In terms of safety factor the orbit shift can be written

$$|\Delta| = r_{L\theta}\ \frac{r}{R} = r_{L\phi}\frac{B_\phi r}{B_\theta R} = r_L q_s \tag{2.76}$$

(assuming $B_\phi >> B_\theta$).

2.6 The Mirror Effect of Parallel Field Gradients: $\mathbf{E} = 0$, $\nabla B \parallel \mathbf{B}$

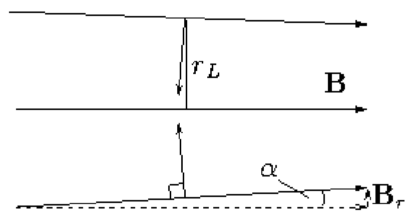

Figure 2.13: Basis of parallel mirror force

In the above situation there is a net force along **B**.
Force is

$$\begin{aligned} < F_\| > &= -|q\mathbf{v}\wedge\mathbf{B}|\ \sin\alpha = \ -|q|v_\perp B\sin\alpha & (2.77)\\ \sin\alpha &= \frac{-B_r}{B} & (2.78)\end{aligned}$$

Calculate B_r as function of B_z from $\nabla.\mathbf{B} = 0$.

$$\nabla.\mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}B_z = 0\ . \tag{2.79}$$

Hence

$$rB_r = -\int r\frac{\partial B_z}{\partial z}dr \tag{2.80}$$

Suppose r_L is small enough that $\frac{\partial B_z}{\partial z} \simeq$ const.

$$[rB_r]_0^{r_L} \simeq \int_0^{r_L} rdr\ \frac{\partial B_z}{\partial z} = -\frac{1}{2}\ r_L^2\frac{\partial B_z}{\partial z} \tag{2.81}$$

So

$$B_r(r_L) = -\frac{1}{2}r_L\frac{\partial B_z}{\partial z} \tag{2.82}$$

$$\sin\alpha = -\frac{B_r}{B} = +\frac{r_L}{2}\frac{1}{2}\frac{\partial B_z}{\partial z} \tag{2.83}$$

Hence

$$< F_\parallel >= -|q| \; \frac{v_\perp r_L}{2}\frac{\partial B_z}{\partial z} = -\frac{\frac{1}{2}mv_\perp^2}{B}\frac{\partial B_z}{\partial z}. \tag{2.84}$$

As particle enters increasing field region it experiences a net parallel *retarding* force.

Define *Magnetic Moment*

$$\mu \equiv \frac{1}{2}mv_\perp^2/B \; . \tag{2.85}$$

Note this is consistent with loop current definition

$$\mu = AI = \pi r_L^2 . \frac{|q|v_\perp}{2\pi r_L} = \frac{|q|r_L v_\perp}{2} \tag{2.86}$$

Force is $F_\parallel = \mu.\nabla_\parallel \mathbf{B}$

This is force on a 'magnetic dipole' of moment μ.

$$F_\parallel = \mu.\nabla_\parallel \mathbf{B} \tag{2.87}$$

Our μ always points along $\mathbf{B}$ but in opposite direction.

2.6.1 Force on an Elementary Magnetic Moment Circuit

Consider a plane rectangular circuit carrying current I having elementary area $dxdy = dA$. Regard this as a vector pointing in the $\mathbf{z}$ direction $\mathbf{dA}$. The force on this circuit in a field $\mathbf{B}(\mathbf{r})$ is $\mathbf{F}$ such that

$$\begin{aligned}
F_x &= Idy[B_z(x+dx) - B_z(x)] = Idydx\frac{\partial B_z}{\partial x} && (2.88)\\
F_y &= -Idx[B_z(y+dy) - B_z(y)] = Idydx\frac{\partial B_z}{\partial y} && (2.89)\\
F_z &= -Idx[B_y(y+dy) - B_y(y)] - Idy[B_x(x+dx) - B_x(x)] && (2.90)\\
&= -Idxdy\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right] = Idydx\frac{\partial B_z}{\partial z} && (2.91)
\end{aligned}$$

(Using $\nabla.\mathbf{B} = 0$).

