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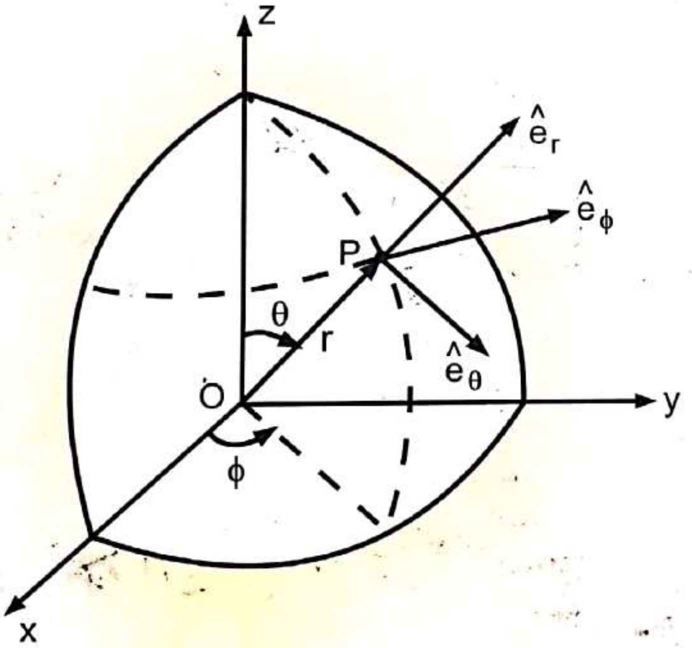
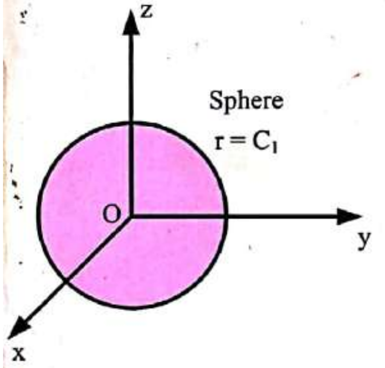
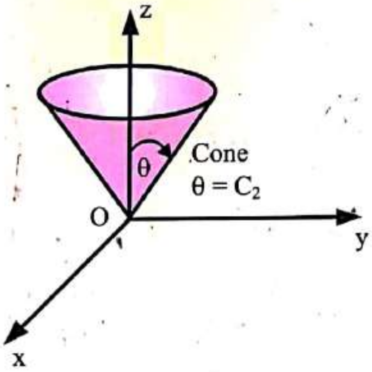
Vector and Tensor Analysis

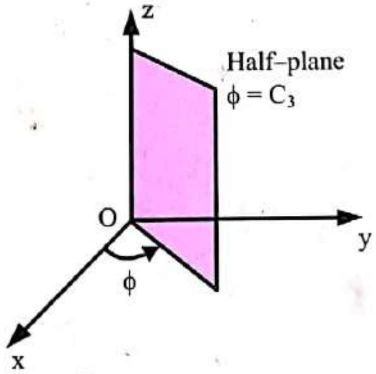
for Scientists and Engineers

Third Edition









Chapter 7

CARTESIAN TENSORS

7.1 INTRODUCTION

Tensor analysis may be regarded as a generalization of vector analysis. It is of great value to a scientist or an engineer in two ways. First, it allows complex mathematical and physical relationships to be expressed in a compact way and simplifies the mechanics of the development of theory. Second, it imparts greater understanding to vector notations and also establishes certain invariance properties with elegance and simplicity. It is of great use in mechanics, fluid dynamics, elasticity, differential geometry, electromagnetic theory, general relativity theory, and numerous other fields of science and engineering. In this chapter, we shall discuss Cartesian tensors i.e. tensors which are expressed in terms of components referred to rectangular Cartesian coordinate systems. At first sight, the notation of Cartesian tensors is somewhat complicated. The aim of this chapter is to provide a familiarity with the notation which will enable the reader to study other texts and applications without difficulty.

Before we start the actual studies of Cartesian tensors, it will be important to give some basic ideas (notations, definitions, transformations, etc.) which are useful in the study of Cartesian tensors.

7.2 SUMMATION CONVENTION

Consider an expression $a_1 x_1 + a_2 x_2 + a_3 x_3$ (1)

which can be written using summation sign as $\sum_{j=1}^{3} a_j x_j$ (2)

Let us omit the summation sign and write it simply as $a_j x_j$ (3)

where it is understood that the repeated index (or suffix) j represents the summation from 1 to 3.

Note that the form (3) is much more convenient than the original form (1). This situation occurs so frequently that it is convenient to adopt a convention which avoids the necessity of writing summation signs. This convention known as the summation convention is as follows:

Whenever a suffix appears twice in the same expression that expression is to be summed over all values of the suffix namely, 1, 2, 3.

DUMMY AND FREE INDICES

An index which is repeated in a given expression so that the summation convention applies, is called a **dummy** index, while an index occurring only once in a given expression is called a **free** index and does not imply any summation. For example, in the expression $A_k B_{jk}$, k is dummy index while j is a free index.

EXAMPLE (1): **Write each of the following using summation convention.**

(i) $a_{11}x_1 + a_{12}x_2 + a_{13}x_3$ (ii) $a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13}$

(iii) $(x_1)^2 + (x_2)^2 + (x_3)^2$ (iv) $\frac{\partial \phi}{\partial x_1}dx_1 + \frac{\partial \phi}{\partial x_2}dx_2 + \frac{\partial \phi}{\partial x_3}dx_3$

SOLUTION: We have

(i) $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = a_{1i}x_i$

(ii) $a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} = a_{1i}b_{1i}$

(iii) $(x_1)^2 + (x_2)^2 + (x_3)^2 = x_1x_1 + x_2x_2 + x_3x_3 = x_ix_i$

(iv) $\frac{\partial \phi}{\partial x_1}dx_1 + \frac{\partial \phi}{\partial x_2}dx_2 + \frac{\partial \phi}{\partial x_3}dx_3 = \frac{\partial \phi}{\partial x_i}dx_i$

EXAMPLE (2): **Write out explicitly the following summations and compare the results :**

(i) $a_i(x_i + y_i)$ (ii) $a_jx_j + a_ky_k$

SOLUTION: We have

(i) $a_i(x_i + y_i) = a_1(x_1 + y_1) + a_2(x_2 + y_2) + a_3(x_3 + y_3)$

$= a_1x_1 + a_1y_1 + a_2x_2 + a_2y_2 + a_3x_3 + a_3y_3$

(ii) $a_jx_j + a_ky_k = a_1x_1 + a_2x_2 + a_3x_3 + a_1y_1 + a_2y_2 + a_3y_3$

The two summations are identical except for the order in which the terms occur.

NOTE: (i) A repeated suffix may be replaced by any other suitable symbol not already in use. For example, $a_jb_j = a_kb_k = a_\alpha b_\alpha$ since in each expression summation over the repeated suffix is implied.

(ii) No suffix may occur more than twice in an expression. For example, $a_{ii}x_i$ is ambiguous because of the differences in the three quantities :

$a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$

$a_{ii}x_j = (a_{11} + a_{22} + a_{33})x_j$

$a_{ij}x_i = a_{1j}x_1 + a_{2j}x_2 + a_{3j}x_3$

On putting $j = i$ in these equations, we obtain entirely different expressions on the R.H.S.

(iii) An expression of the form $a_i(x_i + y_i)$ is considered well – defined, for it is obtained by composition of the meaningful expressions a_iz_i and $x_i + y_i = z_i$. In other words, the index i is regarded as occurring once in the term $(x_i + y_i)$.

7.3 DOUBLE SUMS

An expression can involve more than one summation indices. For example, $a_{ij}x_ix_j$ indicates a summation taking place on both i and j simultaneously. If an expression has two summation (dummy) indices, there will be a total of 3^2 terms in the sum; if there are three indices, there will be 3^3 terms; and so on.

EXAMPLE (3): **Write the terms in the expression $a_{ij} x_i x_j$; $i, j = 1, 2, 3$.**

SOLUTION: The given expression represents the double sum and has 9 terms in it. Its expansion can be written logically by first summing over i, and then over j. Since i varies from 1 to 3, therefore holding j fixed, the given expression is the sum of three terms. That is

$$a_{ij} x_i x_j = a_{1j} x_1 x_j + a_{2j} x_2 x_j + a_{3j} x_3 x_j$$

Now each term on the R.H.S. has the repeated index j which implies summation. Hence

$$a_{ij} x_i x_j = a_{11} x_1 x_1 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + a_{21} x_2 x_1 + a_{22} x_2 x_2 + a_{23} x_2 x_3 + a_{31} x_3 x_1 + a_{32} x_3 x_2 + a_{33} x_3 x_3$$

The result is the same if one sums over j first, and then over i.

EXAMPLE (4): **Write the following expression using summation convention.**

$$\mathbf{a_{11} b_{11} + a_{21} b_{12} + a_{31} b_{13} + a_{12} b_{21} + a_{22} b_{22} + a_{32} b_{23} + a_{13} b_{31} + a_{23} b_{32} + a_{33} b_{33}}$$

SOLUTION: The given expression can be written as

$$(a_{11} b_{11} + a_{21} b_{12} + a_{31} b_{13}) + (a_{12} b_{21} + a_{22} b_{22} + a_{32} b_{23}) + (a_{13} b_{31} + a_{23} b_{32} + a_{33} b_{33})$$

$$= a_{i1} b_{1i} + a_{i2} b_{2i} + a_{i3} b_{3i} = a_{ij} b_{ji}$$

7.4 SUBSTITUTIONS

Suppose it is required to substitute $y_i = a_{ij} x_j$ in the equation $Q = b_{ij} y_i x_j$. Simple substitution would lead to an absurd expression like $Q = b_{ij} a_{ij} x_j x_j$.

The correct procedure is first to identify any dummy indices in the expression to be substituted that coincide with indices occurring in the main expression. Changing these dummy indices to characters not found in the main expression, one may then carry out the substitution in the usual manner as follows :

Step (1) $y_i = a_{ij} x_j$, $Q = b_{ij} y_i x_j$. We see that the dummy index j is duplicated.

Step (2) Change the dummy index from j to r, to get $y_i = a_{ir} x_r$.

Step (3) Substitute and rearrange to get $Q = b_{ij} (a_{ir} x_r) x_j = a_{ir} b_{ij} x_r x_j$.

EXAMPLE (5): **If $y_i = a_{ij} x_j$, express the quadratic form $Q = g_{ij} y_i y_j$ in terms of x – variables.**

SOLUTION: First write $y_i = a_{ir} x_r$, $y_j = a_{js} x_s$

Then by substitution, $Q = g_{ij} (a_{ir} x_r) (a_{js} x_s) = g_{ij} a_{ir} a_{js} x_r x_s$

7.5 ALGEBRA AND THE SUMMATION CONVENTION

Certain routine algebraic manipulations in tensors can be easily justified by properties of ordin[ary] sums. However, some care should be taken. The following are several valid identities ; they will be u[sed] repeatedly from now on.

(1) $a_{ij} (x_j + y_j) \equiv a_{ij} x_j + a_{ij} y_j$

(2) $a_{ij} x_i y_j \equiv a_{ij} y_j x_i$

(3) $a_{ij} x_i x_j \equiv a_{ji} x_i x_j$

(4) $(a_{ij} + a_{ji}) x_i x_j \equiv 2 a_{ji} x_i x_j$

(5) $(a_{ij} - a_{ji}) x_i x_j \equiv 0$

The following non – identities should be carefully noted :

(1) $a_{ij}(x_i+y_j) \not\equiv a_{ij}x_i+a_{ij}y_j$ (2) $a_{ij}x_iy_j \not\equiv a_{ij}y_ix_j$

(3) $(a_{ij}+a_{ji})x_iy_j \not\equiv 2a_{ij}x_iy_j$

EXAMPLE (6): **Show that , generally,** $a_{ijk}(x_i+y_j)z_k \neq a_{ijk}x_iz_k+a_{ijk}y_jz_k$.

SOLUTION: Simply observe that on the left side there are no free indices , but on the right , j is free for the first term and i is free for the second .

7.6 THE KRONECKER DELTA δ_{ij}

The Kronecker delta or substitution operator written δ_{ij}, is defined as $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

Thus $\delta_{11}=\delta_{22}=\delta_{33}=1$ and $\delta_{12}=\delta_{21}=\delta_{23}=\delta_{32}=\delta_{31}=\delta_{13}=0$

EXAMPLE (7): **Using the definition of Kronecker delta , calculate** $\delta_{ij}x_ix_j$.

SOLUTION: We have $\delta_{ij}x_ix_j = \delta_{1j}x_1x_j+\delta_{2j}x_2x_j+\delta_{3j}x_3x_j$

$$= \delta_{11}x_1x_1+\delta_{12}x_1x_2+\delta_{13}x_1x_3+\delta_{21}x_2x_1+\delta_{22}x_2x_2+\delta_{23}x_2x_3+\delta_{31}x_3x_1+\delta_{32}x_3x_2+\delta_{33}x_3x_3$$

$$= 1x_1x_1+0x_1x_2+0x_1x_3+0x_2x_1+1x_2x_2+0x_2x_3+0x_3x_1+0x_3x_2+1x_3x_3$$

$$= x_1x_1+x_2x_2+x_3x_3 = x_ix_i$$

THEOREM (7.1): **Show that if x_1, x_2, x_3 are independent variables , then** $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$

PROOF: We have , if $i=j$, $\dfrac{\partial x_i}{\partial x_j} = \dfrac{\partial x_i}{\partial x_i} = 1$

If $i \neq j$, $\dfrac{\partial x_i}{\partial x_j} = 0$ since x_i and x_j are independent variables .

Thus $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

THEOREM (7.2): **Prove that** $\delta_{ij}A_j = A_i$.

PROOF: We know that the index j represents summation , therefore

$$\delta_{ij}A_j = \delta_{i1}A_1+\delta_{i2}A_2+\delta_{i3}A_3 \quad (i=1,2,3)$$

when $i=1$, $\delta_{1j}A_j = \delta_{11}A_1+\delta_{12}A_2+\delta_{13}A_3 = A_1$, and

when $i=2$, $\delta_{2j}A_j = \delta_{21}A_1+\delta_{22}A_2+\delta_{23}A_3 = A_2$, and

when $i=3$, $\delta_{3j}A_j = \delta_{31}A_1+\delta_{32}A_2+\delta_{33}A_3 = A_3$.

Thus in all cases : $\delta_{ij}A_j = A_i$

That is , δ_{ij} operating on A_j has substituted the free index i for the index j in A_j which gives a justification of the term **substitution operator** .

NOTE: This result is of fundamental importance and will be used frequently in our later discussion .

THEOREM (7.3): **Prove that $\delta_{ik}\delta_{jk} = \delta_{ij}$.**

PROOF: We have $\delta_{ik}\delta_{jk} = \delta_{i1}\delta_{j1} + \delta_{i2}\delta_{j2} + \delta_{i3}\delta_{j3} \quad (i, j = 1, 2, 3)$

when $j = 1$, $\delta_{ik}\delta_{1k} = \delta_{i1}$ and

when $j = 2$, $\delta_{ik}\delta_{2k} = \delta_{i2}$ and

when $j = 3$, $\delta_{ik}\delta_{3k} = \delta_{i3}$

It therefore follows that $\delta_{ik}\delta_{jk} = \delta_{ij}$.

EXAMPLE (8): **Show that**

(i) $\delta_{ii} = 3$ (ii) $\delta_{ik}\delta_{ik} = 3$

(iii) $\delta_{ij}\delta_{jk}\delta_{ki} = 3$ (iv) $\delta_{ij}\delta_{k\ell}A_{ik} = A_{j\ell}$

(v) $\delta_{ij}\,\delta_{jk}\,A_{ik} = A_{ii}$

SOLUTION: We have (i) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$

(ii) $\delta_{ik}\delta_{ik} = \delta_{1k}\delta_{1k} + \delta_{2k}\delta_{2k} + \delta_{3k}\delta_{3k}, \quad k = 1, 2, 3$

$= (\delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13}) + (\delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23}) + (\delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33})$

$= \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} = (1)(1) + (1)(1) + (1)(1) = 1 + 1 + 1 = 3$

(iii) $\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ik}\delta_{ki} = \delta_{ii} = 3$

(iv) $\delta_{ij}\delta_{k\ell}A_{ik} = \delta_{ij}A_{i\ell} = A_{j\ell}$

(v) $\delta_{ij}\,\delta_{jk}\,A_{ik} = \delta_{ik}\,A_{ik} = A_{ii}$.

