) = 4 \quad \text{or} \quad z\bar{z} + 5\bar{z} + 5z + 9 = 0$$

i.e., $(z+5)(\bar{z}+5) = 16$ or $|z+5| = 4$.

(b) The given inequality is equivalent to $|z-3| < 2|z+3|$ or $\sqrt{(x-3)^2+y^2} < 2\sqrt{(x+3)^2+y^2}$. Squaring and simplifying, this becomes $x^2+y^2+10x+9 > 0$ or $(x+5)^2+y^2 > 16$, i.e., $|z+5| > 4$.

The required set thus consists of all points external to the circle of Fig. 1-36.

1.49. Given the sets A and B represented by $|z-1|<2$ and $|z-2i|<1.5$, respectively. Represent geometrically (a) $A \cap B$, (b) $A \cup B$.

Solution

The required sets of points are shown shaded in Figs. 1-37 and 1-38, respectively.

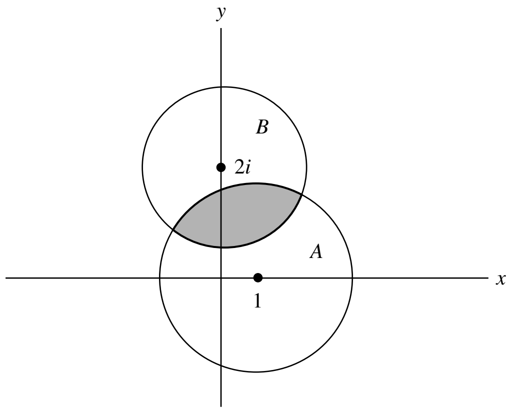

Fig. 1-37

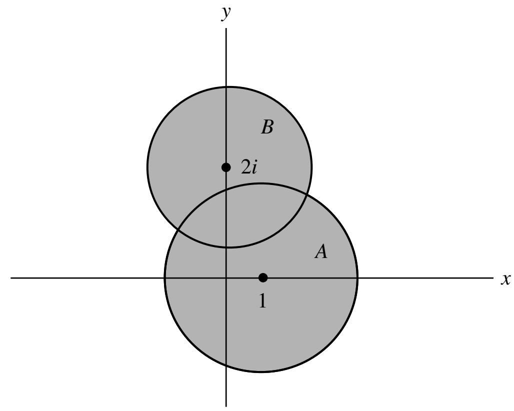

Fig. 1-38

1.50. Solve $z^2(1-z^2)=16$.

Solution

Method 1. The equation can be written $z^4 - z^2 + 16 = 0$, i.e., $z^4 + 8z^2 + 16 - 9z^2 = 0$, $(z^2+4)^2 - 9z^2 = 0$ or $(z^2+4+3z)(z^2+4-3z)=0$. Then, the required solutions are the solutions of $z^2+3z+4=0$ and $z^2-3z+4=0$, or

$$-\frac{3}{2} \pm \frac{\sqrt{7}}{2} i \quad \text{and} \quad \frac{3}{2} \pm \frac{\sqrt{7}}{2} i$$

.

Method 2. Letting $w=z^2$, the equation can be written $w^2 - w + 16 = 0$ and $w = \frac{1}{2} \pm \frac{3}{2}\sqrt{7}i$. To obtain solutions of $z^2 = \frac{1}{2} \pm \frac{3}{2}\sqrt{7}i$, the methods of Problem 1.30 can be used.

1.51. Let z_1, z_2, z_3 represent vertices of an equilateral triangle. Prove that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

Solution

From Fig. 1-39, we see that

$$z_2 - z_1 = e^{\pi i/3}(z_3 - z_1)$$
$$z_1 - z_3 = e^{\pi i/3}(z_2 - z_3)$$

Then, by division,

$$\frac{z_2 - z_1}{z_1 - z_3} = \frac{z_3 - z_1}{z_2 - z_3}$$

or

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

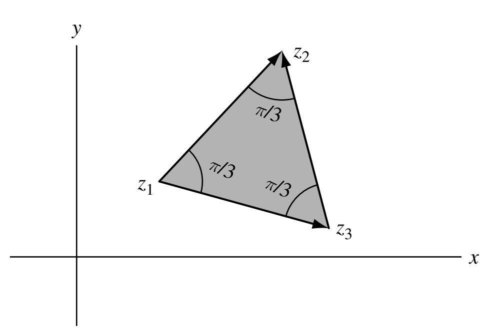

Fig. 1-39

1.52. Prove that for $m = 2, 3, \ldots$

$$\sin\frac{\pi}{m}\sin\frac{2\pi}{m}\sin\frac{3\pi}{m}\cdots\sin\frac{(m-1)\pi}{m} = \frac{m}{2^{m-1}}$$

Solution

The roots of $z^m = 1$ are $z = 1, e^{2\pi i/m}, e^{4\pi i/m}, \ldots, e^{2(m-1)\pi i/m}$. Then, we can write

$$z^m - 1 = (z-1)(z - e^{2\pi i/m})(z - e^{4\pi i/m})\cdots(z - e^{2(m-1)\pi i/m})$$

Dividing both sides by $z - 1$ and then letting $z = 1$ [realizing that $(z^m - 1)/(z-1) = 1 + z + z^2 + \cdots + z^{m-1}$], we find

$$m = (1 - e^{2\pi i/m})(1 - e^{4\pi i/m})\cdots(1 - e^{2(m-1)\pi i/m}) \tag{1}$$

Taking the complex conjugate of both sides of (1) yields

$$m = (1 - e^{-2\pi i/m})(1 - e^{-4\pi i/m})\cdots(1 - e^{-2(m-1)\pi i/m}) \tag{2}$$

Multiplying (1) by (2) using $(1 - e^{2k\pi i/m})(1 - e^{-2k\pi i/m}) = 2 - 2\cos(2k\pi/m)$, we have

$$m^2 = 2^{m-1}\left(1 - \cos\frac{2\pi}{m}\right)\left(1 - \cos\frac{4\pi}{m}\right)\cdots\left(1 - \cos\frac{2(m-1)\pi}{m}\right) \tag{3}$$

Since $1 - \cos(2k\pi/m) = 2\sin^2(k\pi/m)$, (3) becomes

$$m^2 = 2^{2m-2}\sin^2\frac{\pi}{m}\sin^2\frac{2\pi}{m}\cdots\sin^2\frac{(m-1)\pi}{m} \tag{4}$$

Then, taking the positive square root of both sides yields the required result.

SUPPLEMENTARY PROBLEMS

Fundamental Operations with Complex Numbers

1.53. Perform each of the indicated operations:

(a) $(4-3i) + (2i-8)$, (d) $(i-2)\{2(1+i) - 3(i-1)\}$, (g) $\dfrac{(2+i)(3-2i)(1+2i)}{(1-i)^2}$

(b) $3(-1+4i) - 2(7-i)$, (e) $\dfrac{2-3i}{4-i}$, (h) $(2i-1)^2\left\{\dfrac{4}{1-i} + \dfrac{2-i}{1+i}\right\}$

(c) $(3+2i)(2-i)$, (f) $(4+i)(3+2i)(1-i)$ (i) $\dfrac{i^4 + i^9 + i^{16}}{2 - i^5 + i^{10} - i^{15}}$

1.54. Suppose $z_1 = 1 - i$, $z_2 = -2 + 4i$, $z_3 = \sqrt{3} - 2i$. Evaluate each of the following:

(a) $z_1^2 + 2z_1 - 3$ (d) $|z_1\bar{z}_2 + z_2\bar{z}_1|$ (g) $\overline{(z_2 + z_3)(z_1 - z_3)}$

(b) $|2z_2 - 3z_1|^2$ (e) $\left|\dfrac{z_1 + z_2 + 1}{z_1 - z_2 + i}\right|$ (h) $|z_1^2 + \bar{z}_2^2|^2 + |\bar{z}_3^2 - z_2^2|^2$

(c) $(z_3 - \bar{z}_3)^5$ (f) $\dfrac{1}{2}\left(\dfrac{z_3}{\bar{z}_3} + \dfrac{\bar{z}_3}{z_3}\right)$ (i) $\text{Re}\{2z_1^3 + 3z_2^2 - 5z_3^2\}$

1.55. Prove that (a) $(\overline{z_1 z_2}) = \bar{z}_1 \bar{z}_2$, (b) $(\overline{z_1 z_2 z_3}) = \bar{z}_1 \bar{z}_2 \bar{z}_3$. Generalize these results.

1.56. Prove that (a) $(\overline{z_1/z_2}) = \bar{z}_1/\bar{z}_2$, (b) $|z_1/z_2| = |z_1|/|z_2|$ if $z_2 \neq 0$.

1.57. Find real numbers x and y such that $2x - 3iy + 4ix - 2y - 5 - 10i = (x + y + 2) - (y - x + 3)i$.

1.58. Prove that (a) $\text{Re}\{z\} = (z + \bar{z})/2$, (b) $\text{Im}\{z\} = (z - \bar{z})/2i$.

1.59. Suppose the product of two complex numbers is zero. Prove that at least one of the numbers must be zero.

1.60. Let $w = 3iz - z^2$ and $z = x + iy$. Find $|w|^2$ in terms of x and y.

Graphical Representation of Complex Numbers. Vectors.

1.61. Perform the indicated operations both analytically and graphically.

(a) $(2 + 3i) + (4 - 5i)$

(b) $(7 + i) - (4 - 2i)$

(c) $3(1 + 2i) - 2(2 - 3i)$

(d) $3(1 + i) + 2(4 - 3i) - (2 + 5i)$

(e) $\frac{1}{2}(4 - 3i) + \frac{3}{2}(5 + 2i)$

1.62. Let z_1, z_2, and z_3 be the vectors indicated in Fig. 1-40. Construct graphically:

(a) $2z_1 + z_3$

(b) $(z_1 + z_2) + z_3$

(c) $z_1 + (z_2 + z_3)$

(d) $3z_1 - 2z_2 + 5z_3$

(e) $\frac{1}{3}z_2 - \frac{3}{4}z_1 + \frac{2}{3}z_3$

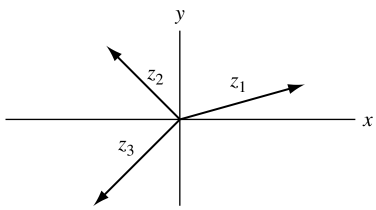

Fig. 1-40

1.63. Let $z_1 = 4 - 3i$ and $z_2 = -1 + 2i$. Obtain graphically and analytically (a) $|z_1 + z_2|$, (b) $|z_1 - z_2|$, (c) $\bar{z}_1 - \bar{z}_2$, (d) $|2\bar{z}_1 - 3\bar{z}_2 - 2|$.

1.64. The position vectors of points A, B, and C of triangle ABC are given by $z_1 = 1 + 2i$, $z_2 = 4 - 2i$, and $z_3 = 1 - 6i$, respectively. Prove that ABC is an isosceles triangle and find the lengths of the sides.

1.65. Let z_1, z_2, z_3, z_4 be the position vectors of the vertices for quadrilateral $ABCD$. Prove that $ABCD$ is a parallelogram if and only if $z_1 - z_2 - z_3 + z_4 = 0$.

1.66. Suppose the diagonals of a quadrilateral bisect each other. Prove that the quadrilateral is a parallelogram.

1.67. Prove that the medians of a triangle meet in a point.

1.68. Let $ABCD$ be a quadrilateral and E, F, G, H the midpoints of the sides. Prove that $EFGH$ is a parallelogram.

1.69. In parallelogram $ABCD$, point E bisects side AD. Prove that the point where BE meets AC trisects AC.

1.70. The position vectors of points A and B are $2 + i$ and $3 - 2i$, respectively. (a) Find an equation for line AB. (b) Find an equation for the line perpendicular to AB at its midpoint.

1.71. Describe and graph the locus represented by each of the following: (a) $|z - i| = 2$, (b) $|z + 2i| + |z - 2i| = 6$, (c) $|z - 3| - |z + 3| = 4$, (d) $z(\bar{z} + 2) = 3$, (e) $\text{Im}\{z^2\} = 4$.

1.72. Find an equation for (a) a circle of radius 2 with center at $(-3, 4)$, (b) an ellipse with foci at $(0, 2)$ and $(0, -2)$ whose major axis has length 10.

1.73. Describe graphically the region represented by each of the following:

(a) $1 < |z+i| \leq 2$, (b) $\text{Re}\{z^2\} > 1$, (c) $|z+3i| > 4$, (d) $|z+2-3i| + |z-2+3i| < 10$.

1.74. Show that the ellipse $|z+3| + |z-3| = 10$ can be expressed in rectangular form as $x^2/25 + y^2/16 = 1$ [see Problem 1.13(b)].

Axiomatic Foundations of Complex Numbers

1.75. Use the definition of a complex number as an ordered pair of real numbers to prove that if the product of two complex numbers is zero, then at least one of the numbers must be zero.

1.76. Prove the commutative laws with respect to (a) addition, (b) multiplication.

1.77. Prove the associative laws with respect to (a) addition, (b) multiplication.

1.78. (a) Find real numbers x and y such that $(c, d) \cdot (x, y) = (a, b)$ where $(c, d) \neq (0, 0)$.

(b) How is (x, y) related to the result for division of complex numbers given on page 2?

1.79. Prove that

$$(\cos\theta_1, \sin\theta_1)(\cos\theta_2, \sin\theta_2)\cdots(\cos\theta_n, \sin\theta_n)$$
$$= (\cos[\theta_1 + \theta_2 + \cdots + \theta_n], \ \sin[\theta_1 + \theta_2 + \cdots + \theta_n])$$

1.80. (a) How would you define $(a, b)^{1/n}$ where n is a positive integer?

(b) Determine $(a, b)^{1/2}$ in terms of a and b.

Polar Form of Complex Numbers

1.81. Express each of the following complex numbers in polar form:

(a) $2-2i$, (b) $-1+\sqrt{3}i$, (c) $2\sqrt{2}+2\sqrt{2}i$, (d) $-i$, (e) -4, (f) $-2\sqrt{3}-2i$, (g) $\sqrt{2}i$, (h) $\sqrt{3}/2 - 3i/2$.

1.82. Show that $2+i = \sqrt{5}e^{i\tan^{-1}(1/2)}$.

1.83. Express in polar form: (a) $-3-4i$, (b) $1-2i$.

1.84. Graph each of the following and express in rectangular form:

(a) $6(\cos 135^\circ + i\sin 135^\circ)$, (b) $12 \text{ cis } 90^\circ$, (c) $4 \text{ cis } 315^\circ$, (d) $2e^{5\pi i/4}$, (e) $5e^{7\pi i/6}$, (f) $3e^{-2\pi i/3}$.

1.85. An airplane travels 150 miles southeast, 100 miles due west, 225 miles 30° north of east, and then 200 miles northeast. Determine (a) analytically and (b) graphically how far and in what direction it is from its starting point.

1.86. Three forces as shown in Fig. 1-41 act in a plane on an object placed at O. Determine (a) graphically and (b) analytically what force is needed to prevent the object from moving. [This force is sometimes called the *equilibrant*.]

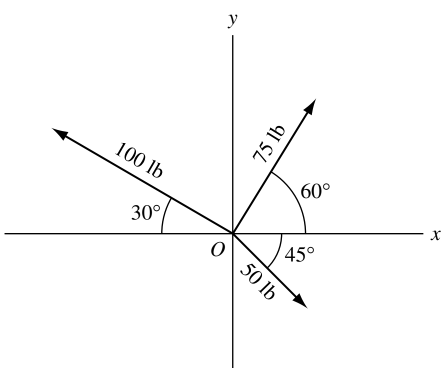

Fig. 1-41

1.87. Prove that on the circle $z = Re^{i\theta}$, $|e^{iz}| = e^{-R\sin\theta}$.

1.88. (a) Prove that $r_1e^{i\theta_1} + r_2e^{i\theta_2} = r_3e^{i\theta_3}$ where

$$r_3 = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}$$

and

$$\theta_3 = \tan^{-1}\left(\frac{r_1\sin\theta_1 + r_2\sin\theta_2}{r_1\cos\theta_1 + r_2\cos\theta_2}\right)$$

(b) Generalize the result in (a).

