Introduction to Proofs

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Proofs

- proof
- **theorem**.
- **Propositions**, **facts** or **results**.)
- We demonstrate that a theorem is true with a **proof.**
- A proof is a valid argument that establishes the truth of a theorem.

Example

- If ab is an even number, then aa or bb is even.

Assume that a or b is even - say it is a (the case where b is even will be identical). That is, $a = 2k$ for some integer k. Then

$$\begin{aligned} ab &= (2k)b \\ &= 2(kb). \end{aligned}$$

Thus ab is even.

Proofs

- **lemma**
- **corollary**
- **conjecture**
- conjecture becomes a theorem.
- Conjectures are some time not theorems.

Methods of Proving Theorems

- **axiom .**
- **Direct Proofs** $p \to q$.
- **DEFINITION**

The integer n is *even* if there exists an integer k such that $n = 2k$, and n is *odd* if there exists an integer k such that $n = 2k + 1$. (Note that every integer is either even or odd, and no integer is both even and odd.) Two integers have the *same parity* when both are even or both are odd; they have *opposite parity* when one is even and the other is odd.

Example

- Give a direct proof of the theorem "If n is an odd integer, then $n2$ is odd."
- Note that this theorem states $\forall nP\ ((n) \rightarrow Q(n))$, where $P(n)$ is "n is an odd integer" and $Q(n)$ is "$n2$ is odd."
- $n = 2k + 1$
- $n2 = (2k + 1)2 = 4k2 + 4k + 1 = 2(2k2 + 2k) + 1$.

indirect proofs.

- **indirect proofs.**
- **Proofs by contraposition** make use of the fact that the conditional statement $p \to q$, $\neg q \to \neg p$. This means that the conditional statement $p \to q$ can be proved by showing that its contrapositive, $\neg q \to \neg p$, is true.

EXAMPLE

- Prove that if n is an integer and $3n + 2$ is odd, then n is odd.
- ***Solution***:
- $3n + 2$ is an odd integer.
- This means that $3n + 2 = 2k + 1$ for some integer k. Can we use this fact to show that n is odd? We see that $3n + 1 = 2k$,.
- The first step in a proof by contraposition is to assume that the conclusion of the conditional statement "If $3n + 2$ is odd, then n is odd" is false; namely, assume that n is even.
- Then, by the definition of an even integer, $n = 2k$ for some integer k. Substituting $2k$ for n, we find that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$. This tells us that $3n + 2$ is even

Proofs by Contradiction

- Because the statement $r \wedge \neg r$ is a contradiction whenever r is a proposition, we can prove that p is true if we can show that $\neg p \rightarrow (r \wedge \neg r)$ is true for some proposition r. Proofs of this type are called **proofs by contradiction**.
- **EXAMPLE** Show that at least four of any 22 days must fall on the same day of the week
- *Solution:* Let p be the proposition "At least four of 22 chosen days fall on the same day of the week." Suppose that $\neg p$ is true. This means that at most three of the 22 days fall on the same day of the week.

Proof Methods and Strategy

- **exhaustive proofs**, or **proofs by**
- **EXAMPLE 1** Prove that $(n + 1)3 \geq 3n$ if n is a positive integer with $n \leq 4$.
- *Solution:* We use a proof by exhaustion. We only need verify the inequality $(n + 1)3 \geq 3n$
- when $n = 1, 2, 3,$ and 4.

PROOF BY CASES

- proof by cases
- **EXAMPLE 3** Prove that if n is an integer, then $n^2 \geq n$.
- *Solution:* We can prove that $n^2 \geq n$ for every integer by considering three cases, when $n = 0$, when $n \geq 1$, and when $n \leq -$.
- *Case (i):* When $n = 0$, True
- *Case (ii):* When $n \geq 1$ True .
- *Case (iii):* In this case $n \leq -1$. However, $n^2 \geq 0$. It follows that $n^2 \geq n$.