7.7 RECTANGULAR COORDINATE SYSTEM

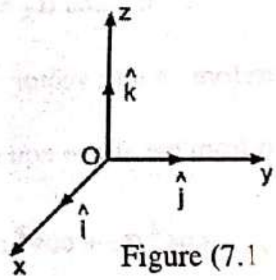
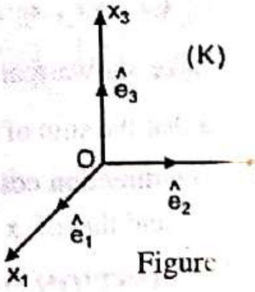


Figure (7.1

From vector analysis, we are familiar with the rectangular coordinate system in which we take Ox, Oy, Oz as the coordinate axes and $\hat{i}, \hat{j}, \hat{k}$ the unit vectors along these coordinate axes respectively, as shown in figure (7.1).



Figure

In tensor analysis, in stead of this system we take the system in which we have Ox_1, Ox_2, Ox_3 as the coordinate axes and $\hat{e}_1, \hat{e}_2, \hat{e}_3$ the unit vectors along these axes respectively as shown in figure (7.2). This system of coordinate axes will be denoted by K.

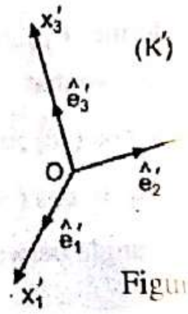


Figure (7.3)

In addition to the system K we need another coordinate system which will be denoted by K'. In the system K', we take Ox_1', Ox_2', Ox_3' as the coordinate axes and $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ the unit vectors along these axes respectively as shown in figure (7.3).

THEOREM (7.4): **Prove that** $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$.

PROOF: From vector analysis, we know that when $j = i$, $\hat{e}_i \cdot \hat{e}_j = \hat{e}_i \cdot \hat{e}_i = 1$, and when $j \neq i$, $\hat{e}_i \cdot \hat{e}_j = 0$.

Therefore $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$.

NOTE: Similarly, we can prove $\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij}$.

7.8 DIRECTION COSINES

Let $\alpha_1, \alpha_2, \alpha_3$ be the angles which the position vector $\vec{r}$ makes with the positive directions of Ox_1, Ox_2, Ox_3 respectively as shown in figure (7.4). Then the three quantities $\cos\alpha_1, \cos\alpha_2$, $\cos\alpha_3$ are called the direction cosines of the vector $\vec{r}$.

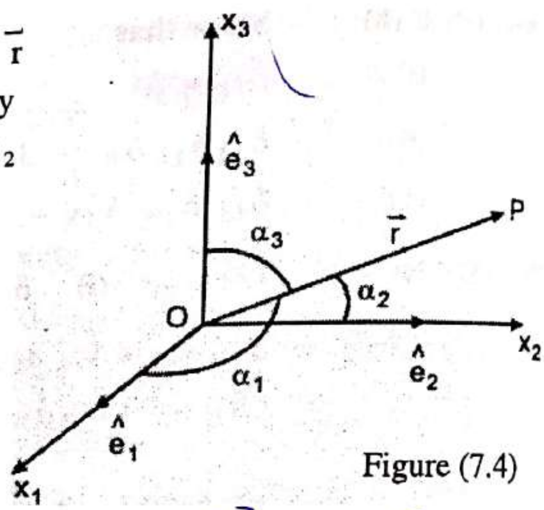


Figure (7.4)

For convenience, we write

$$\ell_1 = \cos\alpha_1, \quad \ell_2 = \cos\alpha_2, \quad \ell_3 = \cos\alpha_3$$

Now $\quad \vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 \qquad (1)$

and $\quad \vec{r} \cdot \hat{e}_1 = x_1 \quad$ or $\quad r\cos\alpha_1 = x_1$

Similarly, $r\cos\alpha_2 = x_2$, and $r\cos\alpha_3 = x_3$

Substitution for x_1, x_2, and x_3 in equation (1), gives

$$\vec{r} = r\cos\alpha_1\hat{e}_1 + r\cos\alpha_2\hat{e}_2 + r\cos\alpha_3\hat{e}_3$$

Therefore, a unit vector in the direction of $\vec{r}$ is $\hat{r} = \frac{\vec{r}}{r} = \cos\alpha_1\hat{e}_1 + \cos\alpha_2\hat{e}_2 + \cos\alpha_3\hat{e}_3$

Also from the above equations, $\cos\alpha_1 = \frac{x_1}{r}$, $\cos\alpha_2 = \frac{x_2}{r}$, and $\cos\alpha_3 = \frac{x_3}{r}$.

Hence $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = \frac{x_1^2 + x_2^2 + x_3^2}{r^2} = \frac{r^2}{r^2} = 1 \quad (\text{since } x_1^2 + x_2^2 + x_3^2 = r^2)$

or $\quad \ell_1^2 + \ell_2^2 + \ell_3^2 = 1$

Thus we have shown that the unit vector in the direction of $\vec{r}$ has its components the direction cosines of $\vec{r}$ and that the sum of the squares of the direction cosines is unity.

NOTE: The direction cosines of the x_1 – axis are 1, 0, 0. Similarly, the direction cosines of x_2 – axis are 0, 1, 0 and that of x_3 – axes are 0, 0, 1.

DEFINITION OF ℓ_{ij}

We define ℓ_{ij} to be the cosine of the angle between the ith – axis of the system K' and jth – axis of the system K, so that

$$\ell_{ij} = \cos(x_i', x_j) = \hat{e}_i' \cdot \hat{e}_j, \quad i, j = 1, 2, 3.$$

For example $\ell_{21} = \cos(x_2', x_1) = \hat{e}_2' \cdot \hat{e}_1 = \cos\theta$ where θ is the angle between x_2' – and x_1 – axes.

7.9 ORTHOGONAL ROTATION OF AXES

Consider two right–handed rectangular coordinate systems $Ox_1x_2x_3$ and $Ox'_1x'_2x'_3$ (i.e. systems K and K$'$) having the same origin O and with unit vectors along the coordinate axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ as shown in figure (7.5) . Rotate the system $Ox_1x_2x_3$ about O (with Ox_1, Ox_2, Ox_3 always fixed relative to each other) so that it coincides with the system $Ox'_1x'_2x'_3$. Such a movement is called a **rotation of axes** .

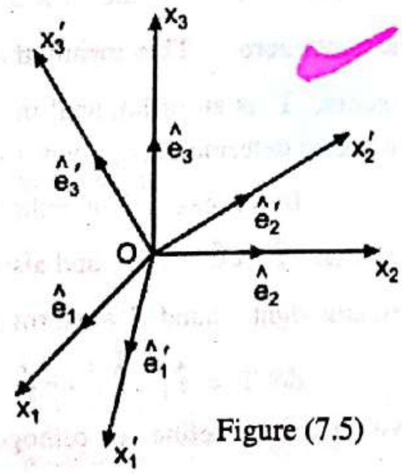


Figure (7.5)

Let the direction cosines of Ox'_1 relative to the axes Ox_1, Ox_2, Ox_3 be $\ell_{11}, \ell_{12}, \ell_{13}$ respectively . Furthermore , denote the direction cosines of Ox'_2 and Ox'_3 by $\ell_{21}, \ell_{22}, \ell_{23}$ and $\ell_{31}, \ell_{32}, \ell_{33}$, respectively . We may conveniently summarize this in the adjacent table . In this table , the direction cosines of Ox'_1 relative to the axes Ox_1, Ox_2, Ox_3 occur in the first row , the direction cosines of Ox'_2 occur in the second row , and those of Ox'_3 in the third row . Furthermore , reading down the three columns in turn , it is seen that we obtain the direction cosines of the axes Ox_1, Ox_2, Ox_3 relative to the axes Ox'_1, Ox'_2, Ox'_3 respectively .

	Ox_1	Ox_2	Ox_3
Ox'_1	ℓ_{11}	ℓ_{12}	ℓ_{13}
Ox'_2	ℓ_{21}	ℓ_{22}	ℓ_{23}
Ox'_3	ℓ_{31}	ℓ_{32}	ℓ_{33}

The table of direction cosines is called the **transformation matrix** and is written as

$$T = [\ell_{ij}] = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}$$

The transpose of T is $T' = [\ell_{ji}] = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix}$

and $TT' = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix}$

The element in the ith – row and jth – column of the product matrix TT' is the inner product of the ith – row of T with the jth – column of T', i.e.

$$[(TT')_{ij}] = \ell_{i1}\ell_{j1} + \ell_{i2}\ell_{j2} + \ell_{i3}\ell_{j3} = \ell_{ik}\,\ell_{jk}$$

$$= (\hat{e}_i'.\hat{e}_k)(\hat{e}_j'.\hat{e}_k) = (\hat{e}_i'.\hat{e}_k)(\hat{e}_k.\hat{e}_j')$$

$$= [(\hat{e}_i'.\hat{e}_k)\hat{e}_k].\hat{e}_j' \quad [\text{since } m(\vec{A}.\vec{B}) = m\vec{A}.\vec{B}]$$

$$= \hat{e}_i'.\hat{e}_j' = \delta_{ij} \quad [\text{since } (\vec{A}.\hat{e}_i)\hat{e}_i = \vec{A}]$$

Hence $TT' = [\delta_{ij}] = I$ (1)

The matrix $T\,T'$ is the 3×3 unit matrix I, its principle diagonal elements being unity and all other elements zero. This means that the transposed matrix T' is the inverse of T and so, as in matrix algebra, T is an orthogonal matrix. Since T, T' have the same determinants, the relation $T\,T' = I$, on taking determinants, gives $(\det T)^2 = 1$ so that $\det T = \pm 1$.

In the case of an orthogonal rotation of rectangular coordinate axes as shown in figure (7.5), we see that $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$ and also $\hat{e}_1' \times \hat{e}_2' = \hat{e}_3'$ meaning that a right – handed system $Ox_1x_2x_3$ remains right – handed when rotated to $Ox_1'x_2'x_3'$. Thus in this case,

$$\det T = \hat{e}_1' \cdot \hat{e}_2' \times \hat{e}_3' = \hat{e}_1' \cdot \hat{e}_1' = 1$$

We therefore define an orthogonal transformation to be one whose matrix $T = [\ell_{ij}]$, where $T\,T' = I$ and for which $\det T = \pm 1$. Such a transformation would leave a right – handed system of axes right – hand and would likewise preserve left – handedness. However, a right – handed orthogonal rotation of coordinate axes specifically requires that $\det T = +1$. This transformation is called **right – handed orthogonal (or proper)** transformation. A transformation which is not right – handed is called a **left – handed (or an improper)** transformation.

Now, when the axes $Ox_1x_2x_3$ and $Ox_1'x_2'x_3'$ coincide, i.e. $x_1' = x_1$, $x_2' = x_2$, $x_3' = x_3$, it is easily seen that the values of the direction cosines in the transformation matrix T are $\ell_{ij} = 1$ when $i = j$ and $\ell_{ij} = 0$ when $i \neq j$; and so for this particular case

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \det(T) = 1$$

Note that if one set of axes is right – handed and the other left – handed, it is impossible to bring them into coincidence by a rotation.

NOTE: Equation (1) can be written in full as :

$$\begin{bmatrix} \ell_{11}^2+\ell_{12}^2+\ell_{13}^2 & \ell_{11}\ell_{21}+\ell_{12}\ell_{22}+\ell_{13}\ell_{23} & \ell_{11}\ell_{31}+\ell_{12}\ell_{32}+\ell_{13}\ell_{33} \\ \ell_{21}\ell_{11}+\ell_{22}\ell_{12}+\ell_{23}\ell_{13} & \ell_{21}^2+\ell_{22}^2+\ell_{23}^2 & \ell_{21}\ell_{31}+\ell_{22}\ell_{32}+\ell_{23}\ell_{32} \\ \ell_{31}\ell_{11}+\ell_{32}\ell_{12}+\ell_{33}\ell_{13} & \ell_{31}\ell_{21}+\ell_{32}\ell_{22}+\ell_{32}\ell_{23} & \ell_{31}^2+\ell_{32}^2+\ell_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which implies the following six equations called the orthonormality conditions :

$$\left.\begin{aligned} \ell_{11}^2+\ell_{12}^2+\ell_{13}^2 &= 1 \\ \ell_{21}^2+\ell_{22}^2+\ell_{23}^2 &= 1 \\ \ell_{31}^2+\ell_{32}^2+\ell_{33}^2 &= 1 \end{aligned}\right\} \quad (2) \qquad \left.\begin{aligned} \ell_{11}\ell_{21}+\ell_{12}\ell_{22}+\ell_{13}\ell_{23} &= 0 \\ \ell_{21}\ell_{31}+\ell_{22}\ell_{32}+\ell_{23}\ell_{33} &= 0 \\ \ell_{31}\ell_{11}+\ell_{32}\ell_{12}+\ell_{33}\ell_{13} &= 0 \end{aligned}\right\} \quad (3)$$

EXAMPLE (9): **Show that the transformation**

(i) $$T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$ **is orthogonal and right – handed .**

On the other hand , the transformation .

(ii) $$T = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$$ **is orthogonal but left – handed .**

SOLUTION: (i) The transpose of the given matrix T is $T' = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix}$

and so $$TT' = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

We note that $$\det T = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = 1$$

and so the corresponding transformation is orthogonal and right – handed .

(ii) The transpose of the given matrix is $T' = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$

and so $$TT' = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

We note that $\det T = \begin{vmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{vmatrix} = -1$

and so the corresponding transformation is orthogonal but left – handed .

7.10 PROPER AND IMPROPER TRANSFORMATIONS

Since for an orthogonal transformation , $\det T = \pm 1$, all coordinate transformations are divided into two classes . One class consists of those transformations for which $\det T = 1$ and are called **proper** transformations ; the other class consists of transformations for which $\det T = -1$ are called **improper** transformations . Under a proper transformation , a right – handed (or left – handed) system remains right – handed (or left – handed) after rotation . Under an improper transformation , a right – handed system is changed into a left – handed system and vice–versa . The transformation for which $T = I$, the unit matrix , is called the **identity** transformation . Obviously , for the identity transformation

$$x'_1 = x_1 , \quad x'_2 = x_2 , \quad x'_3 = x_3$$

or $\quad x'_i = x_i \quad (i = 1, 2, 3)$

The proper transformation can be obtained from the identity transformation by continuously rotating the coordinate axes . On the other hand , the improper transformations cannot be obtained by that process . The improper transformation can be obtained from the identity transformation by two types of discontinuous or discrete operations .

(i) REFLECTION: This is the operation in which the new coordinate system $Ox'_1 x'_2 x'_3$ is obtained from the original system $Ox_1 x_2 x_3$ by inverting (reversing) the direction of one of the axes of the latter , the other two remaining in their original position as shown in figure (7.6) .

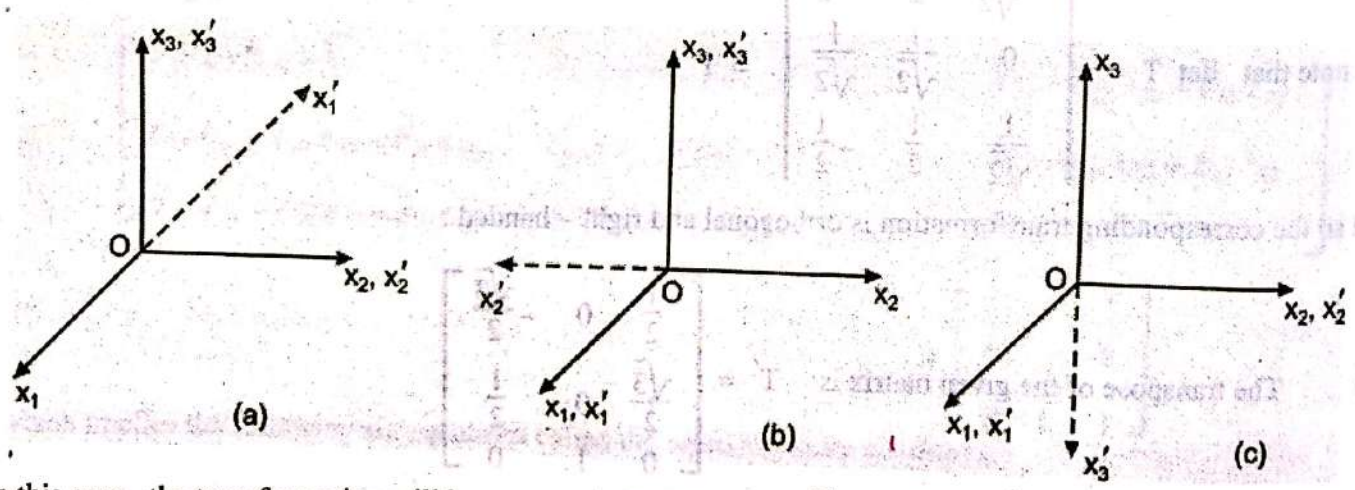


Figure (7.6)

In this case , the transformation will be

$$x'_1 = -x_1 , \quad x'_2 = x_2 , \quad x'_3 = x_3$$

or $\quad x'_1 = x_1 , \quad x'_2 = -x_2 , \quad x'_3 = x_3$

or $\quad x'_1 = x_1 , \quad x'_2 = x_2 , \quad x'_3 = -x_3$

The corresponding transformation matrix T will be

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that for each of these matrices del $T = -1$.