De Moivre's Theorem

1.89. Evaluate each of the following: (a) (5 cis 20°)(3 cis 40°) (b) $(2 \text{ cis } 50°)^6$

(c) $\dfrac{(8 \text{ cis } 40°)^3}{(2 \text{ cis } 60°)^4}$ (d) $\dfrac{(3e^{\pi i/6})(2e^{-5\pi i/4})(6e^{5\pi i/3})}{(4e^{2\pi i/3})^2}$ (e) $\left(\dfrac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4\left(\dfrac{1+i}{1-i}\right)^5$

1.90. Prove that (a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, (b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

1.91. Prove that the solutions of $z^4 - 3z^2 + 1 = 0$ are given by
$z = 2\cos 36°, 2\cos 72°, 2\cos 216°, 2\cos 252°$.

1.92. Show that (a) $\cos 36° = (\sqrt{5}+1)/4$, (b) $\cos 72° = (\sqrt{5}-1)/4$. [*Hint*: Use Problem 1.91.]

1.93. Prove that (a) $\sin 4\theta/\sin\theta = 8\cos^3\theta - 4\cos\theta = 2\cos 3\theta + 2\cos\theta$

(b) $\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1$

1.94. Prove De Moivre's theorem for (a) negative integers, (b) rational numbers.

Roots of Complex Numbers

1.95. Find each of the indicated roots and locate them graphically.

(a) $(2\sqrt{3}-2i)^{1/2}$, (b) $(-4+4i)^{1/5}$, (c) $(2+2\sqrt{3}i)^{1/3}$, (d) $(-16i)^{1/4}$, (e) $(64)^{1/6}$, (f) $(i)^{2/3}$.

1.96. Find all the indicated roots and locate them in the complex plane. (a) Cube roots of 8, (b) square roots of $4\sqrt{2}+4\sqrt{2}i$, (c) fifth roots of $-16+16\sqrt{3}i$, (d) sixth roots of $-27i$.

1.97. Solve the equations (a) $z^4 + 81 = 0$, (b) $z^6 + 1 = \sqrt{3}i$.

1.98. Find the square roots of (a) $5 - 12i$, (b) $8 + 4\sqrt{5}i$.

1.99. Find the cube roots of $-11 - 2i$.

Polynomial Equations

1.100. Solve the following equations, obtaining all roots:
(a) $5z^2 + 2z + 10 = 0$, (b) $z^2 + (i-2)z + (3-i) = 0$.

1.101. Solve $z^5 - 2z^4 - z^3 + 6z - 4 = 0$.

1.102. (a) Find all the roots of $z^4 + z^2 + 1 = 0$ and (b) locate them in the complex plane.

1.103. Prove that the sum of the roots of $a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_n = 0$ where $a_0 \neq 0$ taken r at a time is $(-1)^r a_r/a_0$ where $0 < r < n$.

1.104. Find two numbers whose sum is 4 and whose product is 8.

The *n*th Roots of Unity

1.105. Find all the (a) fourth roots, (b) seventh roots of unity, and exhibit them graphically.

1.106. (a) Prove that $1 + \cos 72° + \cos 144° + \cos 216° + \cos 288° = 0$.

(b) Give a graphical interpretation of the result in (a).

1.107. Prove that $\cos 36° + \cos 72° + \cos 108° + \cos 144° = 0$ and interpret graphically.

1.108. Prove that the sum of the products of all the nth roots of unity taken $2, 3, 4, \ldots, (n-1)$ at a time is zero.

1.109. Find all roots of $(1+z)^5 = (1-z)^5$.

The Dot and Cross Product

1.110. Given $z_1 = 2 + 5i$ and $z_2 = 3 - i$. Find
(a) $z_1 \cdot z_2$, (b) $|z_1 \times z_2|$, (c) $z_2 \cdot z_1$, (d) $|z_2 \times z_1|$, (e) $|z_1 \cdot z_2|$, (f) $|z_2 \cdot z_1|$.

1.111. Prove that $z_1 \cdot z_2 = z_2 \cdot z_1$.

1.112. Suppose $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Prove that
(a) $z_1 \cdot z_2 = r_1 r_2 \cos(\theta_2 - \theta_1)$, (b) $|z_1 \times z_2| = r_1 r_2 \sin(\theta_2 - \theta_1)$.

1.113. Prove that $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$.

1.114. Find the area of a triangle having vertices at $-4 - i$, $1 + 2i$, $4 - 3i$.

1.115. Find the area of a quadrilateral having vertices at $(2, -1)$, $(4, 3)$, $(-1, 2)$, and $(-3, -2)$.

Conjugate Coordinates

1.116. Describe each of the following loci expressed in terms of conjugate coordinates z, $\bar{z}$.
(a) $z\bar{z} = 16$, (b) $z\bar{z} - 2z - 2\bar{z} + 8 = 0$, (c) $z + \bar{z} = 4$, (d) $\bar{z} = z + 6i$.

1.117. Write each of the following equations in terms of conjugate coordinates.
(a) $(x-3)^2 + y^2 = 9$, (b) $2x - 3y = 5$, (c) $4x^2 + 16y^2 = 25$.

Point Sets

1.118. Let S be the set of all points $a + bi$, where a and b are rational numbers, which lie inside the square shown shaded in Fig. 1-42. (a) Is S bounded? (b) What are the limit points of S, if any? (c) Is S closed? (d) What are its interior and boundary points? (e) Is S open? (f) Is S connected? (g) Is S an open region or domain? (h) What is the closure of S? (i) What is the complement of S? (j) Is S countable? (k) Is S compact? (l) Is the closure of S compact?

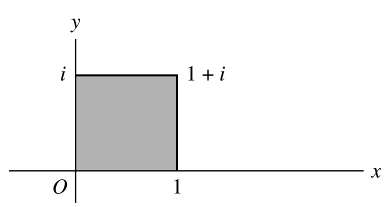

Fig. 1-42

1.119. Answer Problem 1.118 if S is the set of all points inside the square.

1.120. Answer Problem 1.118 if S is the set of all points inside or on the square.

1.121. Given the point sets $A = \{1, i, -i\}$, $B = \{2, 1, -i\}$, $C = \{i, -i, 1+i\}$, $D = \{0, -i, 1\}$. Find:
(a) $A \cup (B \cup C)$, (b) $(A \cap C) \cup (B \cap D)$, (c) $(A \cup C) \cap (B \cup D)$.

1.122. Suppose A, B, C, and D are any point sets. Prove that (a) $A \cup B = B \cup A$, (b) $A \cap B = B \cap A$, (c) $A \cup (B \cup C) = (A \cup B) \cup C$, (d) $A \cap (B \cap C) = (A \cap B) \cap C$, (e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

1.123. Suppose A, B, and C are the point sets defined by $|z + i| < 3$, $|z| < 5$, $|z + 1| < 4$. Represent graphically each of the following:

(a) $A \cap B \cap C$, (b) $A \cup B \cup C$, (c) $A \cap B \cup C$, (d) $C \cap (A \cup B)$, (e) $(A \cup B) \cap (B \cup C)$,
(f) $(A \cap B) \cup (B \cap C) \cup (C \cap A)$, (g) $(A \cap \tilde{B}) \cup (B \cap \tilde{C}) \cup (C \cap \tilde{A})$.

1.124. Prove that the complement of a set S is open or closed according as S is closed or open.

1.125. Suppose $S_1, S_2, \ldots, S_n$ are open sets. Prove that $S_1 \cup S_2 \cup \cdots \cup S_n$ is open.

1.126. Suppose a limit point of a set does not belong to the set. Prove that it must be a boundary point of the set.

Miscellaneous Problems

1.127. Let $ABCD$ be a parallelogram. Prove that $(AC)^2 + (BD)^2 = (AB)^2 + (BC)^2 + (CD)^2 + (DA)^2$.

1.128. Explain the fallacy: $-1 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$. Hence $1 = -1$.

1.129. (a) Show that the equation $z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0$ where a_1, a_2, a_3, a_4 are real constants different from zero, has a pure imaginary root if $a_3^2 + a_1^2 a_4 = a_1 a_2 a_3$.

(b) Is the converse of (a) true?

1.130. (a) Prove that $\cos^n \phi = \dfrac{1}{2^{n-1}}\left\{\cos n\phi + n\cos(n-2)\phi + \dfrac{n(n-1)}{2!}\cos(n-4)\phi + \cdots + R_n\right\}$ where

$$R_n = \begin{cases} \dfrac{n!}{[(n-1)/2]![(n+1)/2]!}\cos\phi & \text{if } n \text{ is odd} \\ \dfrac{n!}{2[(n/2)!]^2} & \text{if } n \text{ is even} \end{cases}$$

(b) Derive a similar result for $\sin^n \phi$.

1.131. Let $z = 6e^{\pi i/3}$. Evaluate $|e^{iz}|$.

1.132. Show that for any real numbers p and m, $e^{2mi\cot^{-1}p}\left\{\dfrac{pi+1}{pi-1}\right\}^m = 1$.

1.133. Let $P(z)$ be any polynomial in z with real coefficients. Prove that $\overline{P(z)} = P(\bar{z})$.

1.134. Suppose z_1, z_2, and z_3 are collinear. Prove that there exist real constants α, β, γ, not all zero, such that $\alpha z_1 + \beta z_2 + \gamma z_3 = 0$ where $\alpha + \beta + \gamma = 0$.

1.135. Given the complex number z, represent geometrically (a) $\bar{z}$, (b) $-z$, (c) $1/z$, (d) z^2.

1.136. Consider any two complex numbers z_1 and z_2 not equal to zero. Show how to represent graphically using only ruler and compass (a) $z_1 z_2$, (b) z_1/z_2, (c) $z_1^2 + z_2^2$, (d) $z_1^{1/2}$, (e) $z_2^{3/4}$.

1.137. Prove that an equation for a line passing through the points z_1 and z_2 is given by

$$\arg\{(z - z_1)/(z_2 - z_1)\} = 0$$

1.138. Suppose $z = x + iy$. Prove that $|x| + |y| \le \sqrt{2}|x + iy|$.

1.139. Is the converse to Problem 1.51 true? Justify your answer.

1.140. Find an equation for the circle passing through the points $1 - i$, $2i$, $1 + i$.

1.141. Show that the locus of z such that $|z - a||z + a| = a^2$, $a > 0$ is a *lemniscate* as shown in Fig. 1-43.

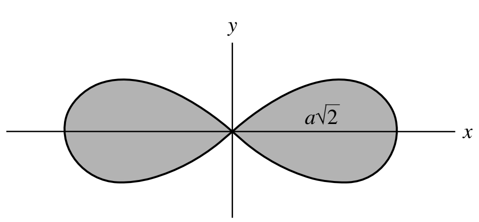

Fig. 1-43

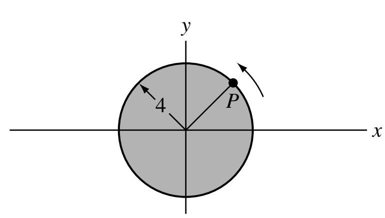

Fig. 1-44

1.142. Let $p_n = a_n^2 + b_n^2$, $n = 1, 2, 3, \ldots$ where a_n and b_n are positive integers. Prove that for every positive integer M, we can always find positive integers A and B such that $p_1 p_2 \cdots p_M = A^2 + B^2$. [Example: If $5 = 2^2 + 1^2$ and $25 = 3^2 + 4^2$, then $5 \cdot 25 = 2^2 + 11^2$.]

1.143. Prove that: (a) $\cos\theta + \cos(\theta+\alpha) + \cdots + \cos(\theta + n\alpha) = \dfrac{\sin\frac{1}{2}(n+1)\alpha}{\sin\frac{1}{2}\alpha}\cos(\theta + \frac{1}{2}n\alpha)$

(b) $\sin\theta + \sin(\theta+\alpha) + \cdots + \sin(\theta + n\alpha) = \dfrac{\sin\frac{1}{2}(n+1)\alpha}{\sin\frac{1}{2}\alpha}\sin(\theta + \frac{1}{2}n\alpha)$

1.144. Prove that (a) $\text{Re}\{z\} > 0$ and (b) $|z-1| < |z+1|$ are equivalent statements.

1.145. A wheel of radius 4 feet [Fig. 1-44] is rotating counterclockwise about an axis through its center at 30 revolutions per minute. (*a*) Show that the position and velocity of any point P on the wheel are given, respectively, by $4e^{i\pi t}$ and $4\pi i e^{i\pi t}$, where t is the time in seconds measured from the instant when P was on the positive x axis. (b) Find the position and velocity when $t = 2/3$ and $t = 15/4$.

1.146. Prove that for any integer $m > 1$,

$$(z+a)^{2m} - (z-a)^{2m} = 4maz \prod_{k=1}^{m-1} \{z^2 + a^2 \cot^2(k\pi/2m)\}$$

where $\prod_{k=1}^{m-1}$ denotes the product of all the factors indicated from $k = 1$ to $m - 1$.

1.147. Suppose points P_1 and P_2, represented by z_1 and z_2 respectively, are such that $|z_1 + z_2| = |z_1 - z_2|$. Prove that (a) z_1/z_2 is a pure imaginary number, (b) $\angle P_1 O P_2 = 90^\circ$.

1.148. Prove that for any integer $m > 1$,

$$\cot\frac{\pi}{2m}\cot\frac{2\pi}{2m}\cot\frac{3\pi}{2m}\cdots\cot\frac{(m-1)\pi}{2m} = 1$$

1.149. Prove and generalize: (a) $\csc^2(\pi/7) + \csc^2(2\pi/7) + \csc^2(4\pi/7) = 2$

(b) $\tan^2(\pi/16) + \tan^2(3\pi/16) + \tan^2(5\pi/16) + \tan^2(7\pi/16) = 28$

1.150. Let masses m_1, m_2, m_3 be located at points z_1, z_2, z_3, respectively. Prove that the center of mass is given by

$$\hat{z} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3}{m_1 + m_2 + m_3}$$

Generalize to n masses.

1.151. Find the point on the line joining points z_1 and z_2 which divides it in the ratio $p:q$.

1.152. Show that an equation for a circle passing through three points z_1, z_2, z_3 is given by

$$\left(\frac{z - z_1}{z - z_2}\right)\Big/\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \left(\frac{\bar{z} - \bar{z}_1}{\bar{z} - \bar{z}_2}\right)\Big/\left(\frac{\bar{z}_3 - \bar{z}_1}{\bar{z}_3 - \bar{z}_2}\right)$$

1.153. Prove that the medians of a triangle with vertices at z_1, z_2, z_3 intersect at the point $\frac{1}{3}(z_1 + z_2 + z_3)$.

1.154. Prove that the rational numbers between 0 and 1 are countable.

[*Hint.* Arrange the numbers as $0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \ldots$.]

