Predicates and Quantifiers

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Predicates

- **Predicate**
- $x > 3$
- **EXAMPLE 1** Let *P(x)* denote the statement "$x > 3$." What are the truth values of *P(4)* and *P(2)*?
- ***Solution:*** Hence, *P(4)*, which is the statement "$4 > 3$," is true. However, *P(2)*, which is the statement "$2 > 3$," is false.

Predicates

- **EXAMPLE 2** Let *A(x)* denote the statement "Computer x is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of *A*(CS1), *A*(CS2), and *A*(MATH1)?
- *Solution:*
- *A*(CS1) is false. Similarly, because CS2 and MATH1 are on the list of computers under attack, we know that *A*(CS2) and *A*(MATH1) are true.

Predicates

- **EXAMPLE 3** Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

- ***Solution:*** To obtain $Q(1, 2)$, set $x = 1$ and $y = 2$ in the statement $Q(x, y)$. Hence, $Q(1, 2)$ is the statement "$1 = 2 + 3$," which is false. The statement $Q(3, 0)$ is the proposition "$3 = 0 + 3$," which is true.

Quantifiers

- Quantification
- We will focus on two types of quantification here:

1. universal quantification,
2. existential quantification,

- **predicate calculus**.

Quantifiers

- Let *Q(x)* be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?
- ***Solution:*** *Q(x)* is not true for every real number x, because, for instance, *Q(3)* is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus $\forall x Q(x)$ is false.

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Quantifiers

- **EXAMPLE 11** What is the truth value of ∀x*P(x)*, where *P(x)* is the statement "x2 < 10" and the domain consists of the positive integers not exceeding 4?
- *Solution:*

it follows that ∀x*P(x)* is false.

THE UNIQUENESSQUANTIFIER

- Denoted by ∃! or ∃1
- The notation $\exists! x P(x)$ [or $\exists 1 x P(x)$] states "There exists a unique x such that $P(x)$ is true.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Nested Quantifiers

- **EXAMPLE 1** Assume that the domain for the variables x and *y* consists of all real numbers. The statement $\forall x \forall y(x + y = y + x)$
- **Solution**

says that $x + y = y + x$ for all real numbers x and *y*. This is the commutative law for addition of real numbers. Likewise, the statement

- **EXAMPLE2** $\forall x \exists y(x + y = 0)$
- **Solution**

says that for every real number x there is a real number *y* such that $x + y = 0$. This states that every real number has an additive inverse.

Nested Quantifiers

- Translate into English the statement
- $\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$, where the domain for both variables consists of all real numbers.
- ***Solution:*** This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$.