(ii) **INVERSION:** This is the operation in which the new coordinate system $Ox'_1 x'_2 x'_3$ is obtained from the original system $Ox_1 x_2 x_3$ by inverting the directions of all the coordinate axes of the latter as shown in figure (7.7). The transformation equations in this case will be

$$x'_1 = -x_1, \quad x'_2 = -x_2, \quad x'_3 = -x_3$$

The corresponding transformation matrix T in this case will be

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that for each of these matrices del $T = -1$.

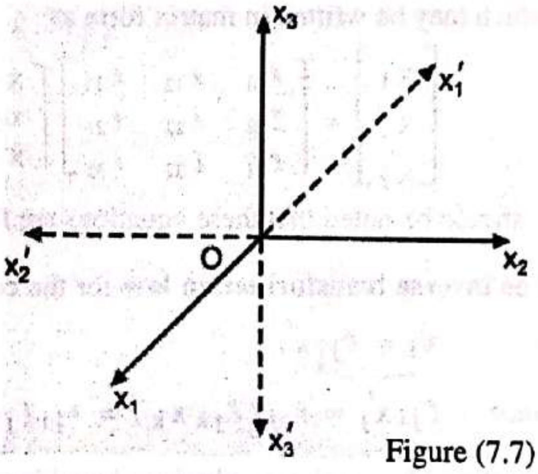


Figure (7.7)

7.11 TRANSFORMATION EQUATIONS

(a) TRANSFORMATION EQUATIONS FOR COORDINATES OF A POINT

Consider two rectangular coordinate systems K and K′ having the same origin O as shown in figure (7.8). We first find the transformation equations expressing the coordinates x'_1, x'_2, x'_3 of an arbitrary point P in the system K′ in terms of its coordinates x_1, x_2, x_3 in the system K, and vice versa. Let $\vec{r}$ and $\vec{r}'$ be the position vectors of any point P in the systems K and K′ respectively. Then $\vec{r}' = \vec{r}$

or $$x'_1 \hat{e}'_1 + x'_2 \hat{e}'_2 + x'_3 \hat{e}'_3 = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3$$

or $$x'_j \hat{e}'_j = x_i \hat{e}_i, \quad i, j = 1, 2, 3 \qquad (1)$$

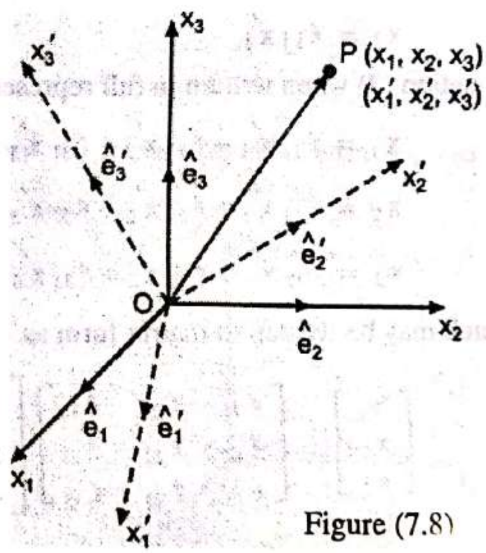


Figure (7.8)

For the components x'_j, take the dot product of equation (1) with $\hat{e}'_j$, we have

$$x'_j (\hat{e}'_j \cdot \hat{e}'_j) = x_i (\hat{e}_i \cdot \hat{e}'_j)$$

$$= (\hat{e}'_j \cdot \hat{e}_i) x_i$$

$$x'_j = \ell_{ji} x_i \qquad (2)$$

where ℓ_{ji} are the direction cosines of the jth – axis of the system K′ with the ith – axis of the system K. Equation (2) is called the equation of transformation for the coordinates of the point P from the system K to the system K′.

Equation (2) when written in full represents the following three equations:

$$x_1' = \ell_{11} x_1 + \ell_{12} x_2 + \ell_{13} x_3$$

$$x_2' = \ell_{21} x_1 + \ell_{22} x_2 + \ell_{23} x_3$$

$$x_3' = \ell_{31} x_1 + \ell_{32} x_2 + \ell_{33} x_3$$

which may be written in matrix form as

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It should be noted that these equations are formed from the transformation matrix .

The **inverse transformation law** for the coordinates of the point P from the system K' to the system K is

$$x_i = \ell_{ji} x_j' \qquad (3)$$

since $\ell_{ji} x_j' = \ell_{ji}(\ell_{jk} x_k) = \ell_{ji}\ell_{jk} x_k = \delta_{ik} x_k = x_i$

Equation (3) can also be obtained by taking the dot product of equation (1) with $\hat{e}_i$, since

$$x_j' (\hat{e}_j' . \hat{e}_i) = x_i (\hat{e}_i . \hat{e}_i)$$

or $\quad x_i = \ell_{ji} x_j'$

Equation (3) can equivalently be written as

$$x_j = \ell_{ij} x_i'$$

Equation (3) when written in full represents the following three equations :

$$x_1 = \ell_{11} x_1' + \ell_{21} x_2' + \ell_{31} x_3'$$

$$x_2 = \ell_{12} x_1' + \ell_{22} x_2' + \ell_{32} x_3'$$

$$x_3 = \ell_{13} x_1' + \ell_{23} x_2' + \ell_{33} x_3'$$

which may be written in matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

(b) TRANSFORMATION EQUATIONS FOR UNIT VECTORS

We next find the transformation equations expressing the unit vectors $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ in the system K' in terms of the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ of the system K and vice versa .

From vector analysis , we know that

$$\vec{A} = (\vec{A} . \hat{e}_1)\hat{e}_1 + (\vec{A} . \hat{e}_2)\hat{e}_2 + (\vec{A} . \hat{e}_3)\hat{e}_3$$

$$= (\vec{A} . \hat{e}_i)\hat{e}_i$$

Setting $\vec{A} = \hat{e}_j'$, we get $\quad \hat{e}_j' = (\hat{e}_j' . \hat{e}_i)\hat{e}_i = \ell_{ji}\hat{e}_i \qquad (4)$

Equation (4) when written in full represents the following three equations :

$$\hat{e}_1' = \ell_{11}\hat{e}_1 + \ell_{12}\hat{e}_2 + \ell_{13}\hat{e}_3$$

$$\hat{e}_2' = \ell_{21}\hat{e}_1 + \ell_{22}\hat{e}_2 + \ell_{23}\hat{e}_3$$

$$\hat{e}_3' = \ell_{31}\hat{e}_1 + \ell_{32}\hat{e}_2 + \ell_{33}\hat{e}_3$$

Similarly, from $\vec{A} = (\vec{A}\cdot\hat{e}_1')\hat{e}_1' + (\vec{A}\cdot\hat{e}_2')\hat{e}_2' + (\vec{A}\cdot\hat{e}_3')\hat{e}_3'$

$$= (\vec{A}\cdot\hat{e}_i')\hat{e}_i'$$

Setting $\vec{A} = \hat{e}_j$, we get $\hat{e}_j = (\hat{e}_j\cdot\hat{e}_i')\hat{e}_i' = \ell_{ij}\hat{e}_i'$ (5)

Equation (5) when written in full represents the following three equations:

$$\hat{e}_1 = \ell_{11}\hat{e}_1' + \ell_{21}\hat{e}_2' + \ell_{31}\hat{e}_3'$$

$$\hat{e}_2 = \ell_{12}\hat{e}_1' + \ell_{22}\hat{e}_2' + \ell_{32}\hat{e}_3'$$

$$\hat{e}_3 = \ell_{13}\hat{e}_1' + \ell_{23}\hat{e}_2' + \ell_{33}\hat{e}_3'$$

Equations (4) and (5) are the required transformation equations .

EXAMPLE (10): **A set of axes $Ox_1'x_2'x_3'$ is initially coincident with a set $Ox_1x_2x_3$. The set $Ox_1'x_2'x_3'$ is then rotated through an angle θ in the positive sense about the x_3– axis . Show that**

$$x_1' = x_1\cos\theta + x_2\sin\theta$$

$$x_2' = -x_1\sin\theta + x_2\cos\theta$$

$$x_3' = x_3$$

SOLUTION: Since the rotation is about x_3 – axis , therefore x_3' – axis coincides with x_3 – axis as shown in the figure (7.9) . If the angle $x_1'Ox_1 = x_2'Ox_2 = \theta$, then

$$\ell_{11} = \cos(x_1'Ox_1) = \cos\theta$$

$$\ell_{12} = \cos(x_1'Ox_2) = \cos(90-\theta) = \sin\theta$$

$$\ell_{13} = \cos(x_1'Ox_3) = \cos 90^o = 0$$

$$\ell_{21} = \cos(x_2'Ox_1) = \cos(90+\theta) = -\sin\theta$$

$$\ell_{22} = \cos(x_2'Ox_2) = \cos\theta$$

$$\ell_{23} = \cos(x_2'Ox_3) = \cos 90^o = 0$$

$$\ell_{31} = \cos(x_3'Ox_1) = \cos 90^o = 0$$

$$\ell_{32} = \cos(x_3'Ox_2) = \cos 90^o = 0$$

$$\ell_{33} = \cos(x_3'Ox_3) = \cos 0^o = 1$$

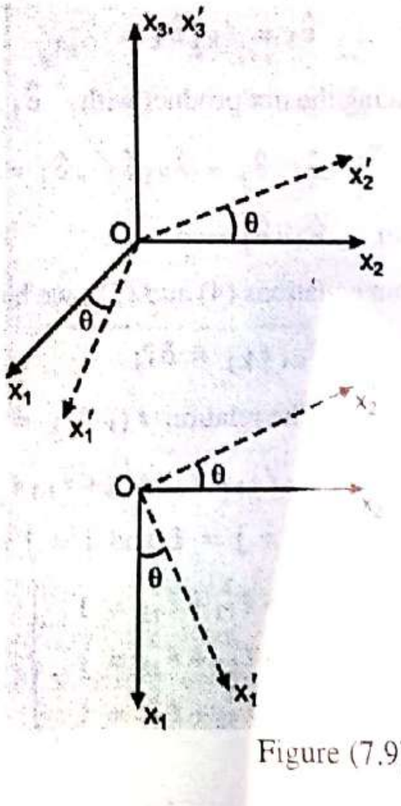


Figure (7.9)

The transformation matrix is given by

$$[\ell_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and thus the transformation equations for the coordinates become

$$x_1' = x_1 \cos\theta + x_2 \sin\theta$$

$$x_2' = -x_1 \sin\theta + x_2 \cos\theta$$

$$x_3' = x_3$$

Note that effectively we are dealing with a transformation of axes in two–dimensions only and we can write the transformation matrix as

$$[\ell_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

7.12 ORTHONORMALITY CONDITIONS

THEOREM (7.5): **Prove that** $\ell_{ik}\ell_{jk} = \delta_{ij} = \ell_{ki}\ell_{kj}$.

PROOF: We know that $\hat{e}_j' = \ell_{ji}\hat{e}_i$

or $\hat{e}_j' = \ell_{jk}\hat{e}_k$

Taking the dot product with $\hat{e}_i'$, we get

$$\hat{e}_i' \cdot \hat{e}_j' = \ell_{jk}\hat{e}_i' \cdot \hat{e}_k = \ell_{ik}\ell_{jk} \qquad (1)$$

Also, $\hat{e}_i' \cdot \hat{e}_j' = \delta_{ij}$ (2)

From equations (1) and (2), we have

$$\ell_{ik}\ell_{jk} = \delta_{ij} \qquad (3)$$

Similarly, we know that $\hat{e}_j = \ell_{ij}\hat{e}_i'$

or $\hat{e}_i = \ell_{ki}\hat{e}_k'$

Taking the dot product with $\hat{e}_j$, we get

$$\hat{e}_i \cdot \hat{e}_j = \ell_{ki}\hat{e}_k' \cdot \hat{e}_j = \ell_{ki}\ell_{kj} \qquad (4)$$

Also $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ (5)

From equations (4) and (5), we have

$$\ell_{ki}\ell_{kj} = \delta_{ij} \qquad (6)$$

NOTE: The relation $\ell_{ik}\ell_{jk} = \delta_{ij}$ implies six orthonormality conditions. Write the relation as

$$\ell_{i1}\ell_{j1} + \ell_{i2}\ell_{j2} + \ell_{i3}\ell_{j3} = \delta_{ij} \qquad (A)$$

If we take $i = j = 1$ and $i = j = 2$ and $i = j = 3$ in turn in relation (A), we get

$$\left.\begin{aligned} \ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 &= 1 \\ \ell_{21}^2 + \ell_{22}^2 + \ell_{32}^2 &= 1 \\ \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 &= 1 \end{aligned}\right\}$$

If we take $i = 1, j = 2$ and $i = 2, j = 3$ and $i = 3, j = 1$ in turn in relation (A), we get

$$\left.\begin{aligned} \ell_{11}\ell_{21}+\ell_{12}\ell_{22}+\ell_{13}\ell_{23} &= 0 \\ \ell_{21}\ell_{31}+\ell_{22}\ell_{32}+\ell_{23}\ell_{33} &= 0 \\ \ell_{31}\ell_{11}+\ell_{32}\ell_{12}+\ell_{33}\ell_{13} &= 0 \end{aligned}\right\} \qquad (2)$$

Similarly, the relation $\ell_{ki}\ell_{kj} = \delta_{ij}$ implies alternative form of orthonormality conditions. Write the relation as

$$\ell_{1i}\ell_{1j}+\ell_{2i}\ell_{2j}+\ell_{3i}\ell_{3j} = \delta_{ij} \qquad (B)$$

If we take $i = j = 1$ and $i = j = 2$ and $i = j = 3$ in turn in relation (B), we get

$$\left.\begin{aligned} \ell_{11}^2+\ell_{21}^2+\ell_{31}^2 &= 1 \\ \ell_{12}^2+\ell_{22}^2+\ell_{32}^2 &= 1 \\ \ell_{13}^2+\ell_{23}^2+\ell_{33}^2 &= 1 \end{aligned}\right\} \qquad (3)$$

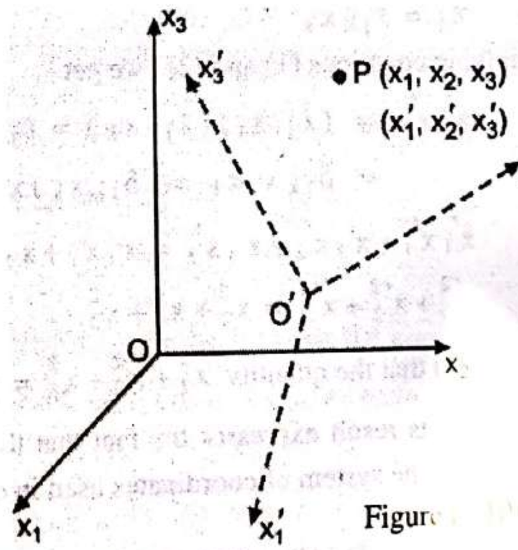
If we take $i = 1, j = 2$ and $i = 2, j = 3$ and $i = 3, j = 1$ in turn in relation (B), we get

$$\left.\begin{aligned} \ell_{11}\ell_{12}+\ell_{21}\ell_{22}+\ell_{31}\ell_{32} &= 0 \\ \ell_{12}\ell_{13}+\ell_{22}\ell_{23}+\ell_{32}\ell_{33} &= 0 \\ \ell_{13}\ell_{11}+\ell_{23}\ell_{21}+\ell_{33}\ell_{31} &= 0 \end{aligned}\right\} \qquad (4)$$

It should be observed how the orthonormality conditions given in equations (1), (2), (3), and (4) are formed from the transformation matrix.