1.155. Prove that all the real rational numbers are countable.

1.156. Prove that the irrational numbers between 0 and 1 are not countable.

1.157. Represent graphically the set of values of z for which (a) $|z| > |z - 1|$, (b) $|z + 2| > 1 + |z - 2|$.

1.158. Show that (a) $\sqrt[3]{2}+\sqrt{3}$ and (b) $2-\sqrt{2}i$ are algebraic numbers. [See Problem 1.47.]

1.159. Prove that $\sqrt{2}+\sqrt{3}$ is an irrational number.

1.160. Let $ABCD\cdots PQ$ represent a regular polygon of n sides inscribed in a circle of unit radius. Prove that the product of the lengths of the diagonals $AC, AD, \ldots, AP$ is $\frac{1}{4}n\csc^2(\pi/n)$.

1.161. Suppose $\sin\theta \neq 0$. Prove that (a) $\dfrac{\sin n\theta}{\sin\theta} = 2^{n-1}\prod_{k=1}^{n-1}\{\cos\theta - \cos(k\pi/n)\}$

(b) $\dfrac{\sin(2n+1)\theta}{\sin\theta} = (2n+1)\prod_{k=1}^{n}\left\{1-\dfrac{\sin^2\theta}{\sin^2 k\pi/(2n+1)}\right\}.$

1.162. Prove $\cos 2n\theta = (-1)^n\prod_{k=1}^{n}\left\{1-\dfrac{\cos^2\theta}{\cos^2(2k-1)\pi/4n}\right\}.$

1.163. Suppose the product of two complex numbers z_1 and z_2 is real and different from zero. Prove that there exists a real number p such that $z_1 = p\bar{z}_2$.

1.164. Let z be any point on the circle $|z-1|=1$. Prove that $\arg(z-1) = 2\arg z = \frac{2}{3}\arg(z^2-z)$ and give a geometrical interpretation.

1.165. Prove that under suitable restrictions (a) $z^m z^n = z^{m+n}$, (b) $(z^m)^n = z^{mn}$.

1.166. Prove (a) $\text{Re}\{z_1z_2\} = \text{Re}\{z_1\}\text{Re}\{z_2\} - \text{Im}\{z_1\}\text{Im}\{z_2\}$

(b) $\text{Im}\{z_1z_2\} = \text{Re}\{z_1\}\text{Im}\{z_2\} + \text{Im}\{z_1\}\text{Re}\{z_2\}$.

1.167. Find the area of the polygon with vertices at $2+3i$, $3+i$, $-2-4i$, $-4-i$, $-1+2i$.

1.168. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be any complex numbers. Prove *Schwarz's inequality*,

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \left(\sum_{k=1}^{n}|a_k|^2\right)\left(\sum_{k=1}^{n}|b_k|^2\right)$$

ANSWERS TO SUPPLEMENTARY PROBLEMS

1.53. (a) $-4-i$, (b) $-17+14i$, (c) $8+i$, (d) $-9+7i$, (e) $11/17-(10/17)i$, (f) $21+i$, (g) $-15/2+5i$, (h) $-11/2-(23/2)i$, (i) $2+i$

1.54. (a) $-1-4i$, (b) 170, (c) $1024i$, (d) 12, (e) 3/5, (f) $-1/7$, (g) $-7+3\sqrt{3}+\sqrt{3}i$, (h) $765+128\sqrt{3}$, (i) -35

1.57. $x=1$, $y=-2$

1.60. $x^4+y^4+2x^2y^2-6x^2y-6y^3+9x^2+9y^2$

1.61. (a) $6-2i$, (b) $3+3i$, (c) $-1+12i$, (d) $9-8i$, (e) $19/2+(3/2)i$

1.63. (a) $\sqrt{10}$, (b) $5\sqrt{2}$, (c) $5+5i$, (d) 15

1.64. 5, 5, 8

1.70. (a) $z-(2+i) = t(1-3i)$ or $x=2+t$, $y=1-3t$ or $3x+y=7$

(b) $z-(5/2-i/2) = t(3+i)$ or $x=3t+5/2$, $y=t-1/2$ or $3-3y=4$

1.71. (a) circle, (b) ellipse, (c) hyperbola, (d) $z=1$ and $x=-3$, (e) hyperbola

1.72. (a) $|z+3-4i|=2$ or $(x+3)^2+(y-4)^2=4$, (b) $|z+2i|+|z-2i|=10$

1.73. (a) $1<|z+i|\leq 2$, (b) $\text{Re}\{z^2\}>1$, (c) $|z+3i|>4$, (d) $|z+2-3i|+|z-2+3i|<10$

1.81. (a) $2\sqrt{2}$ cis 315° or $2\sqrt{2}e^{7\pi i/4}$, (b) 2 cis 120° or $2e^{2\pi i/3}$, (c) 4 cis 45° or $4e^{\pi i/4}$, (d) cis 270° or $e^{3\pi i/2}$, (e) 4 cis 180° or $4e^{\pi i}$, (f) 4 cis 210° or $4e^{7\pi i/6}$, (g) $\sqrt{2}$ cis 90° or $\sqrt{2}e^{\pi i/2}$, (h) $\sqrt{3}$ cis 300° or $\sqrt{3}e^{5\pi i/3}$

1.83. (a) $5\exp[i(\pi + \tan^{-1}(4/3)$, (b) $\sqrt{5}\exp[-i\tan^{-1}2]$

1.84. (a) $-3\sqrt{2} + 3\sqrt{2}i$, (b) $12i$, (c) $2\sqrt{2} - 2\sqrt{2}i$, (d) $-\sqrt{2} - \sqrt{2}i$, (e) $-5\sqrt{3}/2 - (5/2)i$, (f) $-3\sqrt{3}/2 - (3/2)i$

1.85. 375 miles, 23° north of east (approx.)

1.89. (a) $15/2 + (15\sqrt{3}/2)i$, (b) $32 - 32\sqrt{3}i$, (c) $-16 - 16\sqrt{3}i$, (d) $3\sqrt{3}/2 - (3\sqrt{3}/2)i$, (e) $-\sqrt{3}/2 - (1/2)i$

1.95. (a) 2 cis 165°, 2 cis 345°; (b) $\sqrt{2}$ cis 27°, $\sqrt{2}$ cis 99°, $\sqrt{2}$ cis 171°, $\sqrt{2}$ cis 243°, $\sqrt{2}$ cis 315°; (c) $\sqrt[3]{4}$ cis 20°, $\sqrt[3]{4}$ cis 140°, $\sqrt[3]{4}$ cis 260°; (d) 2 cis 67.5°, 2 cis 157.5°, 2 cis 247.5°, 2 cis 337.5°; (e) 2 cis 0°, 2 cis 60°, 2 cis 120°, 2 cis 180°, 2 cis 240°, 2 cis 300°; (f) cis 60°, cis 180°, cis 300°

1.96. (a) 2 cis 0°, 2 cis 120°, 2 cis 240°; (b) $\sqrt{8}$ cis 22.5°, $\sqrt{8}$ cis 202.5°; (c) 2 cis 48°, 2 cis 120°, 2 cis 192°, 2 cis 264°, 2 cis 336°; (d) $\sqrt{3}$ cis 45°, $\sqrt{3}$ cis 105°, $\sqrt{3}$ cis 165°, $\sqrt{3}$ cis 225°, $\sqrt{3}$ cis 285°, $\sqrt{3}$ cis 345°

1.97. (a) 3 cis 45°, 3 cis 135°, 3 cis 225°, 3 cis 315°

(b) $\sqrt[6]{2}$ cis 40°, $\sqrt[6]{2}$ cis 100°, $\sqrt[6]{2}$ cis 160°, $\sqrt[6]{2}$ cis 220°, $\sqrt[6]{2}$ cis 280°, $\sqrt[6]{2}$ cis 340°

1.98. (a) $3 - 2i$, $-3 + 2i$, (b) $\sqrt{10} + \sqrt{2}i$, $-\sqrt{10} - \sqrt{2}i$

1.99. $1 + 2i$, $\frac{1}{2} - \sqrt{3} + (1 + \frac{1}{2}\sqrt{3})i$, $-\frac{1}{2} - \sqrt{3} + \left(\frac{1}{2}\sqrt{3} - 1\right)i$

1.100. (a) $(-1 \pm 7i)/5$, (b) $1 + i$, $1 - 2i$

1.101. 1, 1, 2, $-1 \pm i$

1.102. $\frac{1}{2}(1 \pm i\sqrt{3})$, $\frac{1}{2}(-1 \pm i\sqrt{3})$

1.104. $2 + 2i$, $2 - 2i$

1.105. (a) $e^{2\pi ik/4} = e^{2\pi ik/2}$, $k = 0, 1, 2, 3$, (b) $e^{2\pi ik/7}$, $k = 0, 1, \ldots, 6$

1.109. $\theta_i(\omega - 1)/(\omega + 1)$, $(\omega^2 - 1)/(\omega^2 + 1)$, $(\omega^3 - 1)/(\omega^3 + 1)$, $(\omega^4 - 1)/(\omega^4 + 1)$, where $\omega = e^{2\pi i/5}$

1.110. (a) 1, (b) 178, (c) 1, (d) 17, (e) 1, (f) 1

1.114. 17

1.115. 18

1.116. (a) $x^2 + y^2 = 16$, (b) $x^2 + y^2 - 4x + 8 = 0$, (c) $x = 2$, (d) $y = -3$

1.117. (a) $(z - 3)(\bar{z} - 3) = 9$, (b) $(2i - 3)z + (2i + 3)\bar{z} = 10i$, (c) $3(z^2 + \bar{z}^2) - 10z\bar{z} + 25 = 0$

1.118. (a) Yes. (b) Every point inside or on the boundary of the square is a limit point. (c) No. (d) All points of the square are boundary points; there are no interior points. (e) No. (f) No. (g) No. (h) The closure of S is the set of all points inside and on the boundary of the square. (i) The complement of S is the set of all points that are not equal to $a + bi$ when a and b [where $0 < a < 1$, $0 < b < 1$] are rational. (j) Yes. (k) No. (l) Yes.

1.119. (a) Yes. (b) Every point inside or on the square is a limit point. (c) No. (d) Every point inside is an interior point, while every point on the boundary is a boundary point. (e) Yes. (f) Yes. (g) Yes. (h) The closure of S is the set of all points inside and on the boundary of the square. (i) The complement of S is the set of all points exterior to the square or on its boundary. (j) No. (k) No. (l) Yes.

1.120. (a) Yes. (b) Every point of S is a limit point. (c) Yes. (d) Every point inside the square is an interior point, while every point on the boundary is a boundary point. (e) No. (f) Yes. (g) No. (h) S itself. (i) All points exterior to the square. (j) No. (k) Yes. (l) Yes.

1.121. (a) $\{2, 1, -i, i, 1 + i\}$, (b) $\{1, i, -i\}$, (c) $\{1, -i\}$

1.131. $e^{-3\sqrt{3}}$

1.139. Yes

1.140. $|z + 1| = \sqrt{5}$ or $(x + 1)^2 + y^2 = 5$

1.151. $(qz_1 + pz_2)/(q + p)$

1.167. 47/2

CHAPTER 2

Functions, Limits, and Continuity

2.1 Variables and Functions

A symbol, such as z, which can stand for any one of a set of complex numbers is called a *complex variable*.

Suppose, to each value that a complex variable z can assume, there corresponds one or more values of a complex variable w. We then say that w is a *function* of z and write $w = f(z)$ or $w = G(z)$, etc. The variable z is sometimes called an *independent variable*, while w is called a *dependent variable*. The *value of a function* at $z = a$ is often written $f(a)$. Thus, if $f(z) = z^2$, then $f(2i) = (2i)^2 = -4$.

2.2 Single and Multiple-Valued Functions

If only one value of w corresponds to each value of z, we say that w is a *single-valued* function of z or that $f(z)$ is single-valued. If more than one value of w corresponds to each value of z, we say that w is a *multiple-valued* or *many-valued* function of z.

A multiple-valued function can be considered as a collection of single-valued functions, each member of which is called a *branch* of the function. It is customary to consider one particular member as a *principal branch* of the multiple-valued function and the value of the function corresponding to this branch as the *principal value*.

EXAMPLE 2.1

(a) If $w = z^2$, then to each value of z there is only one value of w. Hence, $w = f(z) = z^2$ is a single-valued function of z.

(b) If $w^2 = z$, then to each value of z there are two values of w. Hence, $w^2 = z$ defines a multiple-valued (in this case two-valued) function of z.

Whenever we speak of *function*, we shall, unless otherwise stated, assume *single-valued function*.

2.3 Inverse Functions

If $w = f(z)$, then we can also consider z as a function, possibly multiple-valued, of w, written $z = g(w) = f^{-1}(w)$. The function f^{-1} is often called the *inverse* function corresponding to f. Thus, $w = f(z)$ and $w = f^{-1}(z)$ are *inverse functions* of each other.

2.4 Transformations

If $w = u + iv$ (where u and v are real) is a single-valued function of $z = x + iy$ (where x and y are real), we can write $u + iv = f(x + iy)$. By equating real and imaginary parts, this is seen to be equivalent to

$$u = u(x, y), \quad v = v(x, y) \tag{2.1}$$

Thus given a point (x, y) in the z plane, such as P in Fig. 2-1, there corresponds a point (u, v) in the w plane, say P' in Fig. 2-2. The set of equations (2.1) [or the equivalent, $w = f(z)$] is called a *transformation*. We say that point P is *mapped* or *transformed* into point P' by means of the transformation and call P' the *image* of P.

EXAMPLE 2.2 If $w = z^2$, then $u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy$ and the transformation is $u = x^2 - y^2$, $v = 2xy$. The image of a point (1, 2) in the z plane is the point $(-3, 4)$ in the w plane.

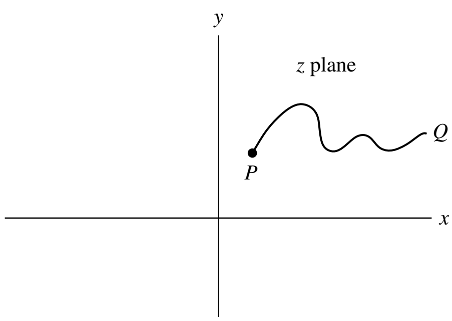

Fig. 2-1

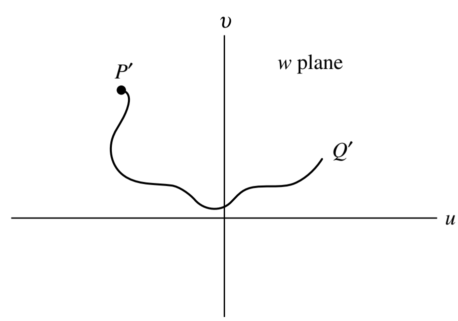

Fig. 2-2

In general, under a transformation, a set of points such as those on curve PQ of Fig. 2-1 is mapped into a corresponding set of points, called the *image*, such as those on curve $P'Q'$ in Fig. 2-2. The particular characteristics of the image depend of course on the type of function $f(z)$, which is sometimes called a *mapping function*. If $f(z)$ is multiple-valued, a point (or curve) in the z plane is mapped in general into more than one point (or curve) in the w plane.

2.5 Curvilinear Coordinates

Given the transformation $w = f(z)$ or, equivalently, $u = u(x, y)$, $v = v(x, y)$, we call (x, y) the rectangular coordinates corresponding to a point P in the z plane and (u, v) the *curvilinear coordinates* of P.