Hence, summarizing: $\mathbf{F} = Idydx\nabla B_z$. Now define $\mu = I\mathbf{dA} = Idydx\hat{\mathbf{z}}$ and take it constant. Then clearly the force can be written

$$\mathbf{F} = \nabla(\mathbf{B}.\mu) \quad [\text{Strictly} = (\nabla\mathbf{B}).\mu] \tag{2.92}$$

μ is the (vector) magnetic moment of the circuit.

The shape of the circuit does not matter since any circuit can be considered to be composed of the sum of many rectangular circuits. So in general

$$\mu = I\mathbf{dA} \tag{2.93}$$

and force is

$$\mathbf{F} = \nabla(\mathbf{B}.\mu) \qquad (\mu \text{ constant}), \tag{2.94}$$

We shall show in a moment that $|\mu|$ is constant for a circulating particle, regard as an elementary circuit. Also, μ for a particle always points in the **-B** direction. [Note that this means that the effect of particles on the field is to *decrease* it.] Hence the force may be written

$$\mathbf{F} = -\mu\nabla B \tag{2.95}$$

This gives us both:

- *Magnetic Mirror Force:*

$$F_{\parallel} = -\mu\nabla_{\parallel}B \tag{2.96}$$

  and

- *Grad B Drift:*

$$\mathbf{v}_{\nabla B} = \frac{1}{q}\,\frac{\mathbf{F}\wedge\mathbf{B}}{B^2} = \frac{\mu}{q}\frac{\mathbf{B}\wedge\nabla B}{B^2}. \tag{2.97}$$

2.6.2 μ is a constant of the motion

'Adiabatic Invariant'

Proof from $F_{\parallel}$

Parallel equation of motion

$$m\frac{dv_{\parallel}}{dt} = F_{\parallel} = -\mu\frac{dB}{dz} \tag{2.98}$$

So

$$mv_{\parallel}\frac{dv_{\parallel}}{dt} = \frac{d}{dt}(\frac{1}{2}mv_{\parallel}^2) = -\mu v_z\frac{dB}{dz} = -\mu\frac{dB}{dt} \tag{2.99}$$

or

$$\frac{d}{dt}(\frac{1}{2}mv_{\parallel}^2) + \mu\frac{dB}{dt} = 0 \quad . \tag{2.100}$$

Conservation of Total KE

$$\frac{d}{dt}(\frac{1}{2}mv_{\parallel}^2 + \frac{1}{2}mv_{\perp}^2) = 0 \tag{2.101}$$

$$= \quad \frac{d}{dt}(\frac{1}{2}mv_{\parallel}^2 + \mu B) = 0 \tag{2.102}$$

Combine

$$\frac{d}{dt}(\mu B) - \mu\frac{dB}{dt} = 0 \tag{2.103}$$

$$= \frac{d\mu}{dt} = 0 \qquad \text{As required} \tag{2.104}$$

Angular Momentum

of particle about the guiding center is

$$r_L m v_\perp = \frac{m v_\perp}{|q| B} m v_\perp = \frac{2m}{|q|} \frac{\frac{1}{2} m v_\perp^2}{B} \tag{2.105}$$

$$= \frac{2m}{|q|}\mu \quad . \tag{2.106}$$

Conservation of magnetic moment is basically conservation of angular momentum about the guiding center.

Proof direct from Angular Momentum

Consider angular momentum about G.C. Because θ is ignorable (locally) Canonical angular momentum is conserved.

$$p = [\mathbf{r} \wedge (m\mathbf{v} + q\mathbf{A})]_z \qquad \text{conserved.} \tag{2.107}$$

Here $\mathbf{A}$ is the vector potential such that $\mathbf{B} = \nabla \wedge \mathbf{A}$

the definition of the vector potential means that

$$B_z = \frac{1}{r} \frac{\partial(r A_\theta))}{\partial r} \tag{2.108}$$

$$\Rightarrow r_L A_\theta(r_L) = \int_0^{r_L} r.B_z dr = \frac{r_L^2}{2} B_z = \frac{\mu m}{|q|} \tag{2.109}$$

Hence

$$p = \frac{-q}{|q|} r_L v_\perp m + q \frac{m\mu}{|q|} \tag{2.110}$$

$$= -\frac{q}{|q|} m\mu. \tag{2.111}$$

So p = const $\leftrightarrow$ μ = constant.