7.13 TRANSLATION AND ROTATION

Let the coordinates of a point P be (x_1, x_2, x_3) in the system $Ox_1x_2x_3$ and the coordinates of the point O' in this system be (a_1, a_2, a_3). Shift the system $Ox_1x_2x_3$ to the positions $O'x'_1x'_2x'_3$ through O' as new origin to form a new rectangular coordinate system as shown in figure (7.10). Let the coordinate of P be (x'_1, x'_2, x'_3) in this new system. We know that under the translation only, the axes remain parallel through O', so that the coordinates of P are $(x_i - a_i)$, $i = 1, 2, 3$. Under orthogonal transformation, these become x'_j so that



Figure

$$\begin{aligned} x'_j &= \ell_{ji}(x_i - a_i) \\ &= \ell_{ji}x_i + d_j \end{aligned} \qquad (1)$$

where $d_j = -\ell_{ji}a_i$

Equation (1) illustrates the transformation of coordinates under translation of axes followed by rotation but preserving the right – handed rectangular character of the axes.

7.14 INVARIANCE WITH RESPECT TO ROTATION OF AXES

Consider two rectangular coordinate systems K and K' having the same origin O but with axes rotated with respect to each other as shown in figure (7.8). Let (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) be the coordinates of an arbitrary point P in the systems K and K' respectively. Let $\phi(x_1, x_2, x_3)$ be the value of the scalar point function ϕ at P in the system K and $\phi'(x'_1, x'_2, x'_3)$ be the value of this function at the same point in the system K'.

If $\phi(x_1, x_2, x_3) = \phi'(x'_1, x'_2, x'_3)$ where x_1, x_2, x_3 and x'_1, x'_2, x'_3 are related by the transformation equations (2) or (3) on pages (405) and (406), then $\phi(x_1, x_2, x_3)$ is called an **invariant** with respect to the coordinate transformation or rotation of axes.

Similarly, a vector point function $\vec{A}(x_1, x_2, x_3)$ is called **invariant** with respect to rotation of axes if $\vec{A}(x_1, x_2, x_3) = \vec{A}'(x'_1, x'_2, x'_3)$. This will be true if

$$A_1(x_1, x_2, x_3)\hat{e}_1 + A_2(x_1, x_2, x_3)\hat{e}_2 + A_3(x_1, x_2, x_3)\hat{e}_3$$
$$= A'_1(x'_1, x'_2, x'_3)\hat{e}_1{}' + A'_2(x'_1, x'_2, x'_3)\hat{e}_2{}' + A'_3(x'_1, x'_2, x'_3)\hat{e}_3{}'$$

EXAMPLE (11): **Show that the quantity $x_1^2 + x_2^2 + x_3^2 = x_i x_i$ is invariant under a rotation of axes.**

SOLUTION: Let (x_1, x_2, x_3) be the coordinates of a point P in the system K and (x'_1, x'_2, x'_3) be the coordinates of the same point in the system K'. Then we know that the transformation equations are:

$$x'_j = \ell_{ji} x_i \qquad (1)$$

Also $$x'_j = \ell_{jk} x_k \qquad (2)$$

Multiplying equations (1) and (2), we get

$$x'_j x'_j = (\ell_{ji} x_i)(\ell_{jk} x_k) = \ell_{ji}\ell_{jk} x_i x_k$$
$$= \delta_{ik} x_i x_k = (\delta_{ik} x_k) x_i = x_i x_i$$

or $$x'_1 x'_1 + x'_2 x'_2 + x'_3 x'_3 = x_1 x_1 + x_2 x_2 + x_3 x_3$$

or $$x'^2_1 + x'^2_2 + x'^2_3 = x_1^2 + x_2^2 + x_3^2$$

which shows that the quantity $x_1^2 + x_2^2 + x_3^2 = x_i x_i$ is invariant with respect to rotation of axes.

NOTE: This result expresses the fact that the distance between the origin O and a point P does not depend upon the system of coordinates used in calculating the distance.

7.15 SCALAR INVARIANT OPERATORS

Let $\mathcal{D}$ denote a linear partial differential operator which involves only the rectangular Cartesian coordinates x_1, x_2, x_3 as independent variables. For example, we might have

$$\mathcal{D} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial x_3} \quad \text{or} \quad \mathcal{D} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} + \frac{\partial^2}{\partial x_2 \partial x_3}$$

The operator $\mathcal{D}$ is called a scalar **invariant operator** if its form is unchanged under a rotation of the coordinate axes. Thus, for example, if the first of the operators above is invariant (it is not, infact), then upon changing to new axes $Ox'_1 x'_2 x'_3$ it would become

$$\mathcal{D}' = \frac{\partial}{\partial x'_1} + \frac{\partial}{\partial x'_2} + 2\frac{\partial}{\partial x'_3}$$

The following theorem concerning scalar invariant operators will be required.

THEOREM (7.6): Let $\mathcal{D}$ be a scalar invariant operator, and define its operation on a vector field $\vec{A}$ by $\mathcal{D}\vec{A} = \mathcal{D}(A_1, A_2, A_3) = (\mathcal{D}A_1, \mathcal{D}A_2, \mathcal{D}A_3)$

Then prove that $\mathcal{D}\vec{A}$ is a vector field.

PROOF: Let A_i and A'_j be the components of $\overline{A}$ in the system $Ox_1 x_2 x_3$ and $Ox'_1 x'_2 x'_3$ respectively, then

$$A'_j = \ell_{ji} A_i$$

Using the property of invariance of the form of $\mathcal{D}$ and also the linearity property of $\mathcal{D}$, we have

$$\mathcal{D}' A'_j = \mathcal{D}(\ell_{ji} A_i) = \ell_{ji}\, \mathcal{D}A_i$$

showing that the components of $\mathcal{D}\vec{A}$ transform according to the vector law under the rotation of the axes.

Thus it follows that $\mathcal{D}\vec{A}$ is a vector field.

THE LAPLACIAN OPERATOR ∇^2

The most important of the scalar invariant operators is the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$$

Formally, the Laplacian operator is the square of the del operator.

THEOREM (7.7): Prove that the Laplacian operator $\nabla^2 = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$ is invariant under a rotation of the axes.

PROOF: The invariance of the Laplacian operator follows from the fact that the components of del–operator transforms as a vector. Under a rotation of coordinate axes $Ox'_1 x'_2 x'_3$, we have

$$\frac{\partial}{\partial x'_j} = \ell_{ji}\frac{\partial}{\partial x_i} \quad \text{and} \quad \frac{\partial}{\partial x'_j} = \ell_{jk}\frac{\partial}{\partial x_k}$$

Thus $$\frac{\partial}{\partial x'_j}\frac{\partial}{\partial x'_j} = \left(\ell_{ji}\frac{\partial}{\partial x_i}\right)\left(\ell_{jk}\frac{\partial}{\partial x_k}\right)$$

$$= \ell_{ji}\ell_{jk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} = \delta_{ik}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}$$

which shows that the Laplacian operator ∇^2 is invariant under a rotation of the axes.

7.16 THE ALTERNATING SYMBOL ϵ_{ijk}

The alternating symbol written ϵ_{ijk} is defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is a cyclic permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ is an anticyclic permutation of } 1, 2, 3 \\ 0 & \text{if any two of } i, j, k \text{ are equal} \end{cases}$$

Thus $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$

$\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$

$\epsilon_{223} = \epsilon_{131} = \epsilon_{313} = \ldots = 0$

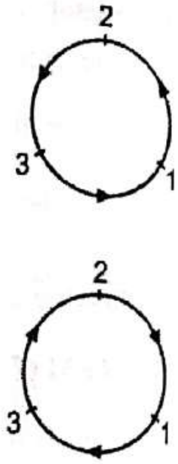


Figure (7.11)

RELATIONSHIP BETWEEN ALTERNATING SYMBOL AND KRONECKER DELTA

THEOREM (7.8): **Prove that $\epsilon_{ijk}\,\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$**

PROOF: We know that

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2 \qquad (1)$$

Using the definition of the alternating symbol we can write equation (1) as

$$\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k = \epsilon_{ijk}$$

Thus
$$\begin{aligned} \epsilon_{ijk}\,\epsilon_{\ell mk} &= (\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k)(\hat{e}_\ell \times \hat{e}_m \cdot \hat{e}_k) \\ &= (\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k)\hat{e}_k \cdot (\hat{e}_\ell \times \hat{e}_m) \\ &= (\hat{e}_i \times \hat{e}_j) \cdot (\hat{e}_\ell \times \hat{e}_m) \quad [\text{since } (\vec{A} \cdot \hat{e}_k)\hat{e}_k = \vec{A}\,] \\ &= (\hat{e}_i \times \hat{e}_j) \times \hat{e}_\ell \cdot \hat{e}_m \\ &= [(\hat{e}_i \cdot \hat{e}_\ell)\hat{e}_j - (\hat{e}_j \cdot \hat{e}_\ell)\hat{e}_i] \cdot \hat{e}_m \\ &= (\hat{e}_i \cdot \hat{e}_\ell)(\hat{e}_j \cdot \hat{e}_m) - (\hat{e}_j \cdot \hat{e}_\ell)(\hat{e}_i \cdot \hat{e}_m) \\ &= \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \end{aligned}$$

ALTERNATIVE METHOD

We have to prove

$$\epsilon_{ij1}\,\epsilon_{\ell m1} + \epsilon_{ij2}\,\epsilon_{\ell m2} + \epsilon_{ij3}\,\epsilon_{\ell m3} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \qquad (A)$$

Case (1) **(a) when $i = j$ but $\ell \neq m$** **(b) when $i \neq j$ but $\ell = m$**

For these cases we may easily verify from equation (A) that L.H.S. = 0 = R.H.S.

Case (2) When $i \neq j$, $\ell \neq m$ and the pairs (i, j) and (ℓ, m) are different from each other.

For example, $(i, j) = (1, 2)$ and $(\ell, m) = (1, 3)$, then from equation (A)

$$\epsilon_{121}\,\epsilon_{131} + \epsilon_{122}\,\epsilon_{132} + \epsilon_{123}\,\epsilon_{133} = \delta_{11}\delta_{23} - \delta_{13}\delta_{21}$$

or L.H.S. = 0 = R.H.S.

Case (3) When $i \neq j$, $\ell \neq m$ but the pairs (i, j) and (ℓ, m) are identical. Thus i, j and ℓ, m can have the following pairs of values in any order :

$$1,2;\quad 1,3;\quad 2,3;\quad 2,1;\quad 3,1;\quad 3,2$$

The first pair gives rise to the following possibilities :

$$i = 1,\ j = 2,\quad \ell = 1,\ m = 2$$

$$i = 1,\ j = 2,\quad \ell = 2,\ m = 1$$

$$i = 2,\ j = 1,\quad \ell = 1,\ m = 2$$

$$i = 2,\ j = 1,\quad \ell = 2,\ m = 1$$

For these possibilities, equation (A) gives

L.H.S. = 1 = R.H.S.

L.H.S. = − 1 = R.H.S.

L.H.S. = − 1 = R.H.S.

L.H.S. = 1 = R.H.S.

The result may easily be seen to be true for the other five pairs also .

Hence equation (A) holds for all possible values of i, j, ℓ and m.

NOTE: Each side of the theorem is a tensor of order 4, therefore equation (A) is equivalent to a set of $3^4 = 81$ scalar equations .

EXAMPLE (12): **Prove that**

(i) $\epsilon_{ijk}\delta_{jk} = 0$ (ii) $\epsilon_{ijk}\epsilon_{ijk} = 6$

(iii) $\epsilon_{ijk}\epsilon_{\ell jk} = 2\delta_{i\ell}$ (iv) $\epsilon_{ijk}\epsilon_{\ell mk}\delta_{jm} = 2\delta_{i\ell}$

(v) $\epsilon_{iks}\epsilon_{mps} = \epsilon_{sik}\epsilon_{smp} = \epsilon_{ksi}\epsilon_{psm}$ (vi) $\frac{1}{2}\epsilon_{ijk}\epsilon_{ij\ell}A_\ell = A_k$

SOLUTION: We know that

(i) $\epsilon_{ijk}\delta_{jk} = \epsilon_{ijj} = \epsilon_{i11} + \epsilon_{i22} + \epsilon_{i33} = 0 + 0 + 0 = 0$

(ii) We know that

$$\epsilon_{ijk}\epsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell} \quad \text{(A)}$$

Setting $\ell = i$ and $m = j$ in the above relation

$$\epsilon_{ijk}\epsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} = (3)(3) - \delta_{ii} = 9 - 3 = 6$$

(iii) Put $m = j$ in relation (A) we get

$$\epsilon_{ijk}\epsilon_{\ell jk} = \delta_{i\ell}\delta_{jj} - \delta_{ij}\delta_{j\ell} = 3\delta_{i\ell} - \delta_{i\ell} = 2\delta_{i\ell}$$

(iv)
$$\begin{aligned}\epsilon_{ijk}\epsilon_{\ell mk}\delta_{jm} &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\delta_{jm}\\ &= \delta_{i\ell}\delta_{jm}\delta_{jm} - \delta_{im}\delta_{j\ell}\delta_{jm}\\ &= \delta_{i\ell}\delta_{mm} - \delta_{im}\delta_{m\ell} = 3\delta_{i\ell} - \delta_{i\ell} = 2\delta_{i\ell}\end{aligned}$$

(v) $\epsilon_{ijs}\epsilon_{mps} = \epsilon_{sik}\epsilon_{smp} = \epsilon_{ksi}\epsilon_{psm}$

A cyclic permutation of the suffixes in ϵ_{ijk} does not change its value .

(vi) $\frac{1}{2}\epsilon_{ijk}\epsilon_{ij\ell}A_\ell = \frac{1}{2}(2\delta_{k\ell})A_\ell = A_k$ (since $\epsilon_{ijk}\epsilon_{\ell jk} = 2\delta_{i\ell}$ from equation (3))

7.17 TENSORS

We know that a scalar is a quantity whose specification (in any coordinate system) requires just one number . On the other hand , a vector is a quantity whose specification in any coordinate system requires three numbers called its **components** . Scalars and vectors are both special cases of a more general object called a **tensor** of order n , whose specification in any given coordinate system requires 3^n numbers, again called the components of the tensor . In fact , scalars are tensors of order 0 , with $3^0 = 1$ component , and vectors are tensors of order 1 with $3^1 = 3$ components . The order or rank of a tensor is effectively the number of suffixes used in it .

ZEROTH – ORDER TENSORS (OR SCALARS)

By a scalar (or zeroth order tensor) is meant a quantity uniquely specified in any coordinate system by a single real number (the component or value of the scalar) which is invariant under changes of the coordinate system i.e. which does not change when the coordinate system is changed . Thus if ϕ is the value of a scalar in the coordinate system K and ϕ' its value in another coordinate system K', then

$$\phi = \phi'$$

or $$\phi(x_1, x_2, x_3) = \phi'(x'_1, x'_2, x'_3)$$

Thus a tensor of order zero is called a **scalar invariant** .