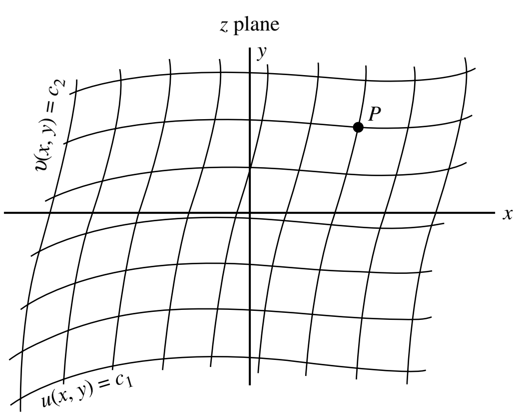

Fig. 2-3

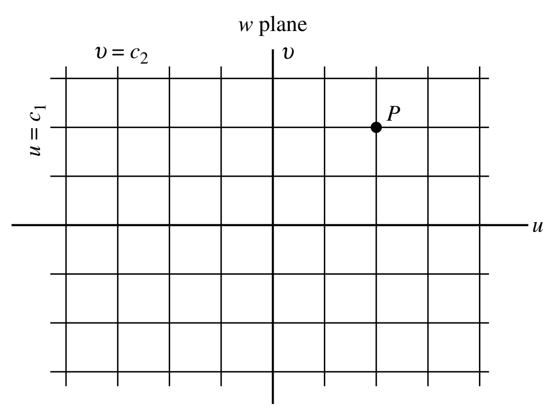

Fig. 2-4

The curves $u(x, y) = c_1$, $v(x, y) = c_2$, where c_1 and c_2 are constants, are called *coordinate curves* [see Fig. 2-3] and each pair of these curves intersects in a point. These curves map into mutually orthogonal lines in the w plane [see Fig. 2-4].

2.6 The Elementary Functions

1. **Polynomial Functions** are defined by

$$w = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = P(z) \tag{2.2}$$

where $a_0 \neq 0, a_1, \ldots, a_n$ are complex constants and n is a positive integer called the *degree* of the polynomial $P(z)$.

The transformation $w = az + b$ is called a *linear transformation*.

2. **Rational Algebraic Functions** are defined by

$$w = \frac{P(z)}{Q(z)} \tag{2.3}$$

where $P(z)$ and $Q(z)$ are polynomials. We sometimes call (2.3) a *rational transformation*. The special case $w = (az + b)/(cz + d)$ where $ad - bc \neq 0$ is often called a *bilinear* or *fractional linear transformation*.

3. **Exponential Functions** are defined by

$$w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) \tag{2.4}$$

where e is the *natural base of logarithms*. If a is real and positive, we define

$$a^z = e^{z \ln a} \tag{2.5}$$

where $\ln a$ is the *natural logarithm of a*. This reduces to (4) if $a = e$.

Complex exponential functions have properties similar to those of real exponential functions. For example, $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$, $e^{z_1}/e^{z_2} = e^{z_1 - z_2}$.

4. **Trigonometric Functions.** We define the trigonometric or circular functions $\sin z, \cos z$, etc., in terms of exponential functions as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}, \qquad \csc z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}, \qquad \cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

Many of the properties familiar in the case of real trigonometric functions also hold for the complex trigonometric functions. For example, we have:

$$\sin^2 z + \cos^2 z = 1, \qquad 1 + \tan^2 z = \sec^2 z, \qquad 1 + \cot^2 z = \csc^2 z$$

$$\sin(-z) = -\sin z, \qquad \cos(-z) = \cos z, \qquad \tan(-z) = -\tan z$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

5. **Hyperbolic Functions** are defined as follows:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$
$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}, \qquad \operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}}$$
$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}, \qquad \coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

The following properties hold:

$$\cosh^2 z - \sinh^2 z = 1, \qquad 1 - \tanh^2 z = \operatorname{sech}^2 z, \qquad \coth^2 z - 1 = \operatorname{csch}^2 z$$
$$\sinh(-z) = -\sinh z, \qquad \cosh(-z) = \cosh z, \qquad \tanh(-z) = -\tanh z$$
$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$
$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$
$$\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$$

The following relations exist between the trigonometric or circular functions and the hyperbolic functions:

$$\sin iz = i \sinh z, \qquad \cos iz = \cosh z, \qquad \tan iz = i \tanh z$$
$$\sinh iz = i \sin z, \qquad \cosh iz = \cos z, \qquad \tanh iz = i \tan z$$

6. **Logarithmic Functions.** If $z = e^w$, then we write $w = \ln z$, called the *natural logarithm* of z. Thus the natural logarithmic function is the inverse of the exponential function and can be defined by

$$w = \ln z = \ln r + i(\theta + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots$$

where $z = re^{i\theta} = re^{i(\theta + 2k\pi)}$. Note that $\ln z$ is a multiple-valued (in this case, infinitely-many-valued) function. The *principal-value* or *principal branch* of $\ln z$ is sometimes defined as $\ln r + i\theta$ where $0 \le \theta < 2\pi$. However, any other interval of length 2π can be used, e.g., $-\pi < \theta \le \pi$, etc.

The logarithmic function can be defined for real bases other than e. Thus, if $z = a^w$, then $w = \log_a z$ where $a > 0$ and $a \ne 0, 1$. In this case, $z = e^{w \ln a}$ and so, $w = (\ln z)/(\ln a)$.

7. **Inverse Trigonometric Functions.** If $z = \sin w$, then $w = \sin^{-1} z$ is called the *inverse sine of* z or *arc sine of* z. Similarly, we define other inverse trigonometric or circular functions $\cos^{-1} z$, $\tan^{-1} z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases, we omit an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm:

$$\sin^{-1} z = \frac{1}{i} \ln\left(iz + \sqrt{1 - z^2}\right), \qquad \csc^{-1} z = \frac{1}{i} \ln\left(\frac{i + \sqrt{z^2 - 1}}{z}\right)$$
$$\cos^{-1} z = \frac{1}{i} \ln\left(z + \sqrt{z^2 - 1}\right), \qquad \sec^{-1} z = \frac{1}{i} \ln\left(\frac{1 + \sqrt{1 - z^2}}{z}\right)$$
$$\tan^{-1} z = \frac{1}{2i} \ln\left(\frac{1 + iz}{1 - iz}\right), \qquad \cot^{-1} z = \frac{1}{2i} \ln\left(\frac{z + i}{z - i}\right)$$

8. **Inverse Hyperbolic Functions.** If $z = \sinh w$, then $w = \sinh^{-1} z$ is called the *inverse hyperbolic sine of* z. Similarly, we define other inverse hyperbolic functions $\cosh^{-1} z$, $\tanh^{-1} z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases, we omit an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm:

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right), \qquad \operatorname{csch}^{-1} z = \ln\left(\frac{1+\sqrt{z^2+1}}{z}\right)$$

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2-1}\right), \qquad \operatorname{sech}^{-1} z = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \qquad \coth^{-1} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$$

9. **The Function** z^α, where α may be complex, is defined as $e^{\alpha \ln z}$. Similarly, if $f(z)$ and $g(z)$ are two given functions of z, we can define $f(z)^{g(z)} = e^{g(z)\ln f(x)}$. In general, such functions are multiple-valued.
10. **Algebraic and Transcendental Functions.** If w is a solution of the polynomial equation

$$P_0(z)w^n + P_1(z)w^{n-1} + \cdots + P_{n-1}(z)w + P_n(z) = 0 \tag{2.6}$$

where $P_0 \neq 0$, $P_1(z), \ldots, P_n(z)$ are polynomials in z and η is a positive integer, then $w = f(z)$ is called an *algebraic function* of z.

EXAMPLE 2.3 $w = z^{1/2}$ is a solution of the equation $w^2 - z = 0$ and so is an algebraic function of z.

Any function that cannot be expressed as a solution of (6) is called a *transcendental function*. The logarithmic, trigonometric, and hyperbolic functions and their corresponding inverses are examples of transcendental functions.

The functions considered in 1–9 above, together with functions derived from them by a finite number of operations involving addition, subtraction, multiplication, division and roots are called *elementary functions*.

2.7 Branch Points and Branch Lines

Suppose that we are given the function $w = z^{1/2}$. Suppose further that we allow z to make a complete circuit (counterclockwise) around the origin starting from point A [Fig. 2-5]. We have $z = re^{i\theta}$, $w = \sqrt{r}e^{i\theta/2}$ so that at A, $\theta = \theta_1$ and $w = \sqrt{r}e^{i\theta_1/2}$. After a complete circuit back to A, $\theta = \theta_1 + 2\pi$ and $w = \sqrt{r}e^{i(\theta_1+2\pi)/2} = -\sqrt{r}e^{i\theta_1/2}$. Thus, we have not achieved the same value of w with which we started. However, by making a second complete circuit back to A, i.e., $\theta = \theta_1 + 4\pi$, $w = \sqrt{r}e^{i(\theta_1+4\pi)/2} = \sqrt{r}e^{i\theta_1/2}$ and we then do obtain the same value of w with which we started.

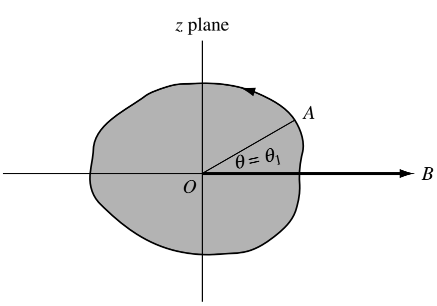

Fig. 2-5

We can describe the above by stating that if $0 \leq \theta < 2\pi$, we are on one branch of the multiple-valued function $z^{1/2}$, while if $2\pi \leq \theta < 4\pi$, we are on the other branch of the function.

It is clear that each branch of the function is single-valued. In order to keep the function single-valued, we set up an artificial barrier such as OB where B is at infinity [although any other line from O can be used], which we agree not to cross. This barrier [drawn heavy in the figure] is called a *branch line* or *branch cut*, and point O is called a *branch point*. It should be noted that a circuit around any point other than $z = 0$ does not lead to different values; thus, $z = 0$ is the only finite branch point.

2.8 Riemann Surfaces

There is another way to achieve the purpose of the branch line described above. To see this, we imagine that the z plane consists of two sheets superimposed on each other. We now cut the sheets along OB and imagine that the lower edge of the bottom sheet is joined to the upper edge of the top sheet. Then, starting in the bottom sheet and making one complete circuit about O, we arrive in the top sheet. We must now imagine the other cut edges joined together so that, by continuing the circuit, we go from the top sheet back to the bottom sheet.

The collection of two sheets is called a *Riemann surface* corresponding to the function $z^{1/2}$. Each sheet corresponds to a branch of the function and on each sheet the function is single-valued.

The concept of Riemann surfaces has the advantage that the various values of multiple-valued functions are obtained in a continuous fashion.

The ideas are easily extended. For example, for the function $z^{1/3}$ the Riemann surface has 3 sheets; for $\ln z$, the Riemann surface has infinitely many sheets.

2.9 Limits

Let $f(z)$ be defined and single-valued in a neighborhood of $z = z_0$ with the possible exception of $z = z_0$ itself (i.e., in a deleted δ neighborhood of z_0). We say that the number l is the *limit* of $f(z)$ as z approaches z_0 and write $\lim_{z\to z_0} f(z) = l$ if for any positive number ϵ (however small), we can find some positive number δ (usually depending on ϵ) such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

In such a case, we also say that $f(z)$ approaches l as z approaches z_0 and write $f(z) \to l$ as $z \to z_0$. The limit must be independent of the manner in which z approaches z_0.

Geometrically, if z_0 is a point in the complex plane, then $\lim_{z\to z_0} f(z) = l$ if the difference in absolute value between $f(z)$ and l can be made as small as we wish by choosing points z sufficiently close to z_0 (excluding $z = z_0$ itself).

EXAMPLE 2.4 Let

$$f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$$

Then, as z gets closer to i (i.e., z approaches i), $f(z)$ gets closer to $i^2 = -1$. We thus *suspect* that $\lim_{z\to i} f(z) = -1$. To *prove* this, we must see whether the above definition of limit is satisfied. For this proof, see Problem 2.23.

Note that $\lim_{z\to i} f(z) \neq f(i)$, i.e., the limit of $f(z)$ as $z \to i$ is not the same as the value of $f(z)$ at $z = i$, since $f(i) = 0$ by definition. The limit would, in fact, be -1 even if $f(z)$ were not defined at $z = i$.

When the limit of a function exists, it is unique, i.e., it is the only one (see Problem 2.26). If $f(z)$ is multiple-valued, the limit as $z \to z_0$ may depend on the particular branch.

2.10 Theorems on Limits

THEOREM 2.1. Suppose $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$. Then

1. $\lim_{z\to z_0}\{f(z) + g(z)\} = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z) = A + B$
2. $\lim_{z\to z_0}\{f(z) - g(z)\} = \lim_{z\to z_0} f(z) - \lim_{z\to z_0} g(z) = A - B$

3. $\lim_{z\to z_0}\{f(z)g(z)\} = \{\lim_{z\to z_0} f(z)\}\{\lim_{z\to z_0} g(z)\} = AB$
4. $\lim\limits_{z\to z_0}\dfrac{f(z)}{g(z)} = \dfrac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)} = \dfrac{A}{B}$ if $B\neq 0$

2.11 Infinity

By means of the transformation $w = 1/z$, the point $z = 0$ (i.e., the origin) is mapped into $w = \infty$, called the *point at infinity in the w plane*. Similarly, we denote by $z = \infty$, *the point at infinity in the z plane*. To consider the behavior of $f(z)$ at $z = \infty$, it suffices to let $z = 1/w$ and examine the behavior of $f(1/w)$ at $w = 0$.

We say that $\lim_{z\to\infty} f(z) = l$ or $f(z)$ approaches l as z approaches infinity, if for any $\epsilon > 0$, we can find $M > 0$ such that $|f(z) - l| < \epsilon$ whenever $|z| > M$.

We say that $\lim_{z\to z_0} f(z) = \infty$ or $f(z)$ approaches infinity as z approaches z_0, if for any $N > 0$, we can find $\delta > 0$ such that $|f(z)| > N$ whenever $0 < |z - z_0| < \delta$.

2.12 Continuity

Let $f(z)$ be defined and single-valued in a neighborhood of $z = z_0$ as well as at $z = z_0$ (i.e., in a δ neighborhood of z_0). The function $f(z)$ is said to be *continuous* at $z = z_0$ if $\lim_{z\to z_0} f(z) = f(z_0)$. Note that this implies three conditions that must be met in order that $f(z)$ be continuous at $z = z_0$:

1. $\lim_{z\to z_0} f(z) = l$ must exist
2. $f(z_0)$ must exist, i.e., $f(z)$ is defined at z_0
3. $l = f(z_0)$

Equivalently, if $f(z)$ is continuous at z_0, we can write this in the suggestive form

$$\lim_{z\to z_0} f(z) = f\left(\lim_{z\to z_0} z\right).$$

EXAMPLE 2.5

(a) Suppose

$$f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$$

Then, $\lim_{z\to i} f(z) = -1$. But $f(i) = 0$. Hence, $\lim_{z\to i} f(z) \neq f(i)$ and the function is not continuous at $z = i$.

(b) Suppose $f(z) = z^2$ for all z. Then $\lim_{z\to i} f(z) = f(i) = -1$ and $f(z)$ is continuous at $z = i$.

Points in the z plane where $f(z)$ fails to be continuous are called *discontinuities* of $f(z)$, and $f(z)$ is said to be *discontinuous* at these points. If $\lim_{z\to z_0} f(z)$ exists but is not equal to $f(z_0)$, we call z_0 a *removable discontinuity* since by redefining $f(z_0)$ to be the same as $\lim_{z\to z_0} f(z)$, the function becomes continuous.

Alternative to the above definition of continuity, we can define $f(z)$ as continuous at $z = z_0$ if for any $\epsilon > 0$, we can find $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. Note that this is simply the definition of limit with $l = f(z_0)$ and removal of the restriction that $z \neq z_0$.

To examine the continuity of $f(z)$ at $z = \infty$, we let $z = 1/w$ and examine the continuity of $f(1/w)$ at $w = 0$.