Conservation of μ is basically conservation of angular momentum of particle about G.C.

2.6.3 Mirror Trapping

$F_\parallel$ may be enough to reflect particles back. But may not!

Let's calculate whether it will:

Suppose reflection occurs.

At reflection point $v_{\parallel r} = 0$.

Energy conservation

$$\frac{1}{2} m (v_{\perp 0}^2 + v_{\parallel 0}^2) = \frac{1}{2} m v_{\perp r}^2 \tag{2.112}$$

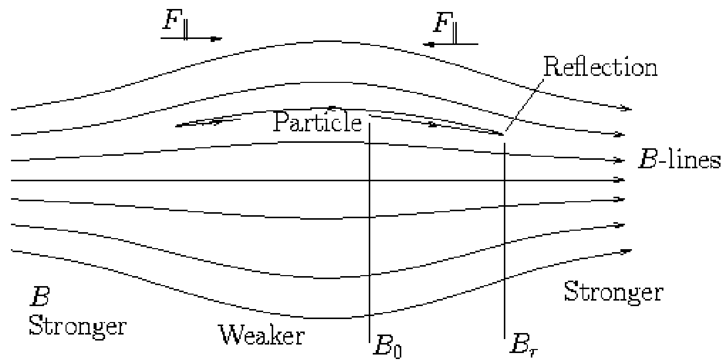

Figure 2.14: Magnetic Mirror

μ conservation

$$\frac{\frac{1}{2}mv_{\perp 0}^2}{B_0} = \frac{\frac{1}{2}mv_{\perp r}^2}{B_r} \tag{2.113}$$

Hence

$$v_{\perp 0}^2 + v_{\parallel 0}^2 = \frac{B_r}{B_0} v_{\perp 0}^2 \tag{2.114}$$

$$\frac{B_0}{B_r} = \frac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{\parallel o}^2} \tag{2.115}$$

2.6.4 Pitch Angle θ

$$\tan\theta = \frac{v_\perp}{v_\parallel} \tag{2.116}$$

$$\frac{B_0}{B_r} = \frac{v_{\perp 0}^2}{v_{\perp 0}^2 + v_{\parallel 0}^2} = \sin^2\theta_0 \tag{2.117}$$

So, given a pitch angle θ_0, reflection takes place where $B_0/B_r = \sin^2\theta_0$.
If θ_0 is too small no reflection can occur.
Critical angle θ_c is obviously

$$\theta_c = \sin^{-1}(B_0/B_1)^{\frac{1}{2}} \tag{2.118}$$

Loss Cone is all $\theta < \theta_c$.
Importance of Mirror Ratio: $R_m = B_1/B_0$.

2.6.5 Other Features of Mirror Motions

Flux enclosed by gyro orbit is constant.

$$\Phi = \pi r_L^2 B = \frac{\pi m^2 v_\perp^2}{q^2 B^2} B \tag{2.119}$$

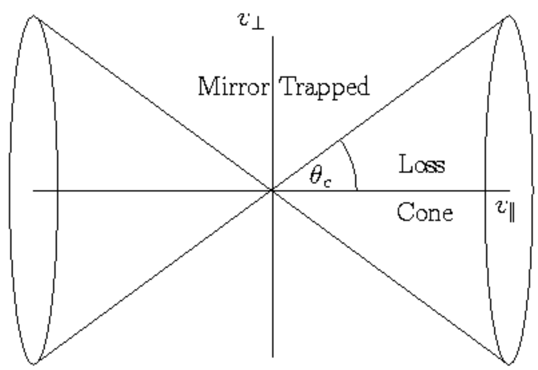

Figure 2.15: Critical angle θ_c divides velocity space into a loss-cone and a region of mirror-trapping

$$= \frac{2\pi m}{q^2}\frac{\frac{1}{2}mv_\perp^2}{B} \tag{2.120}$$

$$= \frac{2\pi m}{q^2}\mu = \text{constant}. \tag{2.121}$$

Note that if B changes 'suddenly' μ might not be conserved.