FIRST ORDER TENSORS (OR VECTORS)

An entity (quantity) representable by a set A_i of three (i.e. 3^1) numbers (called components) relatively to a coordinate system K is called a first order tensor , if its components transform under changes of the coordinate system according to the law

$$A'_j = \ell_{ji} A_i \tag{1}$$

where A'_j are the components of the quantity in the coordinate system K' and ℓ_{ji} is the cosine of the angle between the jth – axis of K' and the ith – axis of K . Equation (1) is called the **transformation law** for the components of first order tensor i.e. vector . Equation (1) represents the following three equations :

$$\begin{aligned} A'_1 &= \ell_{11}A_1 + \ell_{12}A_2 + \ell_{13}A_3 \\ A'_2 &= \ell_{21}A_1 + \ell_{22}A_2 + \ell_{23}A_3 \\ A'_3 &= \ell_{31}A_1 + \ell_{32}A_2 + \ell_{33}A_3 \end{aligned} \tag{2}$$

which may be written in matrix form as

$$\begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \tag{3}$$

The **inverse transformation law** for the components of the first order tensor in the system K in terms of its components in the system K' is

$$A_i = \ell_{ji} A'_j \qquad (4)$$

since $\ell_{ji} A'_j = \ell_{ji}(\ell_{jk} A_k) = \ell_{ji}\ell_{jk} A_k = \delta_{ik} A_k = A_i$

Equation (4) may equivalently be written as

$$A_j = \ell_{ij} A'_i \qquad (5)$$

Equation (4) or (5) represents the following three equations :

$$\begin{aligned} A_1 &= \ell_{11} A'_1 + \ell_{21} A'_2 + \ell_{31} A'_3 \\ A_2 &= \ell_{12} A'_1 + \ell_{22} A'_2 + \ell_{32} A'_3 \\ A_3 &= \ell_{13} A'_1 + \ell_{23} A'_2 + \ell_{33} A'_3 \end{aligned} \qquad (6)$$

which may be written in matrix form as

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \begin{bmatrix} A'_1 \\ A'_2 \\ A'_3 \end{bmatrix} \qquad (7)$$

SECOND ORDER TENSORS

A quantity representable by a two suffix set A_{ij} of nine (i.e. 3^2) numbers (called components) relatively to a coordinate system K is called a second order tensor , if its components transform under changes of the coordinate system according to the law

$$A'_{mn} = \ell_{mi}\ell_{nj} A_{ij} \qquad (1)$$

where A'_{mn} are the components of the quantity in the coordinate system K' and ℓ_{mi} is the cosine of the angle between the mth – axis of K' and the ith – axis of K . (Similarly for ℓ_{nj})

The **inverse transformation law** expressing the components of the second order tensor in the system K in terms of its components in the system K' is :

$$A_{ij} = \ell_{mi}\ell_{nj} A'_{mn} \qquad (3)$$

since
$$\begin{aligned} \ell_{mi}\ell_{nj} A'_{mn} = \ell_{mi}\ell_{nj}(\ell_{mr}\ell_{ns} A_{rs}) &= (\ell_{mi}\ell_{mr})(\ell_{nj}\ell_{ns}) A_{rs} \\ &= \delta_{ir}\delta_{js} A_{rs} = (\delta_{ir} A_{rs})\delta_{js} \\ &= A_{is}\delta_{js} = A_{ij} \end{aligned}$$

Equation (3) can be written equivalently as $A_{mn} = \ell_{im}\ell_{jn} A'_{ij}$

NOTE: (i) The nine components of a second order tensor can be written in the form of a 3

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

or briefly $[A_{ij}]$, where A_{ij} is the element in the ith – row and jth – column of the above matrix .

(ii) Equation (1) when written in matrix form becomes

$$\begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ A'_{21} & A'_{22} & A'_{23} \\ A'_{31} & A'_{32} & A'_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} \quad (2)$$

or $[A'] = [T][A][T']$

Equation (2) is easier to use than equation (1) itself

(iii) Given the components of a second order tensor in the coordinate system K, we can use equation (1) to determine its components in another coordinate system K'. In particular, if all the components of a tensor vanish in one coordinate system, they also vanish in any other coordinate system.

EXAMPLE (13): **Prove that if A_i and B_j are two first order tensors i.e. vectors, then their product $A_i B_j$ $(i, j = 1, 2, 3)$ is a second order tensor.**

SOLUTION: Let $C_{ij} = A_i B_j$ (1)

then we have to prove that C_{ij} $(i, j = 1, 2, 3)$ are the components of a second order tensor. Since A_i and B_j are the first order tensors, their equations of transformation from the system K to K' are

$$A'_m = \ell_{mi} A_i \quad (2)$$

$$B'_n = \ell_{nj} B_j \quad (3)$$

Multiplying equations (2) and (3), we obtain

$$A'_m B'_n = \ell_{mi} \ell_{nj} A_i B_j \quad (4)$$

or $$C'_{mn} = \ell_{mi} \ell_{nj} C_{ij} \quad (5)$$

where $C'_{mn} = A'_m B'_n$

Equation (5) shows that $C_{ij} = A_i B_j$ are the components of a second order tensor.

9-10-2019

THEOREM (7.9): **Prove that the Kronecker delta δ_{ij} is a Cartesian tensor of rank 2.**

PROOF: Let δ_{ij} and δ'_{mn} be the components of the Kronecker delta in the systems K and K' respectively. Then $\delta_{ij} = \hat{e}_i . \hat{e}_j$ and $\delta'_{mn} = \hat{e}_m{}' . \hat{e}_n{}'$

Also we know that $\hat{e}_m{}' = \ell_{mi} \hat{e}_i$ and $\hat{e}_n{}' = \ell_{nj} \hat{e}_j$

so $$\delta'_{mn} = \hat{e}_m{}' . \hat{e}_n{}'$$
$$= (\ell_{mi} . \hat{e}_i) . (\ell_{nj} \hat{e}_j)$$
$$= \ell_{mi} \ell_{nj} (\hat{e}_i . \hat{e}_j)$$
$$= \ell_{mi} \ell_{nj} \delta_{ij}$$

which shows that δ_{ij} is a second order Cartesian tensor.

NOTE: (i) The nine components of the Kronecker delta tensor δ_{ij} can be written in the form of 3×3 matrix as $[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) Since x_i are the components of a first order tensor (i.e. position vector) it follows that $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ is a tensor of order 2 .

THIRD ORDER TENSORS

A quantity representable by a set of three suffixes A_{ijk} of 27 (i.e. 3^3) numbers (called components) relatively to a coordinate system K is called a third order tensor , if its components transform under changes of the coordinate system according to the law

$$A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}A_{ijk} \quad (1)$$

where A'_{mnp} are the components of a quantity in the coordinate system K' and ℓ_{mi} is the cosine of the angle between the mth – axis of K' and the ith – axis of K . (Similarly , for ℓ_{nj} and ℓ_{pk}) .

The **inverse transformation law** of equation (1) expressing the components of the third order tensor in the system K in terms of its components in the system K' is

$$A_{ijk} = \ell_{mi}\ell_{nj}\ell_{pk}A'_{mnp} \quad (2)$$

since
$$\begin{aligned} \ell_{mi}\ell_{nj}\ell_{pk}A'_{mnp} &= \ell_{mi}\ell_{nj}\ell_{pk}(\ell_{mr}\ell_{ns}\ell_{pt}A_{rst}) \\ &= (\ell_{mi}\ell_{mr})(\ell_{nj}\ell_{ns})(\ell_{pk}\ell_{pt})A_{rst} \\ &= (\delta_{ir}\delta_{js}\delta_{kt})A_{rst} \\ &= \delta_{ir}\delta_{js}(\delta_{kt}A_{rst}) \\ &= \delta_{ir}(\delta_{js}A_{rsk}) \\ &= \delta_{ir}A_{rjk} = A_{ijk} \end{aligned}$$

Equation (2) can be written equivalently as

$$A_{mnp} = \ell_{im}\ell_{jn}\ell_{kp}A'_{ijk} . \quad (3)$$

Note that equation (2) represents components of the form

A_{111} , A_{112} , A_{113} , A_{123} , etc.

EXAMPLE (14): **Prove that if A_i , B_j , and C_k are three first order tensors, then their product $A_iB_jC_k$ (i , j , k = 1 , 2 , 3) is a tensor of order 3 while $A_iB_jC_j$ (i , j = 1 , 2 , 3) form a first order tensor .**

SOLUTION: Let $D_{ijk} = A_iB_jC_k$ (1)

then we have to prove that D_{ijk} (i , j , k = 1 , 2 , 3) are the components of a third order tensor while D_{ijj} are the components of a first order tensor .

Since A_i , B_j , C_k are first order tensors ; their equations of transformations from the system K to K′ are

$$A'_m = \ell_{mi}A_i \quad (2)$$

$$B'_n = \ell_{nj}B_j$$

$$C'_p = \ell_{pk}C_k$$

Multiplying (2), (3), and (4), we obtain

$$A'_m B'_n C'_p = \ell_{mi}\ell_{nj}\ell_{pk} A_i B_j C_k \tag{5}$$

or $$D'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk} D_{ijk} \tag{6}$$

where $D'_{mnp} = A'_m B'_n C'_p$

Equation (6) shows that $D_{ijk} = A_i B_j C_k$ are the components of a third order tensor.

If we set $n = p$ in equation (6), we have

$$D'_{mnn} = \ell_{mi}\ell_{nj}\ell_{nk} D_{ijk}$$

or $$D'_{mnn} = \ell_{mi}\delta_{jk} D_{ijk} = \ell_{mi} D_{ijj}$$

which shows that $D_{ijj} = A_i B_j C_j$ are the components of a first order tensor.

THEOREM (7.10): Prove that the alternating symbol ϵ_{ijk} is a Cartesian tensor of rank 3.

PROOF: Let ϵ_{ijk} and ϵ'_{pqr} be the components of the alternating symbol in the systems K and K$'$ respectively. Then

$$\left.\begin{aligned} &\hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2, \quad \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \\ &\hat{e}_2{}' \times \hat{e}_3{}' = \hat{e}_1{}', \quad \hat{e}_3{}' \times \hat{e}_1{}' = \hat{e}_2{}', \quad \hat{e}_1{}' \times \hat{e}_2{}' = \hat{e}_3{}' \end{aligned}\right\} \tag{1}$$

Using the definition of the alternating symbol we can write equations (1) as

$$\left.\begin{aligned} &\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k = \epsilon_{ijk} \\ &\hat{e}_p{}' \times \hat{e}_q{}' \cdot \hat{e}_r{}' = \epsilon'_{pqr} \end{aligned}\right\} \tag{2}$$

where $(i, j, k, p, q, r = 1, 2, 3)$.

Now $\hat{e}_p{}' = \ell_{pi}\hat{e}_i$, $\hat{e}_q{}' = \ell_{qj}\hat{e}_j$, and $\hat{e}_r{}' = \ell_{rk}\hat{e}_k$ therefore,

$$\epsilon'_{pqr} = \hat{e}_p{}' \times \hat{e}_q{}' \cdot \hat{e}_r{}'$$

$$= (\ell_{pi}\hat{e}_i) \times (\ell_{qj}\hat{e}_j) \cdot (\ell_{rk}\hat{e}_k) = \ell_{pi}\ell_{qj}\ell_{rk}\,\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k$$

or $$\epsilon'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}\epsilon_{ijk} \tag{3}$$

From equation (3) it is clear that ϵ_{ijk} is a Cartesian tensor of rank 3.

HIGHER ORDER TENSORS

A quantity representable by a set of n suffixes $A_{i_1 i_2 \dots i_n}$ of 3^n numbers (called components) relatively to a coordinate system K is said to be a tensor of order (rank) n, if its components transform under changes of the coordinate system according to the law

$$A'_{j_1 j_2 \dots j_n} = \ell_{j_1 i_1}\ell_{j_2 i_2} \dots \ell_{j_n i_n} A_{i_1 i_2 \dots i_n} \tag{1}$$

where $A'_{j_1 j_2 \dots j_n}$ are the components of the quantity in the coordinate system K$'$ and $\ell_{j_1 i_1}$, etc., have the usual meanings.

NOTE: Given the components of a tensor of order n in the coordinate system K, we can use equation (1) to determine its components in another rectangular coordinate system K'. In particular, if all the components of a tensor vanish in one coordinate system, they also vanish in any other coordinate system.

7.18 ALGEBRA OF TENSORS

ADDITION OF TENSORS

The sum of two or more tensors of the same rank is the tensor whose components are equal to the sum of the corresponding components of the individual tensors. Note that the tensors of different ranks cannot be added. Also addition of tensors is commutative and associative.

SUBTRACTION OF TENSORS

The difference of two tensors of the same rank is also a tensor whose components are equal to the difference of the corresponding components of the two tensors. Note that subtraction of tensors of different orders is not defined.

EXAMPLE (15): If A_{jk} and B_{jk} are tensors of rank 2, prove that

(i) $C_{jk} = A_{jk} + B_{jk}$

(ii) $D_{jk} = A_{jk} - B_{jk}$

are also tensors of rank 2.

SOLUTION: (i) Since A_{jk} and B_{jk} are the second order tensors, their equations of transformation from the system K to K' are

$$A'_{mn} = \ell_{mj}\ell_{nk}A_{jk} \qquad (1)$$

$$B'_{mn} = \ell_{mj}\ell_{nk}B_{jk} \qquad (2)$$

Adding equations (1) and (2) we get

$$A'_{mn} + B'_{mn} = \ell_{mj}\ell_{nk}(A_{jk} + B_{jk})$$

or $\quad C'_{mn} = \ell_{mj}\ell_{nk}C_{jk} \qquad (3)$

where $\quad C'_{mn} = A'_{mn} + B'_{mn}$

Equation (3) shows that $C_{jk} = A_{jk} + B_{jk}$ is also a tensor of rank 2.

(ii) Subtracting equation (1) from equation (2), we obtain

$$A'_{mn} - B'_{mn} = \ell_{mj}\ell_{nk}(A_{jk} - B_{jk})$$

or $\quad D'_{mn} = \ell_{mj}\ell_{nk}D_{jk} \qquad (4)$

where $\quad D''_{mn} = A''_{mn} - B''_{mn}$

From equation (4) it is clear that $D_{jk} = A_{jk} - B_{jk}$ is a tensor of rank 2.

This shows that the sum (or difference) of two second order tensors is a second order tensor. In general, the sum (or difference) of two tensors of order n is another tensor of order n.

MULTIPLICATION OF A TENSOR BY A SCALAR

The multiplication of a tensor of any rank by a scalar yields another tensor of the same rank.

EXAMPLE (16): Prove that if ϕ is a scalar and A_{ij} is a second order tensor, then $C_{ij} = \phi A_{ij}$ is also a second order tensor.

SOLUTION: Since A_{ij} is a second order tensor, its equation of transformation from the system K to K$'$ is

$$A'_{mn} = \ell_{mi}\ell_{nj}A_{ij} \quad (1)$$

Multiplying equation (1) by the scalar ϕ, we get

$$\phi A'_{mn} = \ell_{mi}\ell_{nj}(\phi A_{ij})$$

or $$C'_{mn} = \ell_{mi}\ell_{nj}C_{ij} \quad (2)$$

where $C'_{mn} = \phi A'_{mn}$

Equation (2) shows that $C_{ij} = \phi A_{ij}$ is also a second order tensor.

In general, multiplication of a tensor of order n by a scalar gives another tensor of order n.

(OUTER) MULTIPLICATION OF TENSORS

The product of two or more tensors is the tensor whose components are the product of the components of the given tensors. The order of a tensor product is clearly the sum of the orders of the given tensors.

EXAMPLE (17): If A_{ijk} and B_{mn} are two Cartesian tensors of rank 3 and 2 respectively, prove that $C_{ijkmn} = A_{ijk}B_{mn}$ is also a tensor of rank 5.

SOLUTION: Since A_{ijk} and B_{mn} are tensors, their equations of transformation from the systems K to K$'$ are

$$A'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}A_{ijk} \quad (1)$$

$$B'_{st} = \ell_{sm}\ell_{tn}B_{mn} \quad (2)$$

Multiplying equations (1) and (2), we get

$$A'_{pqr}B'_{st} = \ell_{pi}\ell_{qj}\ell_{rk}\ell_{sm}\ell_{tn}A_{ijk}B_{mn}$$

or $$C'_{pqrst} = \ell_{pi}\ell_{qj}\ell_{rk}\ell_{sm}\ell_{tn}C_{ijkmn} \quad (3)$$

where $C'_{pqrst} = A'_{pqr}B'_{st}$

which shows that $C_{ijkmn} = A_{ijk}B_{mn}$ called the outer product of A_{ijk} and B_{mn} is a tensor of rank $3 + 2 = 5$.

NOTE: (i) The tensor multiplication is non – commutative. For example,

$$C_{ijkmn} = A_{ijk}B_{mn} \neq B_{mn}A_{ijk} = C_{mnijk}$$

(ii) The outer product of two vectors i.e. tensors of the first order, is sometimes called a **dyadic** tensor or just **dyad**.

even → scalar
odd → vector

7.19 CONTRACTION OF TENSORS

The process of setting two indices in a tensor equal and summing over that repeated index is called contraction. For example, the three possible contractions of a third order tensor A_{ijk} are A_{iik}, A_{iji} and A_{ijj}.