Continuity in a Region

A function $f(z)$ is said to be *continuous in a region* if it is continuous at all points of the region.

2.13 Theorems on Continuity

THEOREM 2.2. Given $f(z)$ and $g(z)$ are continuous at $z = z_0$. Then so are the functions $f(z) + g(z)$, $f(z) - g(z)$, $f(z)g(z)$ and $f(z)/g(z)$, the last if $g(z_0) \neq 0$. Similar results hold for continuity in a region.

THEOREM 2.3. Among the functions continuous in every finite region are (a) all polynomials, (b) e^z, (c) $\sin z$ and $\cos z$.

THEOREM 2.4. Suppose $w = f(z)$ is continuous at $z = z_0$ and $z = g(\zeta)$ is continuous at $\zeta = \zeta_0$. If $z_0 = g(\zeta_0)$, then the function $w = f[g(\zeta)]$, called a *function of a function* or *composite function*, is continuous at $\zeta = \zeta_0$. This is sometimes briefly stated as: A continuous function of a continuous function is continuous.

THEOREM 2.5. Suppose $f(z)$ is continuous in a closed and bounded region. Then it is bounded in the region; i.e., there exists a constant M such that $|f(z)| < M$ for all points z of the region.

THEOREM 2.6. If $f(z)$ is continuous in a region, then the real and imaginary parts of $f(z)$ are also continuous in the region.

2.14 Uniform Continuity

Let $f(z)$ be continuous in a region. Then, by definition at each point z_0 of the region and for any $\epsilon > 0$, we can find $\delta > 0$ (which will in general depend on both ϵ and the particular point z_0) such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$. If we can find δ depending on ϵ but not on the particular point z_0, we say that $f(z)$ is *uniformly continuous* in the region.

Alternatively, $f(z)$ is uniformly continuous in a region if for any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(z_1) - f(z_2)| < \epsilon$ whenever $|z_1 - z_2| < \delta$ where z_1 and z_2 are any two points of the region.

THEOREM 2.7. Let $f(z)$ be continuous in a *closed* and *bounded* region. Then it is uniformly continuous there.

2.15 Sequences

A function of a positive integral variable, designated by $f(n)$ or u_n, where $n = 1, 2, 3, \ldots$, is called a *sequence*. Thus, a sequence is a set of numbers $u_1, u_2, u_3, \ldots$ in a definite order of arrangement and formed according to a definite rule. Each number in the sequence is called a *term* and u_n is called the nth *term*. The sequence $u_1, u_2, u_3, \ldots$ is also designated briefly by $\{u_n\}$. The sequence is called *finite* or *infinite* according as there are a finite number of terms or not. Unless otherwise specified, we shall only consider infinite sequences.

EXAMPLE 2.6

(a) The set of numbers $i, i^2, i^3, \ldots, i^{100}$ is a finite sequence; the nth term is given by $u_n = i^n$, $n = 1, 2, \ldots, 100$

(b) The set of numbers $1 + i$, $(1 + i)^2/2!$, $(1 + i)^2/3!$, $\ldots$ is an infinite sequence; the nth term is given by $u_n = (1 + i)^n/n!$, $n = 1, 2, 3, \ldots$.

2.16 Limit of a Sequence

A number l is called the *limit* of an infinite sequence $u_1, u_2, u_3, \ldots$ if for any positive number ϵ we can find a positive number N depending on ϵ such that $|u_n - l| < \epsilon$ for all $n > N$. In such case, we write $\lim_{n\to\infty} u_n = l$. If the limit of a sequence exists, the sequence is called *convergent*; otherwise it is called *divergent*. A sequence can converge to only one limit, i.e., if a limit exists it is unique.

A more intuitive but unrigorous way of expressing this concept of limit is to say that a sequence $u_1, u_2, u_3, \ldots$ has a limit l if the successive terms get "closer and closer" to l. This is often used to provide a "guess" as to the value of the limit, after which the definition is applied to see if the guess is really correct.

2.17 Theorems on Limits of Sequences

THEOREM 2.8. Suppose $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$
2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = A - B$
3. $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = AB$
4. $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B} \quad \text{if } B \neq 0$

Further discussion of sequences is given in Chapter 6.

2.18 Infinite Series

Let $u_1, u_2, u_3, \ldots$ be a given sequence.

Form a new sequence $S_1, S_2, S_3, \ldots$ defined by

$$S_1 = u_1, \quad S_2 = u_1 + u_2, \quad S_3 = u_1 + u_2 + u_3, \ldots, \quad S_n = u_1 + u_2 + \cdots + u_n$$

where S_n, called the nth *partial sum*, is the sum of the first n terms of the sequence $\{u_n\}$.

The sequence $S_1, S_2, S_3, \ldots$ is symbolized by

$$u_1 + u_2 + u_3 + \cdots = \sum_{n=1}^{\infty} u_n$$

which is called an infinite series. If $\lim_{n\to\infty} S_n = S$ exists, the series is called *convergent* and S is its *sum*; otherwise the series is called *divergent*. A necessary condition that a series converges is $\lim_{n\to\infty} u_n = 0$; however, this is not sufficient (see Problems 2.40 and 2.150).

Further discussion of infinite series is given in Chapter 6.

SOLVED PROBLEMS

Functions and Transformations

2.1. Let $w = f(z) = z^2$. Find the values of w that correspond to (a) $z = -2 + i$ and (b) $z = 1 - 3i$, and show how the correspondence can be represented graphically.

Solution

(a) $w = f(-2 + i) = (-2 + i)^2 = 4 - 4i + i^2 = 3 - 4i$
(b) $w = f(1 - 3i) = (1 - 3i)^2 = 1 - 6i + 9i^2 = -8 - 6i$

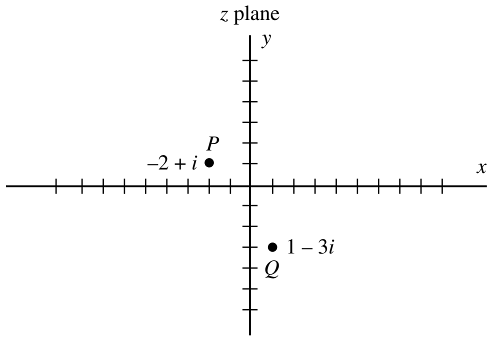

Fig. 2-6

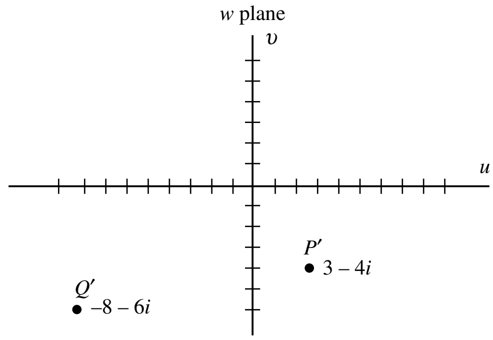

Fig. 2-7

The point $z = -2 + i$, represented by point P in the z plane of Fig. 2-6, has the *image point* $w = 3 - 4i$ represented by P' in the w plane of Fig. 2-7. We say that P is *mapped* into P' by means of the *mapping function* or *transformation* $w = z^2$. Similarly, $z = 1 - 3i$ [point Q of Fig. 2-6] is mapped into $w = -8 - 6i$ [point Q' of Fig. 2-7]. To each point in the z plane, there corresponds one and only one point (image) in the w plane, so that w is a single-valued function of z.

2.2. Show that the line joining the points P and Q in the z plane of Problem 2.1 [Fig. 2-6] is mapped by $w = z^2$ into curve joining points $P'Q'$ [Fig. 2-7] and determine the equation of this curve.

Solution

Points P and Q have coordinates $(-2, 1)$ and $(1, -3)$. Then, the parametric equations of the line joining these points are given by

$$\frac{x - (-2)}{1 - (-2)} = \frac{y - 1}{-3 - 1} = t \qquad \text{or} \qquad x = 3t - 2,\ y = 1 - 4t$$

The equation of the line PQ can be represented by $z = 3t - 2 + i(1 - 4t)$. The curve in the w plane into which this line is mapped has the equation

$$\begin{aligned} w = z^2 &= \{3t - 2 + i(1 - 4t)\}^2 = (3t - 2)^2 - (1 - 4t)^2 + 2(3t - 2)(1 - 4t)i \\ &= 3 - 4t - 7t^2 + (-4 + 22t - 24t^2)i \end{aligned}$$

Then, since $w = u + iv$, the parametric equations of the image curve are given by

$$u = 3 - 4t - 7t^2, \quad v = -4 + 22t - 24t^2$$

By assigning various values to the parameter t, this curve may be graphed.

2.3. A point P moves in a counterclockwise direction around a circle in the z plane having center at the origin and radius 1. If the mapping function is $w = z^3$, show that when P makes one complete revolution, the image P' of P in the w plane makes three complete revolutions in a counterclockwise direction on a circle having center at the origin and radius 1.

Solution

Let $z = re^{i\theta}$. Then, on the circle $|z| = 1$ [Fig. 2-8], $r = 1$ and $z = e^{i\theta}$. Hence, $w = z^3 = (e^{i\theta})^3 = e^{3i\theta}$. Letting (ρ, ϕ) denote polar coordinates in the w plane, we have $w = \rho e^{i\phi} = e^{3i\theta}$ so that $\rho = 1$, $\phi = 3\theta$.

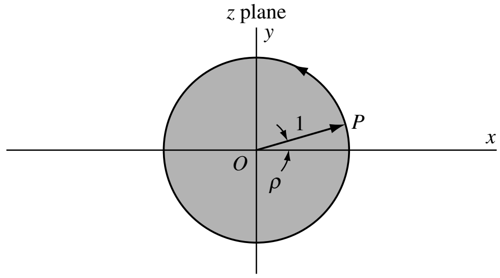

Fig. 2-8

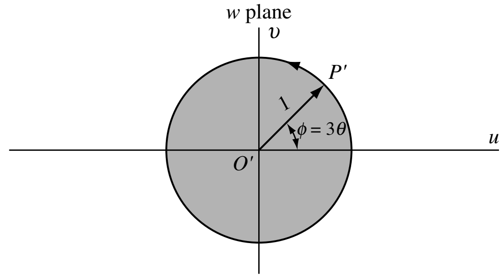

Fig. 2-9

Since $\rho = 1$, it follows that the image point P' moves on a circle in the w plane of radius 1 and center at the origin [Fig. 2-9]. Also, when P moves counterclockwise through an angle θ, P' moves counterclockwise through an angle 3θ. Thus, when P makes one complete revolution, P' makes three complete revolutions. In terms of vectors, it means that vector $O'P'$ is rotating three times as fast as vector OP.

2.4. Suppose c_1 and c_2 are any real constants. Determine the set of all points in the z plane that map into the lines (a) $u = c_1$, (b) $v = c_2$ in the w plane by means of the mapping function $w = z^2$. Illustrate by considering the cases $c_1 = 2, 4, -2, -4$ and $c_2 = 2, 4, -2, -4$.

Solution

We have $w = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$ so that $u = x^2 - y^2$, $v = 2xy$. Then lines $u = c_1$ and $v = c_2$ in the w plane correspond, respectively, to hyperbolas $x^2 - y^2 = c_1$ and $2xy = c_2$ in the z plane as indicated in Figs. 2-10 and 2-11.

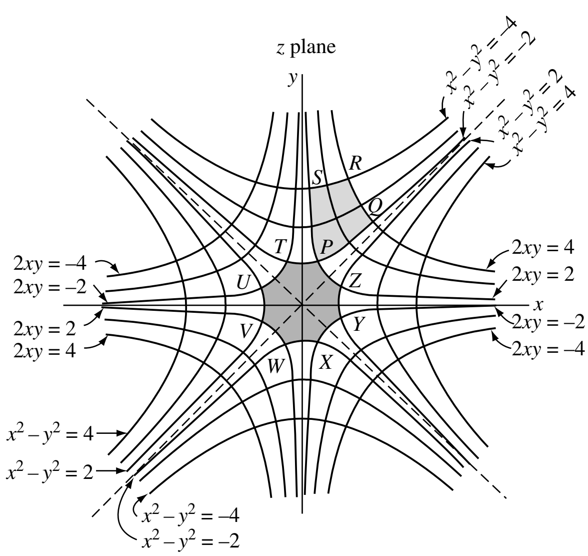

Fig. 2-10

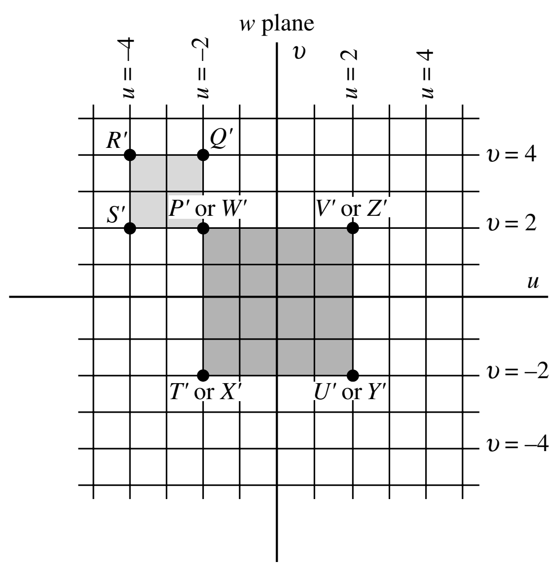

Fig. 2-11

2.5. Referring to Problem 2.4, determine: (a) the image of the region in the first quadrant bounded by $x^2 - y^2 = -2$, $xy = 1$, $x^2 - y^2 = -4$, and $xy = 2$; (b) the image of the region in the z plane bounded by all the branches of $x^2 - y^2 = 2$, $xy = 1$, $x^2 - y^2 = -2$, and $xy = -1$; (c) the curvilinear coordinates of that point in the xy plane whose rectangular coordinates are $(2, -1)$.

Solution

(a) The region in the z plane is indicated by the shaded portion $PQRS$ of Fig. 2-10. This region maps into the required image region $P'Q'R'S'$ shown shaded in Fig. 2-11. It should be noted that curve $PQRSP$ is traversed in a counterclockwise direction and the image curve $P'Q'R'S'P'$ is also traversed in a counterclockwise direction.

(b) The region in the z plane is indicated by the shaded portion $PTUVWXYZ$ of Fig. 2-10. This region maps into the required image region $P'T'U'V'$ shown shaded in Fig. 2-11.

It is of interest to note that when the boundary of the region $PTUVWXYZ$ is traversed only once, the boundary of the image region $P'T'U'V'$ is traversed twice. This is due to the fact that the eight points P and W, T and X, U and Y, V and Z of the z plane map into the four points P' or W', T' or X', U' or Y', V' or Z', respectively.

However, when the boundary of region $PQRS$ is traversed only once, the boundary of the image region is also traversed only once. The difference is due to the fact that in traversing the curve $PTUVWXYZP$, we are encircling the origin $z = 0$, whereas when we are traversing the curve $PQRSP$, we are not encircling the origin.

(c) $u = x^2 - y^2 = (2)^2 - (-1)^2 = 3$, $v = 2xy = 2(2)(-1) = -4$. Then the curvilinear coordinates are $u = 3$, $v = -4$.

Multiple-Valued Functions

2.6. Let $w^5 = z$ and suppose that corresponding to the particular value $z = z_1$, we have $w = w_1$. (a) If we start at the point z_1 in the z plane [see Fig. 2-12] and make one complete circuit counterclockwise around the origin, show that the value of w on returning to z_1 is $w_1 e^{2\pi i/5}$. (b) What are the values of w on returning to z_1, after 2, 3, ... complete circuits around the origin? (c) Discuss parts (a) and (b) if the paths do not enclose the origin.