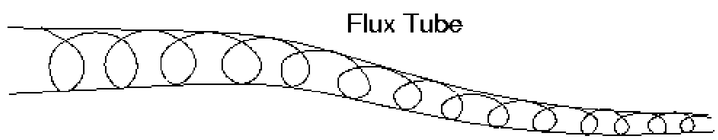

Figure 2.16: Flux tube described by orbit

Basic requirement

$$r_L << B/|\nabla B| \tag{2.122}$$

Slow variation of B (relative to r_L).

2.7 Time Varying B Field (E inductive)

Particle can gain energy from the inductive $\mathbf{E}$ field

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.123}$$

$$\text{or} \oint \mathbf{E}.\mathbf{dl} = -\int_s \dot{\mathbf{B}}.\mathbf{ds} = -\frac{d\Phi}{dt} \tag{2.124}$$

Hence work done on particle in 1 revolution is

$$\delta w = -\oint |q|\mathbf{E}.\mathbf{d}\ell = +|q|\int_s \dot{\mathbf{B}}.\mathbf{ds} = +|q|\frac{d\Phi}{dt} = |q|\dot{B}\pi r_L^2 \tag{2.125}$$

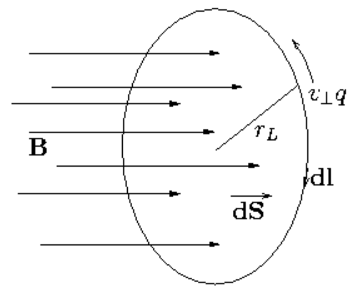

Figure 2.17: Particle orbits round $\mathbf{B}$ so as to perform a line integral of the Electric field

($\mathbf{d}\ell$ and $\mathbf{v}_\perp q$ are in opposition directions).

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = |q|\dot{B}\pi r_L^2 = \frac{2\pi\dot{B}m}{|q|B}\frac{\frac{1}{2}mv_\perp^2}{B} \tag{2.126}$$

$$= \frac{2\pi\dot{B}}{|\Omega|}\mu. \tag{2.127}$$

Hence

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = \frac{|\Omega|}{2\pi}\delta\left(\frac{1}{2}mv_\perp^2\right) = \mu\frac{db}{dt} \tag{2.128}$$

but also

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = \frac{d}{dt}(\mu B)\ . \tag{2.129}$$

Hence

$$\frac{d\mu}{dt} = 0. \tag{2.130}$$

Notice that since $\Phi = \frac{2\pi m}{q^2}\mu$, this is just another way of saying that the flux through the gyro orbit is conserved.

Notice also *energy increase*. Method of ‘heating’. Adiabatic Compression.

2.8 Time Varying E-field (E, B uniform)

Recall the $\mathbf{E}\wedge\mathbf{B}$ drift:

$$\mathbf{v}_{E\wedge B} = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2} \tag{2.131}$$

when E varies so does $\mathbf{v}_{E\wedge B}$. Thus the guiding centre experiences an acceleration

$$\dot{\mathbf{v}}_{E\wedge B} = \frac{d}{dt}\left(\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\right) \tag{2.132}$$

In the frame of the guiding centre which is accelerating, a force is felt.

$$\mathbf{F}_a = -m\frac{d}{dt}\left(\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\right) \qquad \text{(Pushed back into seat! } - \text{ ve.)} \tag{2.133}$$

This force produces another drift

$$\mathbf{v}_D = \frac{1}{q}\frac{\mathbf{F}_a \wedge \mathbf{B}}{B^2} = \frac{m}{qB^2}\frac{d}{dt}\left(\frac{\mathbf{E}\wedge\mathbf{B}}{B^2}\right)\wedge\mathbf{B} \tag{2.134}$$

$$= -\frac{m}{qB}\frac{d}{dt}\Big((\mathbf{E}.\mathbf{B})\,\mathbf{B} - B^2\mathbf{E}\Big) \tag{2.135}$$

$$= \frac{m}{qB^2}\dot{\mathbf{E}}_\perp \tag{2.136}$$

This is called the *'polarization drift'*.