CONTRACTION THEOREM

THEOREM (7.11): The contraction of a tensor of order n ($n \geq 2$) leads to a tensor of order $n-2$.

PROOF: We prove this theorem for $n = 3$ i.e. for the tensor A_{ijk}.

By hypotheses, A_{ijk} is a tensor of order 3. Therefore,

$$A'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}A_{ijk} \qquad (1)$$

Let us contract w.r.t. j and k. Place the corresponding indices q and r equal to each other and sum over this index. Then

$$A'_{pqq} = \ell_{pi}\ell_{qj}\ell_{qk}A_{ijk}$$

$$= \ell_{pi}\delta_{jk}A_{ijk} = \ell_{pi}A_{ijj}$$

which shows that $B_i = A_{ijj}$ is a tensor of rank 1 i.e. a vector.

NOTE: (i) We have seen that contraction can be applied to a tensor of rank 2 or higher.

(ii) We know that contraction of a tensor of order n ($n \geq 2$) leads to a tensor of order $n-2$. This tensor of order ($n-2$) can then be contracted again (provided that $n \geq 4$), giving a tensor of order $n-4$, and so on, until we obtain a tensor of order less than 2. In fact, repeated contraction of a tensor of order n eventually gives a scalar if n is even and a vector if n is odd.

7.20 (INNER) MULTIPLICATION OF TENSORS

The process of multiplying tensors (outer multiplication) and then contracting the product w.r.t. indices belonging to different factors is called inner multiplication and the result is called an inner product of the given tensors. For example the expression $A_i B_{ik}$ is the inner product of the tensors A_i and B_{jk}. Similarly $A_i B_i$ is the inner product of two vectors A_i and B_j (i.e. $\vec{A}$ and $\vec{B}$).

EXAMPLE (18): If A_{ijk} and B_{mn} are two tensors of rank 3 and 2 respectively, show that their inner product $A_{ijk}B_{in}$ is a tensor of rank $3+2-2 = 3$.

SOLUTION: Since A_{ijk} and B_{mn} are tensors, their equations of transformation from the system K to K′ are

$$A'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}A_{ijk} \qquad (1)$$

$$B'_{st} = \ell_{sm}\ell_{tn}B_{mn} \qquad (2)$$

Multiplying equations (1) and (2), we get

$$A'_{pqr}B'_{st} = \ell_{pi}\ell_{qj}\ell_{rk}\ell_{sm}\ell_{tn}A_{ijk}B_{mn} \qquad (3)$$

By contraction (letting $p = s$ in equation (3) and summing), we get

$$A'_{pqr}B'_{pt} = \ell_{pi}\ell_{qj}\ell_{rk}\ell_{pm}\ell_{tn}A_{ijk}B_{mn}$$
$$= \ell_{qj}\ell_{rk}\ell_{tn}(\ell_{pi}\ell_{pm})A_{ijk}B_{mn}$$
$$= \ell_{qj}\ell_{rk}\ell_{tn}\delta_{im}A_{ijk}B_{mn}$$
$$= \ell_{qj}\ell_{rk}\ell_{tn}A_{ijk}B_{in}$$

which shows that $C_{jkn} = A_{ijk}B_{in}$ called the inner product of A_{ijk} and B_{mn} is a tensor of rank 3. By contracting w.r.t. j and n or k and m in the product $A_{ijk}B_{mn}$, we can similarly show that any inner product is a tensor of rank 3.

COROLLARY: The inner product $A_i B_i$ of two vectors A_i and B_j (i.e. $\vec{A}$ and $\vec{B}$) is a tensor of rank zero i.e. a scalar. For this reason $A_i B_i$ is called the scalar or dot product of $\vec{A}$ and $\vec{B}$.

GENERALIZATION

If $A_{i_1 i_2 \cdots i_m}$ and $B_{j_1 j_2 \cdots j_n}$ are two tensors of rank m and n respectively, then any of their inner products is a tensor of rank $m+n-2$.

7.21 QUOTIENT THEOREM

With the help of this theorem we can decide whether a quantity representable by a multi – suffix set is a tensor or not.

THEOREM (7.12): If an inner product of a quantity X with an arbitrary tensor is itself a tensor, then X is also a tensor.

To illustrate this theorem we consider the following example :

EXAMPLE (19): If $A_{ij}B_j$ is a vector where B_j is an arbitrary vector, then prove that the 2 – suffix set A_{ij} is also a tensor of rank 2.

SOLUTION: Let $C_i = A_{ij}B_j$, $C'_p = A'_{pq}B'_q$ where A_{ij}, B_j, C_i and A'_{pq}, B'_q, C'_p are the components of the 2 – suffix set and the two vectors in the systems K and K′ respectively.

Now since C_i is a vector, therefore

$$C'_p = \ell_{pi}C_i$$

or $$A'_{pq}B'_q = \ell_{pi}A_{ij}B_j \qquad (1)$$

Also B_j being an arbitrary, we have $B'_q = \ell_{qj}B_j$

or $$B_j = \ell_{qj}B'_q \qquad (2)$$

From equations (1) and (2), we get

$$A'_{pq}B'_q = \ell_{pi}\ell_{qj}A_{ij}B'_q$$

or $$(A'_{pq} - \ell_{pi}\ell_{qj}A_{ij})B'_q = 0$$

Since the vector B_j is arbitrary, the vector B'_q is also arbitrary so that $B'_q \neq 0$ and the above relation is true only when

$$A'_{pq} - \ell_{pi}\ell_{qj}A_{ij} = 0$$

or $$A'_{pq} = \ell_{pi}\ell_{qj}A_{ij}$$

showing that the 2 – suffix set A_{ij} is a tensor of rank 2.

GENERALIZATION

If $A_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_n} B_{j_1 j_2 \cdots j_n}$ is a tensor of order m, where $B_{j_1 j_2 \cdots j_n}$ is an arbitrary tensor of order n, then prove that $A_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_n}$ is a tensor of order m + n.

7.22 SYMMETRIC AND ANTI – SYMMETRIC TENSORS

A tensor $A_{i_1 i_2 \cdots i_n}$ is said to be symmetric in a pair of indices i_1 and i_2 (say) if

$$A_{i_1 i_2 \cdots i_n} = A_{i_2 i_1 \cdots i_n} \tag{1}$$

while it is said to be anti – symmetric in the indices i_1 and i_2 if

$$A_{i_1 i_2 \cdots i_n} = -A_{i_2 i_1 \cdots i_n} \tag{2}$$

A tensor is said to be symmetric (anti – symmetric) if it is symmetric (anti – symmetric) in all possible pairs of indices. Symmetric and anti – symmetric tensors occur frequently in mathematics and physics. For example, the interia tensor, the stress tensor, the strain tensor and the rate of strain tensor are all symmetric, while the spin tensor is an example of an anti – symmetric tensor.

THEOREM (7.13): Prove that the Kronecker tensor δ_{ij} is a second order symmetric tensor and the alternating tensor ϵ_{ijk} is a third order anti – symmetric tensor.

PROOF: We have $\delta_{ij} = \hat{e}_i \cdot \hat{e}_j = \hat{e}_j \cdot \hat{e}_i = \delta_{ji}$

which shows that δ_{ij} is a symmetric tensor.

Also, $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$

which shows that ϵ_{ijk} is an anti – symmetric tensor.

NOTE: A symmetric second order tensor A_{ij} can be written as a matrix in the form

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

while an anti – symmetric second order tensor has a matrix of the form

$$[A_{ij}] = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}$$

Thus a symmetric second order tensor has only 6 independent components, while an anti – symmetric second order tensor has only 3 independent components. Also in an anti – symmetric tensor the components on the leading diagonal are all zero. $[A_{ii} = -A_{ii}$ or $2A_{ii} = 0$ or $A_{ii} = 0]$

THEOREM (7.14): Prove that every second order tensor can be represented uniquely as the sum of a symmetric and an anti – symmetric tensor .

PROOF: Let A_{ij} be a second order tensor, then we can write

$$A_{ij} = \frac{1}{2}(A_{ij}+A_{ji})+\frac{1}{2}(A_{ij}-A_{ji})$$
$$= B_{ij}+C_{ij}$$

where $B_{ij} = \frac{1}{2}(A_{ij}+A_{ji}) = B_{ji}$ is symmetric

and $C_{ij} = \frac{1}{2}(A_{ij}-A_{ji}) = -\frac{1}{2}(A_{ji}-A_{ij}) = -C_{ji}$, is anti – symmetric .

7.23 INVARIANCE OF SYMMETRIC AND ANTI–SYMMETRIC PROPERTY OF A TENSOR

THEOREM (7.15): If a tensor is symmetric (anti – symmetric) w.r.t. a pair of indices in one coordinate system , then it has the same property in any other coordinate system.

PROOF: We prove this theorem for the tensor A_{ijk}

If A_{ijk} is symmetric in i and j, then

$$A_{ijk} = A_{jik} \quad (1)$$

Also $A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}A_{ijk}$

$= \ell_{mi}\ell_{nj}\ell_{pk}A_{jik}$ [using equation (1)]

$= \ell_{nj}\ell_{mi}\ell_{pk}A_{jik} = A'_{nmp}$

Thus $A'_{mnp} = A'_{nmp}$ showing that the tensor is symmetric w.r.t. the same pair of indices in the new coordinate system as well .

Similarly , in case the tensor A_{ijk} is anti – symmetric in i and j we can show $A'_{mnp} = -A'_{nmp}$.

7.24 FUNDAMENTAL PROPERTY OF TENSOR EQUATIONS

THEOREM (7.16): A tensor equation which holds in one coordinate system holds in every coordinate system i.e. the form of a tensor equation remains the same in every rectangular coordinate system .

PROOF: We prove this property for the simple tensor equation .

Let $A_i B_{ijk} = C_{jk}$ (1)

be a tensor equation where A_i, B_{ijk}, C_{jk} represent the components of the three tensors w.r.t. the system K . We will prove that equation (1) has the same form in another coordinate system K′ .

Multiplying both sides of equation (1) with $\ell_{mj}\ell_{nk}$, we get

$$\ell_{mj}\ell_{nk}A_i B_{ijk} = \ell_{mj}\ell_{nk}C_{jk} \quad (2)$$

where i, j, k are dummies .

THEOREM (7.14): Prove that every second order tensor can be represented uniquely as the sum of a symmetric and an anti – symmetric tensor .

PROOF: Let A_{ij} be a second order tensor, then we can write

$$A_{ij} = \frac{1}{2}(A_{ij}+A_{ji})+\frac{1}{2}(A_{ij}-A_{ji})$$

$$= B_{ij}+C_{ij}$$

where $B_{ij} = \frac{1}{2}(A_{ij}+A_{ji}) = B_{ji}$ is symmetric

and $C_{ij} = \frac{1}{2}(A_{ij}-A_{ji}) = -\frac{1}{2}(A_{ji}-A_{ij}) = -C_{ji}$, is anti – symmetric .

7.23 INVARIANCE OF SYMMETRIC AND ANTI–SYMMETRIC PROPERTY OF A TENSOR

THEOREM (7.15): If a tensor is symmetric (anti – symmetric) w.r.t. a pair of indices in one coordinate system , then it has the same property in any other coordinate system.

PROOF: We prove this theorem for the tensor A_{ijk}

If A_{ijk} is symmetric in i and j , then

$$A_{ijk} = A_{jik} \qquad (1)$$

Also $A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}A_{ijk}$

$$= \ell_{mi}\ell_{nj}\ell_{pk}A_{jik} \qquad \text{[using equation (1)]}$$

$$= \ell_{nj}\ell_{mi}\ell_{pk}A_{jik} = A'_{nmp}$$

Thus $A'_{mnp} = A'_{nmp}$ showing that the tensor is symmetric w.r.t. the same pair of indices in the new coordinate system as well .

Similarly , in case the tensor A_{ijk} is anti – symmetric in i and j we can show $A'_{mnp} = -A'_{nmp}$.

7.24 FUNDAMENTAL PROPERTY OF TENSOR EQUATIONS

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PROOF: We prove this property for the simple tensor equation .

Let $A_i B_{ijk} = C_{jk}$ (1)

be a tensor equation where A_i , B_{ijk}, C_{jk} represent the components of the three tensors w.r.t. the system K . We will prove that equation (1) has the same form in another coordinate system K′ .

Multiplying both sides of equation (1) with $\ell_{mj}\ell_{nk}$, we get

$$\ell_{mj}\ell_{nk}A_i B_{ijk} = \ell_{mj}\ell_{nk}C_{jk} \qquad (2)$$

where i , j , k are dummies .

If A'_p, B'_{pmn}, and C'_{mn} be the components of the same tensors w.r.t. the system K', then

$$C'_{mn} = \ell_{mj}\ell_{nk}C_{jk} \qquad (3)$$

Now since $A_i B_{ijk} = \delta_{iq} A_q B_{ijk}$

$$= \ell_{pi}\ell_{pq}A_q B_{ijk} \qquad (4)$$

Using equations (3) and (4), equation (2) becomes

$$\ell_{mj}\ell_{nk}\ell_{pi}\ell_{pq}A_q B_{ijk} = C'_{mn}$$

But $A'_p = \ell_{pq}A_q$ and $B'_{pmn} = \ell_{pi}\ell_{mj}\ell_{nk}B_{ijk}$

Therefore $A'_p B'_{pmn} = C'_{mn}$ (5)

which is of the same form as equation (1).

COROLLARY: From equations (1) and (5) writing $D_{jk} = A_i B_{ijk} - C_{jk}, D'_{mn} = A'_p B'_{pmn} - C'_{mn}$, we have $D_{jk} = 0$ and $D'_{mn} = 0$ for $m, n = 1, 2, 3$ which shows that if the components of a tensor in one coordinate system are all zero, then the components in every coordinate system are also zero.

ZERO TENSOR

A tensor whose components relatively to one coordinate system and, therefore, also relatively to every coordinate system are all zero is known as a zero tensor.

7.25 ISOTROPIC TENSORS

A tensor is said to be isotropic if its components remain the same in all rectangular Cartesian coordinate systems under orthogonal rotation of axes.

Note that tensors of order zero (i.e. scalars) are all isotropic. Since there are no isotropic tensors of order 1, therefore, we will discuss the isotropic tensors of second and third orders, which are of particular importance in tensor analysis.

THEOREM (7.17): Prove that the Kronecker tensor δ_{ij} is an isotropic tensor of order 2.

PROOF: We know that the equation of transformation for the second order tensor A_{ij} is

$$A'_{mn} = \ell_{mi}\ell_{nj}A_{ij} \qquad (1)$$

Let $A_{ij} = \delta_{ij}$ in equation (1), then

$$A'_{mn} = \ell_{mi}\ell_{nj}\delta_{ij} = \ell_{mi}\ell_{ni} = \delta_{mn}$$

which shows that the components δ_{ij} transform into themselves under the tensor rotation law. Thus δ_{ij} is an isotropic tensor of order 2. This tensor is the most important of all the isotropic tensors. Note that the only isotropic tensor of order 2 is a scalar multiple of δ_{ij}.

THEOREM (7.18): Prove that the alternating tensor ϵ_{ijk} is an isotropic tensor of order 3.

PROOF: The equation of transformation for the third order tensor is

$$A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}A_{ijk} \qquad (1)$$

Let $A_{ijk} = \epsilon_{ijk}$ in equation (1), then

$$A'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}\epsilon_{ijk}$$

$$= \ell_{m1}\ell_{n2}\ell_{p3}+\ell_{m2}\ell_{n3}\ell_{p1}+\ell_{m3}\ell_{n1}\ell_{p2}-\ell_{m1}\ell_{n3}\ell_{p2}-\ell_{m2}\ell_{n1}\ell_{p3}-\ell_{n3}\ell_{n2}\ell_{p1}$$

$$= \begin{vmatrix} \ell_{m1} & \ell_{n1} & \ell_{p1} \\ \ell_{m2} & \ell_{n2} & \ell_{p2} \\ \ell_{m3} & \ell_{n3} & \ell_{p3} \end{vmatrix} \qquad (2)$$

If any pair of suffixes m, n, p are equal, the determinant (2) has two equal columns and hence vanishes. Now m, n, and p can take only the values 1, 2 or 3. If the values of m, n, and p are in cyclic order say m = 1, n = 2, and p = 3, the determinant reduces to the value 1. Since the interchange of any pair of columns in a determinant changes its sign, therefore if the values of m, n, and p are in an anti – cyclic order (say) m = 2, n = 1, and p = 3, then the determinant has the value – 1.