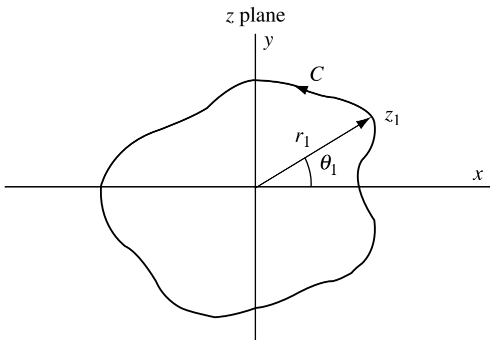

Fig. 2-12

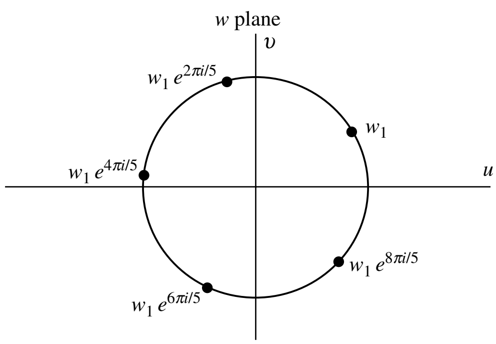

Fig. 2-13

Solution

(a) We have $z = re^{i\theta}$, so that $w = z^{1/5} = r^{1/5} e^{i\theta/5}$. If $r = r_1$ and $\theta = \theta_1$, then $w_1 = r_1^{1/5} e^{i\theta_1/5}$.

As θ increases from θ_1 to $\theta_1 + 2\pi$, which is what happens when one complete circuit counterclockwise around the origin is made, we find

$$w = r_1^{1/5} e^{i(\theta_1 + 2\pi)/5} = r_1^{1/5} e^{i\theta_1/5} e^{2\pi i/5} = w_1 e^{2\pi i/5}$$

(b) After two complete circuits around the origin, we find

$$w = r_1^{1/5} e^{i(\theta_1 + 4\pi)/5} = r_1^{1/5} e^{i\theta_1/5} e^{4\pi i/5} = w_1 e^{4\pi i/5}$$

Similarly, after three and four complete circuits around the origin, we find

$$w = w_1 e^{6\pi i/5} \quad \text{and} \quad w = w_1 e^{8\pi i/5}$$

After five complete circuits, the value of w is $w_1 e^{10\pi i/5} = w_1$, so that the original value of w is obtained after five revolutions about the origin. Thereafter, the cycle is repeated [see Fig. 2-13].

Another Method. Since $w^5 = z$, we have $\arg z = 5 \arg w$ from which

$$\text{Change in } \arg w = \tfrac{1}{5}(\text{Change in } \arg z)$$

Then, if $\arg z$ increases by $2\pi, 4\pi, 6\pi, 8\pi, 10\pi, \ldots,$ $\arg w$ increases by $2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5, 2\pi, \ldots$ leading to the same results obtained in (a) and (b).

(c) If the path does not enclose the origin, then the increase in $\arg z$ is zero and so the increase in $\arg w$ is also zero. In this case, the value of w is w_1, regardless of the number of circuits made.

2.7. (a) In the preceding problem, explain why we can consider w as a collection of five single-valued functions of z.
(b) Explain geometrically the relationship between these single-valued functions.
(c) Show geometrically how we can restrict ourselves to a particular single-valued function.

Solution

(a) Since $w^5 = z = re^{i\theta} = re^{i(\theta+2k\pi)}$ where k is an integer, we have

$$w = r^{1/5} e^{i(\theta+2k\pi)/5} = r^{1/5}\{\cos(\theta + 2k\pi)/5 + i \sin(\theta + 2k\pi)/5\}$$

and so w is a five-valued function of z, the five values being given by $k = 0, 1, 2, 3, 4$.

Equivalently, we can consider w as a collection of five single-valued functions, called *branches* of the multiple-valued function, by properly restricting θ. Thus, for example, we can write

$$w = r^{1/5}(\cos\theta/5 + i \sin\theta/5)$$

where we take the five possible intervals for θ given by $0 \le \theta < 2\pi$, $2\pi \le \theta < 4\pi, \ldots, 8\pi \le \theta < 10\pi$, all other such intervals producing repetitions of these.

The first interval, $0 \le \theta < 2\pi$, is sometimes called the *principal range* of θ and corresponds to the *principal branch* of the multiple-valued function.

Other intervals for θ of length 2π can also be taken; for example, $-\pi \le \theta < \pi$, $\pi \le \theta < 3\pi$, etc., the first of these being taken as the principal range.

(b) We start with the (principal) branch

$$w = r^{1/5}(\cos\theta/5 + i \sin\theta/5)$$

where $0 \le \theta < 2\pi$.

After one complete circuit about the origin in the z plane, θ increases by 2π to give another branch of the function. After another complete circuit about the origin, still another branch of the function is obtained until all five branches have been found, after which we return to the original (principal) branch.

Because different values of $f(z)$ are obtained by successively encircling $z = 0$, we call $z = 0$ a *branch point*.

(c) We can restrict ourselves to a particular single-valued function, usually the principal branch, by insuring that not more than one complete circuit about the branch point is made, i.e., by suitably restricting θ.

In the case of the principal range $0 \le \theta < 2\pi$, this is accomplished by constructing a cut, indicated by *OA* in Fig. 2-14, called a *branch out* or *branch line*, on the positive real axis, the purpose being that we do not allow ourselves to cross this cut (if we do cross the cut, another branch of the function is obtained).

If another interval for θ is chosen, the branch line or cut is taken to be some other line in the z plane emanating from the branch point.

For some purposes, as we shall see later, it is useful to consider the curve of Fig. 2-15 of which Fig. 2-14 is a limiting case.

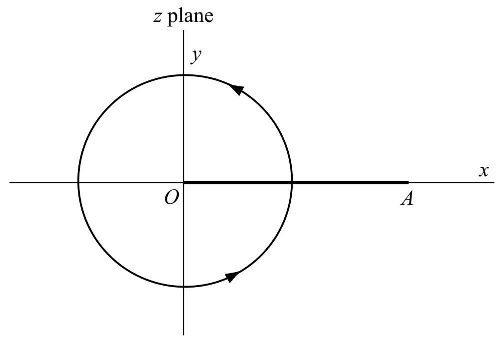

Fig. 2-14

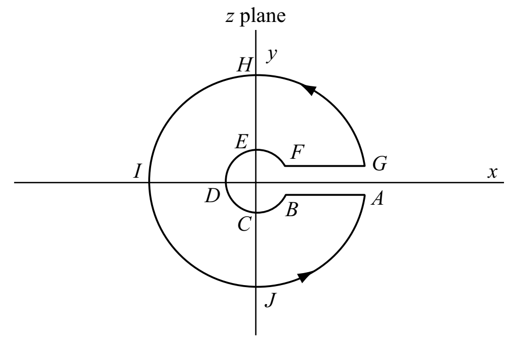

Fig. 2-15

The Elementary Functions

2.8. Prove that (a) $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$, (b) $|e^z| = e^x$, (c) $e^{z+2k\pi i} = e^z$, $k = 0, \pm 1, \pm 2, \ldots$.

Solution

(a) By definition $e^z = e^x(\cos y + i \sin y)$ where $z = x + iy$. Then, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$\begin{aligned} e^{z_1} \cdot e^{z_2} &= e^{x_1}(\cos y_1 + i \sin y_1) \cdot e^{x_2}(\cos y_2 + i \sin y_2) \\ &= e^{x_1} \cdot e^{x_2}(\cos y_1 + i \sin y_1)(\cos y_2 + i \sin y_2) \\ &= e^{x_1+x_2}\{\cos(y_1 + y_2) + i \sin(y_1 + y_2)\} = e^{z_1+z_2} \end{aligned}$$

(b) $|e^z| = |e^x(\cos y + i \sin y)| = |e^x||\cos y + i \sin y| = e^x \cdot 1 = e^x$

(c) By part (a),

$$e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z(\cos 2k\pi + i \sin 2k\pi) = e^z$$

This shows that the function e^z has *period* $2k\pi i$. In particular, it has period $2\pi i$.

2.9. Prove:

(a) $\sin^2 z + \cos^2 z = 1$ (c) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

(b) $e^{iz} = \cos z + i \sin z$, $e^{-iz} = \cos z - i \sin z$ (d) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

Solution

By definition, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$. Then

(a)
$$\begin{aligned} \sin^2 z + \cos^2 z &= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 \\ &= -\left(\frac{e^{2iz} - 2 + e^{-2iz}}{4}\right) + \left(\frac{e^{2iz} + 2 + e^{-2iz}}{4}\right) = 1 \end{aligned}$$

(b)
$$e^{iz} - e^{-iz} = 2i \sin z \qquad (1)$$
$$e^{iz} + e^{-iz} = 2 \cos z \qquad (2)$$

Adding (1) and (2):

$$2e^{iz} = 2\cos z + 2i\sin z \quad \text{and} \quad e^{iz} = \cos z + i\sin z$$

Subtracting (1) from (2):

$$2e^{-iz} = 2\cos z - 2i\sin z \quad \text{and} \quad e^{-iz} = \cos z - i\sin z$$

(c) $$\sin(z_1+z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \frac{e^{iz_1}\cdot e^{iz_2} - e^{-iz_1}\cdot e^{-iz_2}}{2i}$$
$$= \frac{(\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2) - (\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2)}{2i}$$
$$= \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

(d) $$\cos(z_1+z_2) = \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2} = \frac{e^{iz_1}\cdot e^{iz_2} + e^{-iz_1}\cdot e^{-iz_2}}{2}$$
$$= \frac{(\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2) + (\cos z_1 - i\sin z_1)(\cos z_2 - i\sin z_2)}{2}$$
$$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

2.10. Prove that the zeros of (a) $\sin z$ and (b) $\cos z$ are all real and find them.

Solution

(a) If $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = 0$, then $e^{iz} = e^{-iz}$ or $e^{2iz} = 1 = e^{2k\pi i}$, $k = 0, \pm1, \pm2, \ldots$.
Hence, $2iz = 2k\pi i$ and $z = k\pi$, i.e., $z = 0, \pm\pi, \pm2\pi, \pm3\pi, \ldots$ are the zeros.

(b) If $\cos z = \dfrac{e^{iz} + e^{-iz}}{2} = 0$, then $e^{iz} = -e^{-iz}$ or $e^{2iz} = -1 = e^{(2k+1)\pi i}$, $k = 0, \pm1, \pm2, \ldots$.
Hence, $2iz = (2k+1)\pi i$ and $z = (k + \frac{1}{2})\pi$, i.e., $z = \pm\pi/2, \pm3\pi/2, \pm5\pi/2, \ldots$ are the zeros.

2.11. Prove that (a) $\sin(-z) = -\sin z$, (b) $\cos(-z) = \cos z$, (c) $\tan(-z) = -\tan z$.

Solution

(a) $$\sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i} = \frac{e^{-iz} - e^{iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\sin z$$

(b) $$\cos(-z) = \frac{e^{i(-z)} + e^{-i(-z)}}{2} = \frac{e^{-iz} + e^{iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

(c) $\tan(-z) = \dfrac{\sin(-z)}{\cos(-z)} = \dfrac{-\sin z}{\cos z} = -\tan z$, using (a) and (b).

Functions of z having the property that $f(-z) = -f(z)$ are called *odd functions*, while those for which $f(-z) = f(z)$ are called *even functions*. Thus $\sin z$ and $\tan z$ are odd functions, while $\cos z$ is an even function.

2.12. Prove: (a) $1 - \tanh^2 z = \operatorname{sech}^2 z$
(b) $\sin iz = i\sinh z$
(c) $\cos iz = \cosh z$
(d) $\sin(x+iy) = \sin x\cosh y + i\cos x \sinh y$

Solution

(a) By definition, $\cosh z = \dfrac{e^z + e^{-z}}{2}$, $\sinh z = \dfrac{e^z - e^{-z}}{2}$. Then

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = 1$$

Dividing by $\cosh^2 z$, $\dfrac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \dfrac{1}{\cosh^2 z}$ or $1 - \tanh^2 z = \operatorname{sech}^2 z$

(b) $\sin iz = \dfrac{e^{i(iz)} - e^{-i(iz)}}{2i} = \dfrac{e^{-z} - e^{z}}{2i} = i\left(\dfrac{e^{z} - e^{-z}}{2}\right) = i \sinh z$

(c) $\cos iz = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2} = \dfrac{e^{-z} + e^{z}}{2} = \dfrac{e^{z} + e^{-z}}{2} = \cosh z$

(d) From Problem 2.9(c) and parts (b) and (c), we have

$$\sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$

2.13. (a) Suppose $z = e^w$ where $z = r(\cos\theta + i\sin\theta)$ and $w = u + iv$. Show that $u = \ln r$ and $v = \theta + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$ so that $w = \ln z = \ln r + i(\theta + 2k\pi)$. (b) Determine the values of $\ln(1 - i)$. What is the principal value?

Solution

(a) Since $z = r(\cos\theta + i\sin\theta) = e^w = e^{u+iv} = e^u(\cos v + i \sin v)$, we have on equating real and imaginary parts,

$$e^u \cos v = r\cos\theta \tag{1}$$
$$e^u \sin v = r\sin\theta \tag{2}$$

Squaring (1) and (2) and adding, we find $e^{2u} = r^2$ or $e^u = r$ and $u = \ln r$. Then, from (1) and (2), $r\cos v = r\cos\theta$, $r\sin v = r\sin\theta$ from which $v = \theta + 2k\pi$. Hence, $w = u + iv = \ln r + i(\theta + 2k\pi)$.

If $z = e^w$, we say that $w = \ln z$. We thus see that $\ln z = \ln r + i(\theta + 2k\pi)$. An equivalent way of saying the same thing is to write $\ln z = \ln r + i\theta$ where θ can assume infinitely many values which differ by 2π.

Note that *formally* $\ln z = \ln(re^{i\theta}) = \ln r + i\theta$ using laws of real logarithms familiar from elementary mathematics.

(b) Since $1 - i = \sqrt{2}e^{7\pi i/4 + 2k\pi i}$, we have $\ln(1 - i) = \ln\sqrt{2} + \left(\dfrac{7\pi i}{4} + 2k\pi i\right) = \dfrac{1}{2}\ln 2 + \dfrac{7\pi i}{4} + 2k\pi i$.

The principal value is $\dfrac{1}{2}\ln 2 + \dfrac{7\pi i}{4}$ obtained by letting $k = 0$.

2.14. Prove that $f(z) = \ln z$ has a branch point at $z = 0$.

Solution

We have $\ln z = \ln r + i\theta$. Suppose that we start at some point $z_1 \neq 0$ in the complex plane for which $r = r_1$, $\theta = \theta_1$ so that $\ln z_1 = \ln r_1 + i\theta_1$ [see Fig. 2-16]. Then, after making one complete circuit about the origin in the positive or counterclockwise direction, we find on returning to z_1 that $r = r_1$, $\theta = \theta_1 + 2\pi$ so that $\ln z_1 = \ln r_1 + i(\theta_1 + 2\pi)$. Thus, we are on another branch of the function, and so $z = 0$ is a branch point.

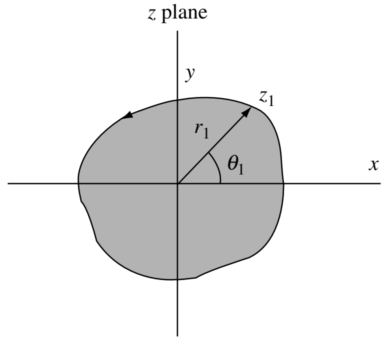

Fig. 2-16

Further complete circuits about the origin lead to other branches and (unlike the case of functions such as $z^{1/2}$ or $z^{1/5}$), we *never* return to the same branch.