$$\mathbf{v}_D = \mathbf{v}_{E\wedge B} \quad + \quad \mathbf{v}_p = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2} \quad + \frac{m}{qB^2}\,\dot{\mathbf{E}}_\perp \tag{2.137}$$

$$= \frac{E\wedge B}{B^2} \quad + \quad \frac{1}{\Omega B}\dot{\mathbf{E}}_\perp \tag{2.138}$$

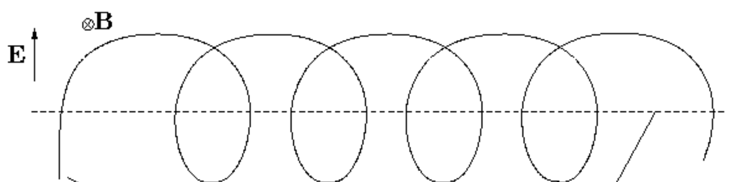

Figure 2.18: Suddenly turning on an electric field causes a shift of the gyrocenter in the direction of force. This is the polarization drift.

Start-up effect: When we 'switch on' an electric field the average position (gyro center) of an initially stationary particle shifts over by $\sim \frac{1}{2}$ the orbit size. The polarization drift is this polarization effect on the medium.
Total *shift* due to $\mathbf{v}_p$ is

$$\Delta\mathbf{r}\int \mathbf{v}_p dt = \frac{m}{qB^2}\int \hat{\mathbf{E}}_\perp dt = \frac{m}{qB^2}\left[\Delta\mathbf{E}_\perp\right] \quad . \tag{2.139}$$

2.8.1 Direct Derivation of $\frac{d\mathbf{E}}{dt}$ effect: 'Polarization Drift'

Consider an oscillatory field $\mathbf{E} = \mathbf{E}e^{-i\omega t}$ $(\perp r_0\mathbf{B})$

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\right) \tag{2.140}$$

$$= q\left(\mathbf{E}e^{-i\omega t} + \mathbf{v}\wedge\mathbf{B}\right) \tag{2.141}$$

Try for a solution in the form

$$\mathbf{v} = \mathbf{v}_D e^{-i\omega t} + \mathbf{v}_L \tag{2.142}$$

where, as usual, $\mathbf{v}_L$ satisfies $m\dot{\mathbf{v}}_L = q\mathbf{v}_L \wedge \mathbf{B}$

Then

$$(1) \qquad m(-i\omega\mathbf{v}_D = q\left(\mathbf{E} + \mathbf{v}_D \wedge \mathbf{B}\right) \qquad x\ell^{-i\omega t} \tag{2.143}$$

Solve for $\mathbf{v}_D$: Take $\wedge\mathbf{B}$ this equation:

$$(2) \qquad -\,mi\omega\left(\mathbf{v}_D \wedge \mathbf{B}\right) = q\left(\mathbf{E} \wedge \mathbf{B} + \left(\mathbf{B}^2.\mathbf{v}|D\right)\mathbf{B} - B^2\mathbf{v}_D\right) \tag{2.144}$$

add $mi\omega \times (1)$ to $q \times (2)$ to eliminate $\mathbf{v}_D \wedge \mathbf{B}$.

$$m^2\omega^2\mathbf{v}_D + q^2(\mathbf{E} \wedge \mathbf{B} - B^2\mathbf{v}_D) = mi\omega q\mathbf{E} \tag{2.145}$$

$$\text{or :} \qquad \mathbf{v}_D\left[1 - \frac{m^2\omega^2}{q^2B^2}\right] = -\frac{mi\omega}{qB^2}\mathbf{E} + \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{2.146}$$

$$\text{i.e.} \qquad \mathbf{v}_D\left[1 - \frac{\omega^2}{\Omega^2}\right] = -\frac{i\omega q}{\Omega B|q|}\mathbf{E} + \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{2.147}$$

Since $-i\omega \leftrightarrow \frac{\partial}{\partial t}$ this is the same formula as we had before: the sum of polarization and $\mathbf{E} \wedge \mathbf{B}$ drifts *except* for the $[1 - \omega^2\Omega^2]$ term.

This term comes from the change in $\mathbf{v}_D$ with time (accel).

Thus our earlier expression was only approximate. A good approx if $\omega << \Omega$.