Thus the following are the combinations of values of m, n, and p for which the value of the determinant is not zero.

Cyclic Order	**Anti – cyclic order**
m = 1, n = 2, p = 3	m = 2, n = 1, p = 3
m = 2, n = 3, p = 1	m = 1, n = 3, p = 2
m = 3, n = 1, p = 2	m = 3, n = 2, p = 1

Similarly, for the other combination of values of m, n, and p, the value of the determinant will be either 1 or – 1. Thus it follows that, for all possible values of m, n, and p, the determinant has the same value as ϵ_{mnp}.

Hence $A'_{mnp} = \epsilon_{mnp}$

which shows that under a rotation of axes, the tensor law is satisfied and each one of the set of the numbers ϵ_{ijk} transform into itself. Hence, ϵ_{ijk} is an isotropic tensor of order 3. Note that the only isotropic tensors of order 3 are scalar multiples of ϵ_{ikm}.

ISOTROPIC TENSORS OF HIGHER ORDER

Similarly, it can be proved that any fourth order isotropic tensor with components $A_{ijk\ell}$ can be expressed as a sum of the products of delta tensor in the form

$$A_{ijk\ell} = \lambda\,\delta_{ij}\delta_{k\ell}+\mu\,\delta_{ik}\delta_{j\ell}+\nu\,\delta_{i\ell}\delta_{jk}$$

where λ, μ, and ν are arbitrary scalar invariants.

The most general isotropic tensor of order 5 is a linear combination of 10 terms of the form $\epsilon_{ijk}\delta_{\ell m}$.

The most general isotropic tensor of order 6 is a linear combination of 15 terms of the form $\delta_{ij}\delta_{k\ell}\delta_{mn}$.

The most general isotropic tensor of order 7 is a linear combination of 105 terms of the form

$\epsilon_{ijk}\delta_{\ell m}\delta_{np}$

22-10-19

7.26 TENSOR CALCULUS

DIFFERENTIATION OF TENSORS

THEOREM (7.19): If $A_{i_1 i_2 \cdots i_n}$ is a tensor of order n, then its partial derivative w.r.t. x_p, i.e. $\frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n}$ is also a tensor of order $n+1$.

PROOF: The law of transformation for the given tensor is

$$A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \ell_{j_2 i_2} \cdots \ell_{j_n i_n} A_{i_1 i_2 \cdots i_n} \tag{1}$$

where all the symbols have the usual meanings. Differentiating both sides of equation (1) w.r.t. x'_k we get

$$\frac{\partial}{\partial x'_k} A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \ell_{j_2 i_2} \cdots \ell_{j_n i_n} \frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n} \frac{\partial x_p}{\partial x'_k} \text{ where } p \text{ is dummy.}$$

Also we know that $x'_k = \ell_{kp} x_p$ or $x_p = \ell_{kp} x'_k$

so that $\frac{\partial x_p}{\partial x'_k} = \ell_{kp}$

Hence $$\frac{\partial}{\partial x'_k} A'_{j_1 j_2 \cdots j_n} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} \frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n} \tag{2}$$

which shows that $\frac{\partial}{\partial x_p} A_{i_1 i_2 \cdots i_n}$ is a tensor of order $n+1$.

NOTE: (i) If the partial derivative of $A_{i_1 i_2 \cdots i_n}$ w.r.t. x_p is denoted by $A_{i_1 i_2 \cdots i_{n,p}}$, then equation (2) can be written in the form

$$A'_{j_1 \cdots j_{n,k}} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} A_{i_1 \cdots i_{n,p}} \tag{3}$$

Differentiating both sides of equation (3) w.r.t. x'_m we can show

$$A'_{j_1 j_2 \cdots j_{n,km}} = \ell_{j_1 i_1} \cdots \ell_{j_n i_n} \ell_{kp} \ell_{mq} A_{i_1 \cdots i_{n,pq}}$$

$\left(\text{where } A_{i_1 \cdots i_{n,pq}} = \frac{\partial^2}{\partial x_q \partial x_p} A_{i_1 \cdots i_n}\right)$ which shows that $A_{i_1 i_2 \cdots i_{n,pq}}$ is a tensor of order $n+2$.

(ii) If ϕ is a scalar, then $\frac{\partial \phi}{\partial x_i}$ or $\phi_{,i}$ is a tensor of order 1 i.e. a vector.

INTEGRATION OF TENSORS

Integration of a tensor with respect to the coordinate direction yields a tensor of one order higher unless integration is combined with a contraction. For example,

$\int A'_{mn} \, dx'_p = \int (\ell_{mi} \ell_{nj} A_{ij}) \ell_{pk} \, dx_k = \ell_{mi} \ell_{nj} \ell_{pk} \left(\int A_{ij} \, dx_k\right)$ and thus $\left(\int A_{ij} \, dx_k\right)$ is a tensor of order 3. However $\left(\int A_{ij} \, dx_j\right)$ is a contraction of $\left(\int A_{ij} \, dx_k\right)$ and is thus of order 1 i.e. one less than A. Integration of a tensor w.r.t. a scalar, for example volume or surface, can be shown to yield a tensor of the same order.

7.27 APPLICATION TO VECTOR ANALYSIS

DOT PRODUCT

Let $\vec{A}$ and $\vec{B}$ be two vectors with components A_1, A_2, A_3 and B_1, B_2, B_3 respectively, then $\vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3 = A_i B_i$.

THEOREM (7.20): Prove that if A_i and B_j are the components of two first order tensors $\vec{A}$ and $\vec{B}$ respectively, then $A_i B_i$ is a zeroth order tensor.

PROOF: Since A_i and B_j are the components of first order tensors, therefore under the transformation law from the system K to K', we have

$$A'_m = \ell_{mi} A_i \tag{1}$$

$$B'_n = \ell_{nj} B_j \tag{2}$$

Multiplying equations (1) and (2), we get

$$A'_m B'_n = \ell_{mi} \ell_{nj} A_i B_j \tag{3}$$

Setting $m = n$ in equation (3), we have

$$A'_m B'_m = \ell_{mi} \ell_{mj} A_i B_j = \delta_{ij} A_i B_j = A_i B_i \tag{4}$$

which shows that $A_i B_i$ is a scalar or zeroth order tensor.

NOTE: (i) Equation (4) can be written as

$$A'_1 B'_1 + A'_2 B'_2 + A'_3 B'_3 = A_1 B_1 + A_2 B_2 + A_3 B_3$$

showing that the scalar product of two vectors is invariant under the orthogonal rotation of axes.

(ii) We have already proved that $A_i B_j$ are the components of a second order tensor, whereas in the above theorem we have seen that $A_i B_i$ is a zeroth order tensor. So the difference between $A_i B_j$ and $A_i B_i$ must be carefully observed.

CROSS PRODUCT

Let $\vec{A}$ and $\vec{B}$ be two vectors with components A_1, A_2, A_3 and B_1, B_2, B_3 respectively, then $\vec{C} = \vec{A} \times \vec{B} = (A_2 B_3 - A_3 B_2, A_3 B_1 - A_1 B_3, A_1 B_2 - A_2 B_1)$.

We now show that the components of $\vec{C} = \vec{A} \times \vec{B}$ are given by

$$C_i = \epsilon_{ijk} A_j B_k \quad \text{for} \quad i = 1, 2, 3 \tag{1}$$

Now $C_1 = \epsilon_{1jk} A_j B_k = \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2 = (\vec{A} \times \vec{B})_1$.

$$C_2 = \epsilon_{2jk} A_j B_k = \epsilon_{231} A_3 B_1 + \epsilon_{213} A_1 B_3 = A_3 B_1 - A_1 B_3 = (\vec{A} \times \vec{B})_2$$

$$C_3 = \epsilon_{3jk} A_j B_k = \epsilon_{312} A_1 B_2 + \epsilon_{321} A_2 B_1 = A_1 B_2 - A_2 B_1 = (\vec{A} \times \vec{B})_3$$

Thus $(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$

NOTE: From equation (1), we have $C_i \hat{e}_i = \epsilon_{ijk} A_j B_k \hat{e}_i$

i.e. $\vec{C} = \epsilon_{ijk} A_j B_k \hat{e}_i$

or $\vec{A} \times \vec{B} = \epsilon_{ijk} A_j B_k \hat{e}_i$ where now the summation is over all the indices.

THEOREM (7.21): Prove that

(i) the components of $\vec{A} \times \vec{B}$ i.e. $C_i = \epsilon_{ijk} A_j B_k$ transform as the components of a vector under a rotation of the coordinate axes.

(ii) $\vec{A} \times \vec{B}$ is invariant under the rotation of coordinate axes.

PROOF: (i) Let ϵ_{ijk}, A_j, B_k be the components of a third order, and two first order tensors in the system $Ox_1x_2x_3$ and $\epsilon'_{mnp}, A'_n, B'_p$ be their corresponding components in the system $Ox'_1x'_2x'_3$. Then the laws of transformation are

$$\epsilon'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}\epsilon_{ijk} \quad (1)$$

$$A'_n = \ell_{nj}A_j = \ell_{nr}A_r \quad (2)$$

$$B'_p = \ell_{pk}B_k = \ell_{ps}B_s \quad (3)$$

From equations (1), (2), and (3), we get

$$\begin{aligned}
\epsilon'_{mnp}A'_nB'_p &= \ell_{mi}\ell_{nj}\ell_{pk}\epsilon_{ijk}\,\ell_{nr}A_r\ell_{ps}B_s \\
&= \ell_{mi}(\ell_{nj}\ell_{nr})(\ell_{pk}\ell_{ps})\epsilon_{ijk}A_rB_s \\
&= \ell_{mi}\delta_{jr}\delta_{ks}\epsilon_{ijk}A_rB_s \\
&= \ell_{mi}\epsilon_{ijk}(\delta_{jr}A_r)(\delta_{ks}B_s) \\
&= \ell_{mi}\epsilon_{ijk}A_jB_k
\end{aligned}$$

$$C'_m = \ell_{mi}C_i \quad (4)$$

where $C'_m = \epsilon'_{mnp}A'_nB'_p$

Equations (4) shows that the components of $\vec{A} \times \vec{B}$ transform as the components of a vector.

(ii) $\vec{A} \times \vec{B} = \epsilon_{ijk}A_jB_k\hat{e}_i$

Now $\epsilon'_{mnp}A'_nB'_p = \ell_{mi}\epsilon_{ijk}A_jB_k \quad (5)$

Also $\hat{e}_m' = \ell_{mi}\hat{e}_i = \ell_{mr}\hat{e}_r \quad (6)$

From equations (5) and (6), we get

$$\begin{aligned}
\epsilon'_{mnp}A'_nB'_p\hat{e}_m' &= \ell_{mi}\epsilon_{ijk}A_jB_k\ell_{mr}\hat{e}_r \\
&= \ell_{mi}\ell_{mr}\epsilon_{ijk}A_jB_k\hat{e}_r \\
&= \delta_{ir}\epsilon_{ijk}A_jB_k\hat{e}_r \\
&= \epsilon_{ijk}A_jB_k(\delta_{ir}\hat{e}_r) = \epsilon_{ijk}A_jB_k\hat{e}_i
\end{aligned}$$

which shows that $\vec{A} \times \vec{B}$ is invariant under the rotation of coordinate axes.

SCALAR TRIPLE PRODUCT

Considering $\vec{A} \cdot \vec{B} \times \vec{C}$ as the scalar product of $\vec{A}$ and $\vec{B} \times \vec{C}$, we get

$$\vec{A} \cdot \vec{B} \times \vec{C} = A_i (B \times C)_i$$

$$= A_i \epsilon_{ijk} B_j C_k = \epsilon_{ijk} A_i B_j C_k \quad (1)$$

THEOREM (7.22): Prove that

(i) $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B}$

(ii) $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C}$

PROOF: (i) Since $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$, therefore,

$$\epsilon_{ijk} A_i B_j C_k = \epsilon_{jki} B_j C_k A_i = \epsilon_{kij} C_k A_i B_j$$

Using equation (1) above, we get

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B}$$

(ii) $\vec{A} \cdot \vec{B} \times \vec{C} = \epsilon_{ijk} A_i B_j C_k$

$$= (\epsilon_{ijk} A_i B_j) C_k = (\vec{A} \times \vec{B})_k C_k = \vec{A} \times \vec{B} \cdot \vec{C}$$

VECTOR TRIPLE PRODUCT 23-10-19

THEOREM (7.23): Prove that $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$

PROOF: We have

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} A_j (\vec{B} \times \vec{C})_k$$

$$= \epsilon_{ijk} A_j \epsilon_{k\ell m} B_\ell C_m = \epsilon_{ijk} \epsilon_{\ell mk} A_j B_\ell C_m$$

$$= (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) A_j B_\ell C_m$$

$$= \delta_{i\ell} \delta_{jm} A_j B_\ell C_m - \delta_{im} \delta_{j\ell} A_j B_\ell C_m$$

$$= A_j B_i C_j - A_j B_j C_i = (A_j C_j) B_i - (A_j B_j) C_i$$

$$= (\vec{A} \cdot \vec{C}) B_i - (\vec{A} \cdot \vec{B}) C_i$$

which gives the three components of the required formula for $i = 1, 2, 3$.

Hence $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$

THE DEL – OPERATOR 5-11-2019 1st Lecture Final

In Cartesian tensors, the del – operator denoted by ∇ is defined as

$$\nabla = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \hat{e}_i \frac{\partial}{\partial x_i}$$

THEOREM (7.24): Prove that

(i) the components of the del–operator $\nabla \left(\text{i.e.} \dfrac{\partial}{\partial x_i} \right)$ transform as the components of a vector under a rotation of the coordinate axes.

(ii) the vector del–operator ∇ is invariant under the rotation of the coordinate axes.

PROOF: (i) Let $\dfrac{\partial}{\partial x_i}$ and $\dfrac{\partial}{\partial x'_j}$ be the components of the del – operator ∇ in the system $Ox_1 x_2 x_3$ and $Ox'_1 x'_2 x'_3$ respectively. Let x_i and x'_j be the coordinates of a point in these systems, then we know that

$$x_1 = \ell_{11} x'_1 + \ell_{21} x'_2 + \ell_{31} x'_3$$

$$x_2 = \ell_{12} x'_1 + \ell_{22} x'_2 + \ell_{32} x'_3$$

$$x_3 = \ell_{13} x'_1 + \ell_{23} x'_2 + \ell_{33} x'_3$$

Using the chain rule, we have

$$\frac{\partial}{\partial x'_1} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x'_1} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x'_1} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x'_1}$$

$$= \ell_{11}\frac{\partial}{\partial x_1} + \ell_{12}\frac{\partial}{\partial x_2} + \ell_{13}\frac{\partial}{\partial x_3}$$

$$= \ell_{1i}\frac{\partial}{\partial x_i} \qquad (1)$$

Similarly, $$\frac{\partial}{\partial x'_2} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x'_2} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x'_2} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x'_2}$$

$$= \ell_{21}\frac{\partial}{\partial x_1} + \ell_{22}\frac{\partial}{\partial x_2} + \ell_{23}\frac{\partial}{\partial x_3}$$

$$= \ell_{2i}\frac{\partial}{\partial x_i} \qquad (2)$$

and $$\frac{\partial}{\partial x'_3} = \frac{\partial}{\partial x_1}\frac{\partial x_1}{\partial x'_3} + \frac{\partial}{\partial x_2}\frac{\partial x_2}{\partial x'_3} + \frac{\partial}{\partial x_3}\frac{\partial x_3}{\partial x'_3}$$

$$= \ell_{31}\frac{\partial}{\partial x_1} + \ell_{32}\frac{\partial}{\partial x_2} + \ell_{33}\frac{\partial}{\partial x_3}$$

$$= \ell_{3i}\frac{\partial}{\partial x_i} \qquad (3)$$

From equations (1), (2), and (3), we get

$$\frac{\partial}{\partial x'_j} = \ell_{ji}\frac{\partial}{\partial x_i} \qquad (4)$$

which shows that under a rotation of the coordinate axes, the components of the del operator ∇ transform as components of a vector. It is often called a vector operator.