It follows that $\ln z$ is an infinitely many-valued function of z with infinitely many branches. That particular branch of $\ln z$ which is real when z is real and positive is called the *principal branch.* To obtain this branch, we require that $\theta = 0$ when $z > 0$. To accomplish this, we can take $\ln z = \ln r + i\theta$ where θ is chosen so that $0 \leq \theta < 2\pi$ or $-\pi \leq \theta < \pi$, etc.

As a generalization, we note that $\ln(z - a)$ has a branch point at $z = a$.

2.15. Consider the transformation $w = \ln z$. Show that (a) circles with center at the origin in the z plane are mapped into lines parallel to the v axis in the w plane, (b) lines or *rays* emanating from the origin in the z plane are mapped into lines parallel to the u axis in the w plane, (c) the z plane is mapped into a strip of width 2π in the w plane. Illustrate the results graphically.

Solution

We have $w = u + iv = \ln z = \ln r + i\theta$ so that $u = \ln r$, $v = \theta$.
Choose the principal branch as $w = \ln r + i\theta$ where $0 \leq \theta < 2\pi$.

(a) Circles with center at the origin and radius α have the equation $|z| = r = \alpha$. These are mapped into lines in the w plane whose equations are $u = \ln a$. In Figs. 2-17 and 2-18, the circles and lines corresponding to $\alpha = 1/2,\ 1,\ 3/2,\ 2$ are indicated.

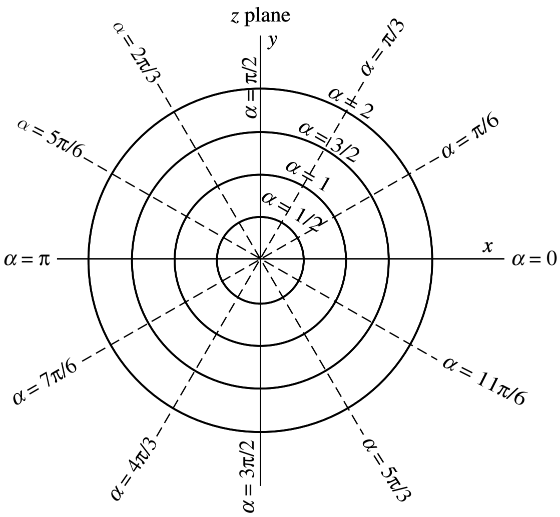

Fig. 2-17

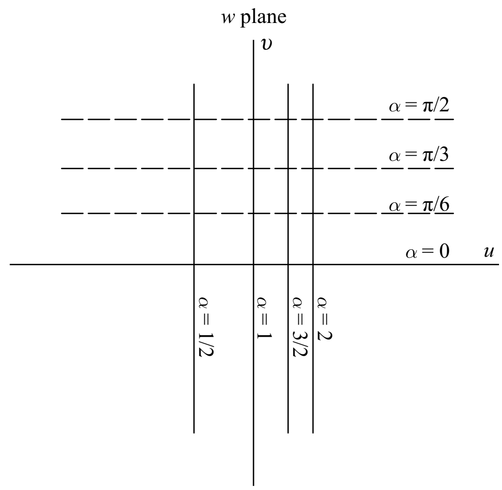

Fig. 2-18

(b) Lines or rays emanating from the origin in the z plane (dashed in Fig. 2-17) have the equation $\theta = \alpha$. These are mapped into lines in the w plane (dashed in Fig. 2-18) whose equations are $v = \alpha$. We have shown the corresponding lines for $\alpha = 0,\ \pi/6,\ \pi/3$, and $\pi/2$.

(c) Corresponding to any given point P in the z plane defined by $z \neq 0$ and having polar coordinates (r, θ) where $0 \leq \theta < 2\pi$, $r > 0$ [as in Fig. 2-19], there is a point P' in the strip of width 2π shown shaded in Fig. 2-20. Thus, the z plane is mapped into this strip. The point $z = 0$ is mapped into a point of this strip sometimes called the *point at infinity.*

If θ is such that $2\pi \leq \theta < 4\pi$, the z plane is mapped into the strip $2\pi \leq v < 4\pi$ of Fig. 2-20. Similarly, we obtain the other strips shown in Fig. 2-20.

It follows that given any point $z \neq 0$ in the z plane, there are infinitely many image points in the w plane corresponding to it.

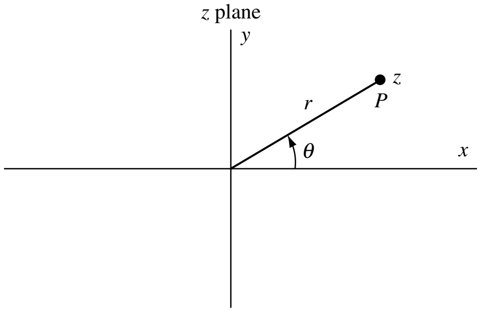

Fig. 2-19

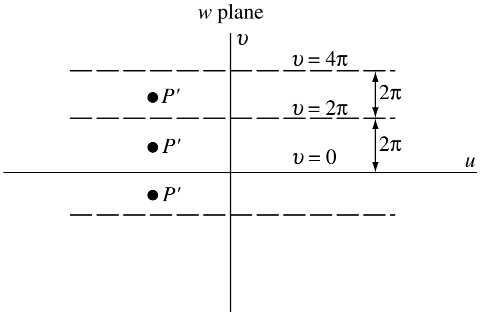

Fig. 2-20

It should be noted that if we had taken θ such that $-\pi \leq \theta < \pi$, $\pi \leq \theta < 3\pi$, etc., the strips of Fig. 2-20 would be shifted vertically a distance π.

2.16. Suppose we choose the principal branch of $\sin^{-1} z$ to be that one for which $\sin^{-1} 0 = 0$. Prove that

$$\sin^{-1} z = \frac{1}{i} \ln\left(iz + \sqrt{1 - z^2}\right)$$

Solution

If $w = \sin^{-1} z$, then $z = \sin w = \dfrac{e^{iw} - e^{-iw}}{2i}$ from which

$$e^{iw} - 2iz - e^{-iw} = 0 \quad \text{or} \quad e^{2iw} - 2ize^{iw} - 1 = 0$$

Solving,

$$e^{iw} = \frac{2iz \pm \sqrt{4 - 4z^2}}{2} = iz \pm \sqrt{1 - z^2} = iz + \sqrt{1 - z^2}$$

since $\pm\sqrt{1 - z^2}$ is implied by $\sqrt{1 - z^2}$. Now, $e^{iw} = e^{i(w - 2k\pi)}$, $k = 0, \pm 1, \pm 2, \ldots$ so that

$$e^{i(w-2k\pi)} = iz + \sqrt{1 - z^2} \quad \text{or} \quad w = 2k\pi + \frac{1}{i} \ln\left(iz + \sqrt{1 - z^2}\right)$$

The branch for which $w = 0$ when $z = 0$ is obtained by taking $k = 0$ from which we find, as required,

$$w = \sin^{-1} z = \frac{1}{i} \ln\left(iz + \sqrt{1 - z^2}\right)$$

2.17. Suppose we choose the principal branch of $\tanh^{-1} z$ to be that one for which $\tanh^{-1} 0 = 0$. Prove that

$$\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1 + z}{1 - z}\right)$$

Solution

If $w = \tanh^{-1} z$, then $z = \tanh w = \dfrac{\sinh w}{\cosh w} = \dfrac{e^w - e^{-w}}{e^w + e^{-w}}$ from which

$$(1 - z)e^w = (1 + z)e^{-w} \quad \text{or} \quad e^{2w} = (1 + z)/(1 - z)$$

Since $e^{2w} = e^{2(w-k\pi i)}$, we have

$$e^{2(w-k\pi i)} = \frac{1+z}{1-z} \qquad \text{or} \qquad w = k\pi i + \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

The principal branch is the one for which $k = 0$ and leads to the required result.

2.18. (a) Suppose $z = re^{i\theta}$. Prove that $z^i = e^{-(\theta+2k\pi)}\{\cos(\ln r) + i\sin(\ln r)\}$ where $k = 0, \pm 1, \pm 2, \ldots$.
(b) Suppose z is a point on the unit circle with center at the origin. Prove that z^i represents infinitely many real numbers and determine the principal value.
(c) Find the principal value of i^i.

Solution

(a) By definition,

$$z^i = e^{i\ln z} = e^{i\{\ln r + i(\theta+2k\pi)\}}$$
$$= e^{i\ln r - (\theta+2k\pi)} = e^{-(\theta+2k\pi)}\{\cos(\ln r) + i\sin(\ln r)\}$$

The principal branch of the many-valued function $f(z) = z^i$ is obtained by taking $k = 0$ and is given by $e^{-\theta}\{\cos(\ln r) + i\sin(\ln r)\}$ where we can choose θ such that $0 \le \theta < 2\pi$.

(b) If z is any point on the unit circle with center at the origin, then $|z| = r = 1$. Hence, by part (a), since $\ln r = 0$, we have $z^i = e^{-(\theta+2k\pi)}$ which represents infinitely many real numbers. The principal value is $e^{-\theta}$ where we choose θ such that $0 \le \theta < 2\pi$.

(c) By definition, $i^i = e^{i\ln i} = e^{i\{i(\pi/2+2k\pi)\}} = e^{-(\pi/2+2k\pi)}$ since $i = e^{i(\pi/2+2k\pi)}$ and $\ln i = i(\pi/2 + 2k\pi)$.
The principal value is given by $e^{-\pi/2}$.
Another Method. By part (b), since $z = i$ lies on the unit circle with center at the origin and since $\theta = \pi/2$, the principal value is $e^{-\pi/2}$.

Branch Points, Branch Lines, Riemann Surfaces

2.19. Let $w = f(z) = (z^2+1)^{1/2}$. (a) Show that $z = \pm i$ are branch points of $f(z)$. (b) Show that a complete circuit around both branch points produces no change in the branches of $f(z)$.

Solution

(a) We have $w = (z^2+1)^{1/2} = \{(z-i)(z+i)\}^{1/2}$. Then, $\arg w = \frac{1}{2}\arg(z-i) + \frac{1}{2}\arg(z+i)$ so that

$$\text{Change in } \arg w = \tfrac{1}{2}\{\text{Change in } \arg(z-i)\} + \tfrac{1}{2}\{\text{Change in } \arg(z+i)\}$$

Let C [Fig. 2-21] be a closed curve enclosing the point i but not the point $-i$. Then, as point z goes once counterclockwise around C,

$$\text{Change in } \arg(z-i) = 2\pi, \quad \text{Change in } \arg(z+i) = 0$$

so that

$$\text{Change in } \arg w = \pi$$

Hence, w does not return to its original value, i.e., a change in branches has occurred. Since a complete circuit about $z = i$ alters the branches of the function, $z = i$ is a branch point. Similarly, if C is a closed curve enclosing the point $-i$ but not i, we can show that $z = -i$ is a branch point.

Another Method.
Let $z - i = r_1 e^{i\theta_1}$, $z + i = r_2 e^{i\theta_2}$. Then

$$w = \{r_1 r_2 e^{i(\theta_1+\theta_2)}\}^{1/2} = \sqrt{r_1 r_2}\, e^{i\theta_1/2} e^{i\theta_2/2}$$

Suppose we start with a particular value of z corresponding to $\theta_1 = \alpha_1$ and $\theta_2 = \alpha_2$. Then $w = \sqrt{r_1 r_2}\, e^{i\alpha_1/2} e^{i\alpha_2/2}$. As z goes once counterclockwise around i, θ_1 increases to $\alpha_1 + 2\pi$ while θ_2

remains the same, i.e., $\theta_2 = \alpha_2$. Hence

$$\begin{aligned} w &= \sqrt{r_1 r_2} e^{i(\alpha_1 + 2\pi)/2} e^{i\alpha_2/2} \\ &= -\sqrt{r_1 r_2} e^{i\alpha_1/2} e^{i\alpha_2/2} \end{aligned}$$

showing that we do not obtain the original value of w, i.e., a change of branches has occurred, showing that $z = i$ is a branch point.

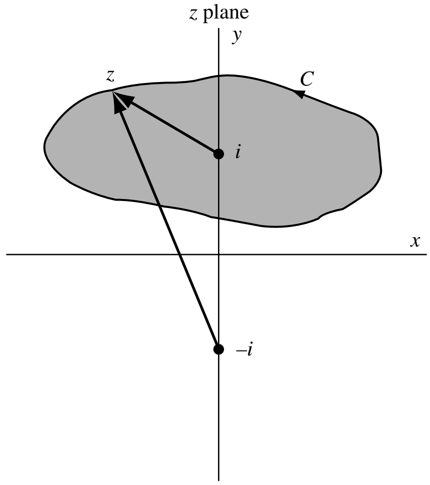

Fig. 2-21

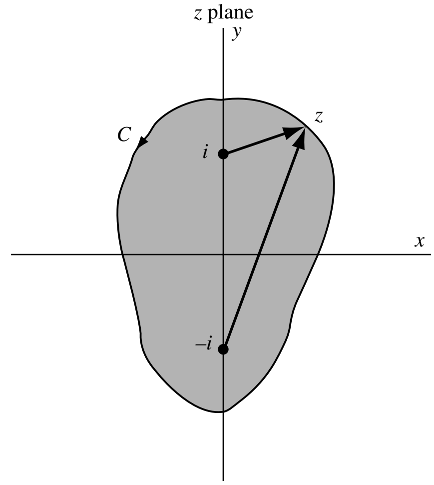

Fig. 2-22

(b) If C encloses both branch points $z = \pm i$ as in Fig. 2-22, then as point z goes counterclockwise around C,

$$\text{Change in } \arg(z - i) = 2\pi$$
$$\text{Change in } \arg(z + i) = 2\pi$$

so that

$$\text{Change in } \arg w = 2\pi$$

Hence a complete circuit around both branch points produces no change in the branches.

Another Method.

In this case, referring to the second method of part (a), θ_1 increases from α_1 to $\alpha_1 + 2\pi$ while θ_2 increases from α_2 to $\alpha_2 + 2\pi$. Thus

$$w = \sqrt{r_1 r_2} e^{i(\alpha_1 + 2\pi)/2} e^{i(\alpha_2 + 2\pi)/2} = \sqrt{r_1 r_2} e^{i\alpha_1/2} e^{i\alpha_2/2}$$

and no change in branch is observed.

2.20. Determine branch lines for the function of Problem 2.19.

Solution

The branch lines can be taken as those indicated with a heavy line in either of Figs. 2-23 or 2-24. In both cases, by not crossing these heavy lines, we ensure the single-valuedness of the function.

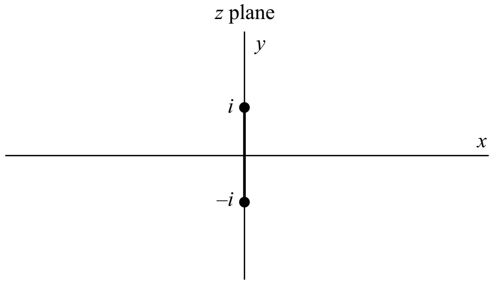

Fig. 2-23

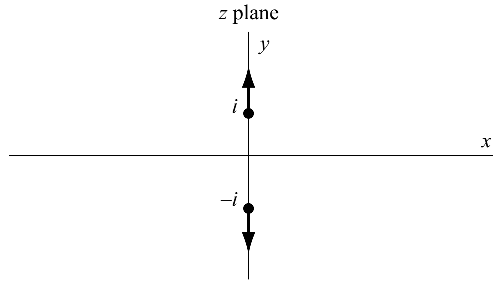

Fig. 2-24

2.21. Discuss the Riemann surface for the function of Problem 2.19.

Solution

We can have different Riemann surfaces corresponding to Figs. 2-23 or 2-24 of Problem 2.20. Referring to Fig. 2-23, for example, we imagine that the z plane consists of two sheets superimposed on each other and cut along the branch line. Opposite edges of the cut are then joined, forming the Riemann surface. On making one complete circuit around $z = i$, we start on one branch and wind up on the other. However, if we make one circuit about both $z = i$ and $z = -i$, we do not change branches at all. This agrees with the results of Problem 2.19.