2.9 Non Uniform E (Finite Larmor Radius)

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E}(\mathbf{r}) + \mathbf{v} \wedge \mathbf{B}\right) \tag{2.148}$$

Seek the usual soltuion $\mathbf{v} = \mathbf{v}_D + \mathbf{v}_g$.

Then average out over a gyro orbit

$$\langle m\frac{dv_D}{dt}\rangle = 0 = \langle q\left(\mathbf{E}(\mathbf{r}) + \mathbf{v} \wedge \mathbf{B}\right)\rangle \tag{2.149}$$

$$= q\left[\langle\mathbf{E}(\mathbf{r})\rangle + \mathbf{v}_D \wedge \mathbf{B}\right] \tag{2.150}$$

Hence drift is obviously

$$\mathbf{v}_D = \frac{\langle\mathbf{E}(\mathbf{r})\rangle \wedge \mathbf{B}}{B^2} \tag{2.151}$$

So we just need to find the *average* E field experienced.

Expand $\mathbf{E}$ as a Taylor series about the G.C.

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 + (\mathbf{r}.\nabla)\,\mathbf{E} + \left(\frac{x^2\partial^2}{2!\partial x^2} + \frac{y^2}{2!}\frac{\partial^2}{\partial y^2}\right)\mathbf{E} + \text{cross terms } + . \tag{2.152}$$

(E.g. cross terms are $xy\frac{\partial^2}{\partial x \partial y}\mathbf{E}$).
Average over a gyro orbit: $\mathbf{r} = r_L(\cos\theta, \sin\theta, 0)$.
Average of cross terms = 0.
Then

$$\langle \mathbf{E}(\mathbf{r}) \rangle = \mathbf{E} + (\langle \mathbf{r}_L \rangle . \nabla)\mathbf{E} + \frac{\langle r_L^2 \rangle}{2!} \nabla^2 \mathbf{E}. \tag{2.153}$$

linear term $\langle r_L \rangle = 0$. So

$$\langle \mathbf{E}(\mathbf{r}) \rangle \simeq \mathbf{E} + \frac{r_L^2}{4} \nabla^2 E \tag{2.154}$$

Hence $\mathbf{E} \wedge \mathbf{B}$ with 1st finite-Larmor-radius correction is

$$\mathbf{v}_{E \wedge B} = \left(1 + \frac{r_L^2}{r} \nabla^2\right) \frac{\mathbf{E} \wedge \mathbf{B}}{B^2}. \tag{2.155}$$

[Note: Grad B drift is a finite Larmor effect already.]

<u>Second and Third Adiabatic Invariants</u>

There are additional approximately conserved quantities like μ in some geometries.

2.10 Summary of Drifts

$$\mathbf{v}_E = \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \quad \text{Electric Field} \tag{2.156}$$

$$\mathbf{v}_F = \frac{1}{q} \frac{\mathbf{F} \wedge \mathbf{B}}{B^2} \quad \text{General Force} \tag{2.157}$$

$$\mathbf{v}_E = \left(1 + \frac{r_L^2}{4} \nabla^2\right) \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \quad \text{Nonuniform E} \tag{2.158}$$

$$\mathbf{v}_{\nabla B} = \frac{m v_\perp^2}{2q} \frac{\mathbf{B} \wedge \nabla B}{B^3} \quad \text{GradB} \tag{2.159}$$

$$\mathbf{v}_R = \frac{m v_\parallel^2}{q} \frac{\mathbf{R}_c \wedge \mathbf{B}}{R_c^2 B^2} \quad \text{Curvature} \tag{2.160}$$

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{1}{q}\left(m v_\parallel^2 + \frac{1}{2} m v_\perp^2\right) \frac{\mathbf{R}_c \wedge \mathbf{B}}{R_c^2 B^2} \quad \text{Vacuum Fields.} \tag{2.161}$$

$$\mathbf{v}_p = \frac{q}{|q|} \frac{\dot{E}_\perp}{|\Omega| B} \quad \text{Polarization} \tag{2.162}$$

<u>Mirror Motion</u>

$$\mu \equiv \frac{m v_\perp^2}{2B} \qquad \text{is constant} \tag{2.163}$$

Force is $\mathbf{F} = -\mu \nabla B$.