2.22 Discuss the Riemann surface for the function $f(z) = \ln z$ [see Problem 2.14].

Solution

In this case, we imagine the z plane to consist of infinitely many sheets superimposed on each other and cut along a branch line emanating from the origin $z = 0$. We then connect each cut edge to the opposite cut edge of an adjacent sheet. Then, every time we make a circuit about $z = 0$, we are on another sheet corresponding to a different branch of the function. The collection of sheets is the Riemann surface. In this case, unlike Problems 2.6 and 2.7, successive circuits never bring us back to the original branch.

Limits

2.23. (a) Suppose $f(z) = z^2$. Prove that $\lim_{z\to z_0} f(z) = z_0^2$.

(b) Find $\lim_{z\to z_0} f(z)$ if $f(z) = \begin{cases} z^2 & z \neq z_0 \\ 0 & z = z_0 \end{cases}$.

Solution

(a) We must show that, given any $\epsilon > 0$, we can find δ (depending in general on ϵ) such that $|z^2 - z_0^2| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

If $\delta \leq 1$, then $0 < |z - z_0| < \delta$ implies that

$$|z^2 - z_0^2| = |z - z_0||z + z_0| < \delta|z - z_0 + 2z_0| < \delta\{|z - z_0| + |2z_0|\} < \delta(1 + 2|z_0|)$$

Take δ as 1 or $\epsilon/(1 + 2|z_0|)$, whichever is smaller. Then, we have $|z^2 - z_0^2| < \epsilon$ whenever $|z - z_0| < \delta$, and the required result is proved.

(b) There is no difference between this problem and that in part (a), since in both cases we exclude $z = z_0$ from consideration. Hence, $\lim_{z\to z_0} f(z) = z_0^2$. Note that the limit of $f(z)$ as $z \to z_0$ has nothing whatsoever to do with the value of $f(z)$ at z_0.

2.24. Interpret Problem 2.23 geometrically.

Solution

(a) The equation $w = f(z) = z^2$ defines a transformation or mapping of points of the z plane into points of the w plane. In particular, let us suppose that point z_0 is mapped into $w_0 = z_0^2$. [See Fig. 2-25 and 2-26.]

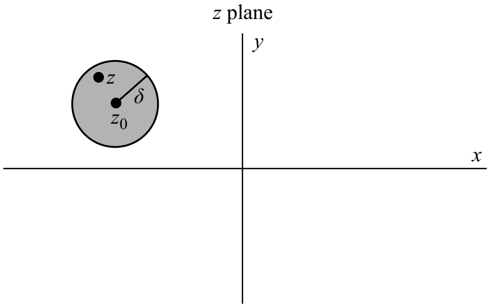

Fig. 2-25

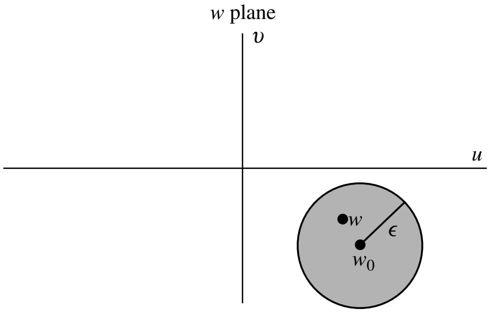

Fig. 2-26

In Problem 2.23(a), we prove that given any $\epsilon > 0$ we can find $\delta > 0$ such that $|w - w_0| < \epsilon$ whenever $|z - z_0| < \delta$. Geometrically, this means that if we wish w to be inside a circle of radius ϵ [see Fig. 2-26] we must choose δ (depending on ϵ) so that z lies inside a circle of radius δ [see Fig. 2-25]. According to Problem 2.23(a), this is certainly accomplished if δ is the smaller of 1 and $\epsilon/(1 + 2|z_0|)$.

(b) In Problem 2.23(a), $w = w_0 = z_0^2$ is the image of $z = z_0$. However, in Problem 2.23(b), $w = 0$ is the image of $z = z_0$. Except for this, the geometric interpretation is identical with that given in part (a).

2.25. Prove that $\lim_{z \to i} \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = 4 + 4i$.

Solution

We must show that for any $\epsilon > 0$, we can find $\delta > 0$ such that

$$\left| \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} - (4 + 4i) \right| < \epsilon \qquad \text{when } 0 < |z - i| < \delta$$

Since $z \neq i$, we can write

$$\frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = \frac{[3z^3 - (2 - 3i)z^2 + (5 - 2i)z + 5i][z - i]}{z - i}$$
$$= 3z^3 - (2 - 3i)z^2 + (5 - 2i)z + 5i$$

on cancelling the common factor $z - i \neq 0$.

Then, we must show that for any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|3z^3 - (2 - 3i)z^2 + (5 - 2i)z - 4 + i| < \epsilon \quad \text{when } 0 < |z - i| < \delta$$

If $\delta \leq 1$, then $0 < |z - i| < \delta$ implies

$$\begin{aligned}
|3z^3 - (2 - 3i)z^2 + (5 - 2i)z - 4 + i| &= |z - i||3z^2 + (6i - 2)z - 1 - 4i| \\
&= |z - i||3(z - i + i)^2 + (6i - 2)(z - i + i) - 1 - 4i| \\
&= |z - i||3(z - i)^2 + (12i - 2)(z - i) - 10 - 6i| \\
&< \delta\{3|z - i|^2 + |12i - 2||z - i| + |-10 - 6i|\} \\
&< \delta(3 + 13 + 12) = 28\delta
\end{aligned}$$

Taking δ as the smaller of 1 and $\epsilon/28$, the required result follows.

Theorems on Limits

2.26. Suppose $\lim_{z \to z_0} f(z)$ exists. Prove that it must be unique.

Solution

We must show that if $\lim_{z \to z_0} f(z) = l_1$ and $\lim_{z \to z_0} f(z) = l_2$, then $l_1 = l_2$.

By hypothesis, given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|f(z) - l_1| < \epsilon/2 \quad \text{when } 0 < |z - z_0| < \delta$$
$$|f(z) - l_2| < \epsilon/2 \quad \text{when } 0 < |z - z_0| < \delta$$

Then

$$|l_1 - l_2| = |l_1 - f(z) + f(z) - l_2| \leq |l_1 - f(z)| + |f(z) - l_2| < \epsilon/2 + \epsilon/2 = \epsilon$$

i.e., $|l_1 - l_2|$ is less than any positive number ϵ (however small) and so must be zero. Thus $l_1 = l_2$.

2.27. Suppose $\lim_{z\to z_0} g(z) = B \neq 0$. Prove that there exists $\delta > 0$ such that

$$|g(z)| > \tfrac{1}{2}|B| \quad \text{for } 0 < |z - z_0| < \delta$$

Solution

Since $\lim_{z\to z_0} g(z) = B$, we can find δ such that $|g(z) - B| < \frac{1}{2}|B|$ for $0 < |z - z_0| < \delta$.

Writing $B = B - g(z) + g(z)$, we have

$$|B| \le |B - g(z)| + |g(z)| < \tfrac{1}{2}|B| + |g(z)|$$

i.e.,

$$|B| < \tfrac{1}{2}|B| + |g(z)| \quad \text{from which } |g(z)| > \tfrac{1}{2}|B|$$

2.28. Given $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$, prove that

(a) $\lim_{z\to z_0} [f(z) + g(z)] = A + B$,

(b) $\lim_{z\to z_0} f(z)g(z) = AB$,

(c) $\lim_{z\to z_0} 1/g(z) = 1/B$ if $B \neq 0$,

(d) $\lim_{z\to z_0} f(z)/g(z) = A/B$ if $B \neq 0$.

Solution

(a) We must show that for any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|[f(z) + g(z)] - (A + B)| < \epsilon \quad \text{when } 0 < |z - z_0| < \delta$$

We have

$$|[f(z) + g(z)] - (A + B)| = |[f(z) - A] + [g(z) - B]| \le |f(z) - A| + |g(z) - B| \tag{1}$$

By hypothesis, given $\epsilon > 0$ we can find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(z) - A| < \epsilon/2 \quad \text{when } 0 < |z - z_0| < \delta_1 \tag{2}$$

$$|g(z) - B| < \epsilon/2 \quad \text{when } 0 < |z - z_0| < \delta_2 \tag{3}$$

Then, from (1), (2), and (3),

$$|[f(z) + g(z)] - (A + B)| < \epsilon/2 + \epsilon/2 = \epsilon \quad \text{when } 0 < |z - z_0| < \delta$$

where δ is chosen as the smaller of δ_1 and δ_2.

(b) We have

$$\begin{aligned} |f(z)g(z) - AB| &= |f(z)\{g(z) - B\} + B\{f(z) - A\}| \le |f(z)||g(z) - B| + |B||f(z) - A| \\ &\le |f(z)||g(z) - B| + (|B| + 1)|f(z) - A| \end{aligned} \tag{4}$$

Since $\lim_{z\to z_0} f(z) = A$, we can find δ_1 such that $|f(z) - A| < 1$ for $0 < |z - z_0| < \delta_1$. Hence, by inequalities 4, page 3, Section 1.5.

$$|f(z) - A| \ge |f(z)| - |A|, \quad \text{i.e., } 1 \ge |f(z)| - |A| \quad \text{or} \quad |f(z)| \le |A| + 1$$

i.e., $|f(z)| < P$ where P is a positive constant.

Since $\lim_{z\to z_0} g(z) = B$, given $\epsilon > 0$, we can find $\delta_2 > 0$ such that $|g(z) - B| < \epsilon/2P$ for $0 < |z - z_0| < \delta_2$.

Since $\lim_{z\to z_0} f(z) = A$, given $\epsilon > 0$, we can find $\delta_3 > 0$ such that $|f(z) - A| < \epsilon/2(|B| + 1)$ for $0 < |z - z_0| < \delta_3$.

Using these in (4), we have

$$|f(z)g(z) - AB| < P\frac{\epsilon}{2P} + (|B| + 1)\frac{\epsilon}{2(|B| + 1)} = \epsilon$$

for $0 < |z - z_0| < \delta$ where δ is the smaller of δ_1, δ_2, δ_3, and the proof is complete.

(c) We must show that, for any $\epsilon > 0$, we can find $\delta > 0$ such that

$$\left|\frac{1}{g(z)} - \frac{1}{B}\right| = \frac{|g(z) - B|}{|B||g(z)|} < \epsilon \quad \text{when } 0 < |z - z_0| < \delta \tag{5}$$

By hypothesis, given any $\epsilon > 0$, we can find $\delta_1 > 0$ such that

$$|g(z) - B| < \tfrac{1}{2}|B|^2\epsilon \quad \text{when } 0 < |z - z_0| < \delta_1$$

By Problem 2.27, since $\lim_{z \to z_0} g(z) = B \neq 0$, we can find $\delta_2 > 0$ such that

$$|g(z)| > \tfrac{1}{2}|B| \quad \text{when } 0 < |z - z_0| < \delta_2$$

Then, if δ is the smaller of δ_1 and δ_2, we can write

$$\left|\frac{1}{g(z)} - \frac{1}{B}\right| = \frac{|g(z) - B|}{|B||g(z)|} < \frac{\frac{1}{2}|B|^2\epsilon}{|B| \cdot \frac{1}{2}|B|} = \epsilon \quad \text{whenever } 0 < |z - z_0| < \delta$$

and the required result is proved.

(d) From parts (b) and (c),

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0}\left\{f(z) \cdot \frac{1}{g(z)}\right\} = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} \frac{1}{g(z)} = A \cdot \frac{1}{B} = \frac{A}{B}$$

This can also be proved directly [see Problem 2.145].

Note. In the proof of (a), we have used the results $|f(z) - A| < \epsilon/2$ and $|g(z) - B| < \epsilon/2$, so that the final result would come out to be $|f(z) + g(z) - (A + B)| < \epsilon$. Of course, the proof would be *just as valid* if we had used 2ϵ [or any other positive multiple of ϵ] in place of ϵ. Similar remarks hold for the proofs of (b), (c), and (d).

2.29. Evaluate each of the following using theorems on limits:

(a) $\lim_{z \to 1+i} (z^2 - 5z + 10)$ (b) $\displaystyle\lim_{z \to -2i} \frac{(2z+3)(z-1)}{z^2 - 2z + 4}$ (c) $\displaystyle\lim_{z \to 2e^{\pi i/3}} \frac{z^3 + 8}{z^4 + 4z^2 + 16}$

Solution

(a)
$$\begin{aligned}\lim_{z \to 1+i} (z^2 - 5z + 10) &= \lim_{z \to 1+i} z^2 + \lim_{z \to 1+i} (-5z) + \lim_{z \to 1+i} 10 \\ &= (\lim_{z \to 1+i} z)(\lim_{z \to 1+i} z) + (\lim_{z \to 1+i} -5)(\lim_{z \to 1+i} z) + \lim_{z \to 1+i} 10 \\ &= (1+i)(1+i) - 5(1+i) + 10 = 5 - 3i\end{aligned}$$

In practice, the intermediate steps are omitted.

(b)
$$\lim_{z \to -2i} \frac{(2z+3)(z-1)}{z^2 - 2z + 4} = \frac{\lim_{z \to -2i}(2z+3)\lim_{z \to -2i}(z-1)}{\lim_{z \to -2i}(z^2 - 2z + 4)} = \frac{(3-4i)(-2i-1)}{4i} = -\frac{1}{2} + \frac{11}{4}i$$

(c) In this case, the limits of the numerator and denominator are each zero and the theorems on limits fail to apply. However, by obtaining the factors of the polynomials, we see that

$$\begin{aligned}\lim_{z \to 2e^{\pi i/3}} \frac{z^3 + 8}{z^4 + 4z^2 + 16} &= \lim_{z \to 2e^{\pi i/3}} \frac{(z+2)(z - 2e^{\pi i/3})(z - 2e^{5\pi i/3})}{(z - 2e^{\pi i/3})(z - 2e^{2\pi i/3})(z - 2e^{4\pi i/3})(z - 2e^{5\pi i/3})} \\ &= \lim_{z \to 2e^{\pi i/3}} \frac{(z+2)}{(z - 2e^{2\pi i/3})(z - 2e^{4\pi i/3})} = \frac{e^{\pi i/3} + 1}{2(e^{\pi i/3} - e^{2\pi i/3})(e^{\pi i/3} - e^{4\pi i/3})} \\ &= \frac{3}{8} - \frac{\sqrt{3}}{8}i\end{aligned}$$

Another Method. Since $z^6 - 64 = (z^2 - 4)(z^4 + 4z^2 + 16)$, the problem is equivalent to finding

$$\lim_{z \to 2e^{\pi i/3}} \frac{(z^2 - 4)(z^3 + 8)}{z^6 - 64} = \lim_{z \to 2e^{\pi i/3}} \frac{z^2 - 4}{z^3 - 8} = \frac{e^{2\pi i/3} - 1}{2(e^{\pi i} - 1)} = \frac{3}{8} - \frac{\sqrt{3}}{8}i$$