(ii) Let $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j'}$ be the components of the vector del – operator ∇ in the system $Ox_1 x_2 x_3$ and $Ox_1' x_2' x_3'$ respectively. Then we know that

$$\frac{\partial}{\partial x_j'} = \ell_{ji} \frac{\partial}{\partial x_i} \qquad (1)$$

Also $\hat{e}_j' = \ell_{ji} \hat{e}_i = \ell_{jk} \hat{e}_k \qquad (2)$

From equations (1) and (2), we get

$$\hat{e}_j' \frac{\partial}{\partial x_j'} = \left(\ell_{jk} \hat{e}_k\right)\left(\ell_{ji} \frac{\partial}{\partial x_i}\right)$$

$$= \ell_{jk} \ell_{ji} \hat{e}_k \frac{\partial}{\partial x_i}$$

$$= \delta_{ki} \hat{e}_k \frac{\partial}{\partial x_i} = \hat{e}_i \frac{\partial}{\partial x_i}$$

or $\hat{e}_1' \frac{\partial}{\partial x_1'} + \hat{e}_2' \frac{\partial}{\partial x_2'} + \hat{e}_3' \frac{\partial}{\partial x_3'} = \hat{e}_1 \frac{\partial}{\partial x_2} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3}$

or $\nabla' = \nabla$

which shows that the vector del – operator ∇ is invariant under the rotation of the coordinate axes.

GRADIENT

5-11-2019 Final 1st Lecture

Let $\phi(x_1, x_2, x_3)$ be a scalar point function, then we know that

$$\nabla \phi = \hat{e}_1 \frac{\partial \phi}{\partial x_1} + \hat{e}_2 \frac{\partial \phi}{\partial x_2} + \hat{e}_3 \frac{\partial \phi}{\partial x_3} \qquad (1)$$

In tensor notation, equation (1) becomes

$$\nabla \phi = \hat{e}_i \frac{\partial \phi}{\partial x_i}$$

The components of $\nabla \phi$ are given by

$$(\nabla \phi)_i = \frac{\partial \phi}{\partial x_i}, \quad i = 1, 2, 3.$$

NOTE: (i) From equation (1) it is clear that the operator ∇ is given by

$$\nabla = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3}$$

$$= \hat{e}_i \frac{\partial}{\partial x_i} \qquad (2)$$

(ii) For any arbitrary vector $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$ we write

$$\vec{A} \cdot \nabla = A_1 \frac{\partial}{\partial x_1} + A_2 \frac{\partial}{\partial x_2} + A_3 \frac{\partial}{\partial x_3}$$

$$= A_j \frac{\partial}{\partial x_j} \qquad (3)$$

THEOREM (7.25): **If $\phi(x_1, x_2, x_3)$ is a scalar point function then $\frac{\partial \phi}{\partial x_i}$ are the components of a first order tensor (vector).**

PROOF: Let (x_1, x_2, x_3) be the coordinates of a point in the system K and (x'_1, x'_2, x'_3) the coordinates of the same point in the system K', then

$$x_i = \ell_{ji} x'_j \tag{1}$$

where ℓ_{ji} is the cosine of the angle between the jth–axis of the system K' and ith–axis of the system K.

Differentiating equation (1) w.r.t. x'_j, we get

$$\frac{\partial x_i}{\partial x'_j} = \ell_{ji} \tag{2}$$

Also from calculus
$$\frac{\partial \phi}{\partial x'_j} = \frac{\partial \phi}{\partial x_1}\frac{\partial x_1}{\partial x'_j} + \frac{\partial \phi}{\partial x_2}\frac{\partial x_2}{\partial x'_j} + \frac{\partial \phi}{\partial x_3}\frac{\partial x_3}{\partial x'_j}$$
$$= \frac{\partial \phi}{\partial x_i}\frac{\partial x_i}{\partial x'_j} \tag{3}$$

From equations (2) and (3), we get

$$\frac{\partial \phi}{\partial x'_j} = \ell_{ji}\frac{\partial \phi}{\partial x_i} \tag{4}$$

Also $\quad A'_j = \ell_{ji} A_i \qquad (5)$

Comparing equations (4) and (5), we see that $\frac{\partial \phi}{\partial x_i}$ are the components of a first order tensor.

THEOREM (7.26): **Prove that**

(i) the components of $\nabla \phi \left(\text{i.e. } \frac{\partial \phi}{\partial x_i} \right)$ transform as the components of a vector law under a rotation of the coordinate axes.

(ii) $\nabla \phi$ is invariant under the rotation of the coordinate axes.

PROOF: (i) Let $\frac{\partial \phi}{\partial x_i}$ and $\frac{\partial \phi}{\partial x'_j}$ be the components of the $\nabla \phi$ in the system $Ox_1x_2x_3$ and $Ox'_1x'_2x'_3$ respectively. Then we know that

$$x_1 = \ell_{11}x'_1 + \ell_{21}x'_2 + \ell_{31}x'_3$$
$$x_2 = \ell_{12}x'_1 + \ell_{22}x'_2 + \ell_{32}x'_3$$
$$x_3 = \ell_{13}x'_1 + \ell_{23}x'_2 + \ell_{33}x'_3$$

Using the chain rule, we have

$$\frac{\partial \phi}{\partial x'_1} = \frac{\partial \phi}{\partial x_1}\frac{\partial x_1}{\partial x'_1} + \frac{\partial \phi}{\partial x_2}\frac{\partial x_2}{\partial x'_1} + \frac{\partial \phi}{\partial x_3}\frac{\partial x_3}{\partial x'_1}$$
$$= \ell_{11}\frac{\partial \phi}{\partial x_1} + \ell_{12}\frac{\partial \phi}{\partial x_2} + \ell_{13}\frac{\partial \phi}{\partial x_3}$$
$$= \ell_{1i}\frac{\partial \phi}{\partial x_i} \tag{1}$$

Similarly, $$\frac{\partial \phi}{\partial x_2'} = \frac{\partial \phi}{\partial x_1}\frac{\partial x_1}{\partial x_2'} + \frac{\partial \phi}{\partial x_2}\frac{\partial x_2}{\partial x_2'} + \frac{\partial \phi}{\partial x_3}\frac{\partial x_3}{\partial x_2'}$$
$$= \ell_{21}\frac{\partial \phi}{\partial x_1} + \ell_{22}\frac{\partial \phi}{\partial x_2} + \ell_{23}\frac{\partial \phi}{\partial x_3}$$
$$= \ell_{2i}\frac{\partial \phi}{\partial x_i} \quad (2)$$

and, $$\frac{\partial \phi}{\partial x_3'} = \frac{\partial \phi}{\partial x_1}\frac{\partial x_1}{\partial x_3'} + \frac{\partial \phi}{\partial x_2}\frac{\partial x_2}{\partial x_3'} + \frac{\partial \phi}{\partial x_3}\frac{\partial x_3}{\partial x_3'}$$
$$= \ell_{31}\frac{\partial \phi}{\partial x_1} + \ell_{32}\frac{\partial \phi}{\partial x_2} + \ell_{33}\frac{\partial \phi}{\partial x_3}$$
$$= \ell_{3i}\frac{\partial \phi}{\partial x_i} \quad (3)$$

From equations (1), (2), and (3), we get
$$\frac{\partial \phi}{\partial x_j'} = \ell_{ji}\frac{\partial \phi}{\partial x_i} \quad (4)$$
which shows that under a rotation of the coordinate axes, the components of the $\nabla \phi$ transform as components of a vector.

(ii) Let $\frac{\partial \phi}{\partial x_i}$ and $\frac{\partial \phi}{\partial x_j'}$ be the components of $\nabla \phi$ in the system $Ox_1x_2x_3$ and $Ox_1'x_2'x_3'$ respectively. Then we know that
$$\frac{\partial \phi}{\partial x_j'} = \ell_{ji}\frac{\partial \phi}{\partial x_i} \quad (5)$$
Also $$\hat{e}_j' = \ell_{ji}\hat{e}_j = \ell_{jk}\hat{e}_k \quad (6)$$
From equations (5) and (6), we get
$$\hat{e}_j'\frac{\partial \phi}{\partial x_j'} = (\ell_{jk}\hat{e}_k)\left(\ell_{ji}\frac{\partial \phi}{\partial x_i}\right)$$
$$= \ell_{jk}\ell_{ji}\hat{e}_k\frac{\partial \phi}{\partial x_i} = \delta_{ki}\hat{e}_k\frac{\partial \phi}{\partial x_i} = \hat{e}_i\frac{\partial \phi}{\partial x_i}$$
or $$\hat{e}_1'\frac{\partial \phi}{\partial x_1'} + \hat{e}_2'\frac{\partial \phi}{\partial x_2'} + \hat{e}_3'\frac{\partial \phi}{\partial x_3'} = \hat{e}_1\frac{\partial \phi}{\partial x_2} + \hat{e}_2\frac{\partial \phi}{\partial x_2} + \hat{e}_3\frac{\partial \phi}{\partial x_3}$$
or $$\nabla \phi' = \nabla \phi$$
which shows that $\nabla \phi$ is invariant under the rotation of the coordinate axes.

DIVERGENCE 5-11-2019 Final 1st Lecture

If (A_1, A_2, A_3) are the components of a vector point function $\overrightarrow{A}$, then by definition
$$\nabla \cdot \overrightarrow{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = \frac{\partial A_i}{\partial x_i} = A_{i,i}$$

THEOREM (7.27): Prove that if A_i are the components of a first order tensor i.e. vector $\overline{A}$ then $\frac{\partial A_i}{\partial x_j}$ is a second order tensor. Hence prove that $\nabla \cdot \overline{A}$ is a scalar quantity.

PROOF: Let A_i and A'_p be the components of $\overline{A}$ in systems K and K′ respectively, then

$$A'_p = \ell_{pi} A_i$$

Differentiating both sides partially w.r.t. x'_q, we get

$$\frac{\partial A'_p}{\partial x'_q} = \ell_{pi} \frac{\partial A_i}{\partial x'_q}$$

But from the chain rule of calculus

$$\frac{\partial A'_p}{\partial x'_q} = \ell_{pi}\left[\frac{\partial A_i}{\partial x_1}\frac{\partial x_1}{\partial x'_q} + \frac{\partial A_i}{\partial x_2}\frac{\partial x_2}{\partial x'_q} + \frac{\partial A_i}{\partial x_3}\frac{\partial x_3}{\partial x'_q}\right]$$

$$= \ell_{pi}\frac{\partial A_i}{\partial x_j}\frac{\partial x_j}{\partial x'_q} \tag{1}$$

Also we know that $x_j = \ell_{qj} x'_q$ therefore $\frac{\partial x_j}{\partial x'_q} = \ell_{qj}$

Thus equation (1) becomes $\frac{\partial A'_p}{\partial x'_q} = \ell_{pi}\ell_{qj}\frac{\partial A_i}{\partial x_j}$

which shows that $\frac{\partial A_i}{\partial x_j}$ are the components of a second order Cartesian tensor.

If contraction is applied to $\frac{\partial A_i}{\partial x_j}$ by putting $i = j$, then $\frac{\partial A_i}{\partial x_i}$ is zeroth order tensor and therefore a scalar which by definition, is the divergence of $\overline{A}$.

THEOREM (7.28): Prove that $\nabla \cdot \overline{A} = \frac{\partial A_i}{\partial x_i}$ is invariant under a rotation of the coordinate axes.

PROOF: Let $\frac{\partial}{\partial x_i}$ and A_i be the components of the del–operator and the vector $\overline{A}$ respectively in the system $Ox_1x_2x_3$, and $\frac{\partial}{\partial x'_j}$ and A'_j be the corresponding components in the system $Ox'_1x'_2x'_3$. Then the laws of transformation become

$$\frac{\partial}{\partial x'_j} = \ell_{ji}\frac{\partial}{\partial x_i} \quad \text{and} \quad A'_j = \ell_{jk} A_k$$

Hence, since the coefficients ℓ_{jk} are independent of x_1, x_2, x_3, therefore

$$\frac{\partial A'_j}{\partial x'_j} = \left(\ell_{ji}\frac{\partial}{\partial x_i}\right)(\ell_{jk}A_k)$$

$$= \ell_{ji}\ell_{jk}\frac{\partial A_k}{\partial x_i} = \delta_{ik}\frac{\partial A_k}{\partial x_i} = \frac{\partial A_i}{\partial x_i}$$

i.e. $\dfrac{\partial A'_1}{\partial x'_1}+\dfrac{\partial A'_2}{\partial x'_2}+\dfrac{\partial A'_3}{\partial x'_3}=\dfrac{\partial A_1}{\partial x_1}+\dfrac{\partial A_2}{\partial x_2}+\dfrac{\partial A_3}{\partial x_3}$

showing that $\nabla \cdot \vec{A} = \dfrac{\partial A_i}{\partial x_i}$ is invariant under a rotation of the coordinate axes .

CURL

5-11-19 1st Lecture Final

If (A_1, A_2, A_3) are the components of a vector point function $\vec{A}$, then

$$\nabla \times \vec{A} = \left(\frac{\partial A_3}{\partial x_2}-\frac{\partial A_2}{\partial x_3}, \frac{\partial A_1}{\partial x_3}-\frac{\partial A_3}{\partial x_1}, \frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2}\right)$$

We now show that the components of $\nabla \times \vec{A}$ are given by

$$C_i = \epsilon_{ijk}\frac{\partial A_k}{\partial x_j} \quad \text{for } i = 1, 2, 3 .$$

Since
$$C_1 = \epsilon_{1jk}\frac{\partial A_k}{\partial x_j} = \epsilon_{123}\frac{\partial A_3}{\partial x_2}+\epsilon_{132}\frac{\partial A_2}{\partial x_3} = \frac{\partial A_3}{\partial x_2}-\frac{\partial A_2}{\partial x_3} = (\nabla \times \vec{A})_1$$
$$C_2 = \epsilon_{2jk}\frac{\partial A_k}{\partial x_j} = \epsilon_{231}\frac{\partial A_1}{\partial x_3}+\epsilon_{213}\frac{\partial A_3}{\partial x_1} = \frac{\partial A_1}{\partial x_3}-\frac{\partial A_3}{\partial x_1} = (\nabla \times \vec{A})_2$$
$$C_3 = \epsilon_{3jk}\frac{\partial A_k}{\partial x_j} = \epsilon_{312}\frac{\partial A_2}{\partial x_1}+\epsilon_{321}\frac{\partial A_1}{\partial x_2} = \frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2} = (\nabla \times \vec{A})_3$$

Hence $(\nabla \times \vec{A})_i = \epsilon_{ijk}\dfrac{\partial A_k}{\partial x_j}$, $i = 1, 2, 3$.

NOTE: The curl itself is given by $\nabla \times \vec{A} = \epsilon_{ijk}\dfrac{\partial A_k}{\partial x_j}\hat{e}_i$
where now the summation is over all the indices .

THEOREM (7.29): Prove that $\nabla \times \vec{A}$ is a vector quantity .

PROOF: We know that $\dfrac{\partial A_p}{\partial x_q}$ is a second order tensor and ϵ_{ijk} is a third order tensor . Therefore their outer product is a fifth order tensor . If we apply contraction by putting $p = k$, we get $\epsilon_{ijk}\dfrac{\partial A_k}{\partial x_q}$ as a third order tensor . Again , if we apply contraction to this third order tensor by putting $q = j$, we obtain $\epsilon_{ijk}\dfrac{\partial A_k}{\partial x_j}$ as a first order tensor (or a vector) which , by definition , are the components of $\nabla \times \vec{A}$. Hence $\nabla \times \vec{A}$ is a vector .

THEOREM (7.30): Prove that

(i) **the components of $\nabla \times \vec{A}$ i.e. $\epsilon_{ijk}\dfrac{\partial A_k}{\partial x_j}$ transform as the components of a vector under a rotation of the coordinate axes .**

(ii) **$\nabla \times \vec{A}$ is invariant vector field under the rotation of the coordinate axes .**

PROOF: (i) Let ϵ_{ijk} and $\frac{\partial A_k}{\partial x_j}$ be the components of a third order and second order tensors respectively in the system $Ox_1x_2x_3$ and ϵ_{mnp} and $\frac{\partial A_p}{\partial x_n}$ their corresponding components in the system $Ox'_1x'_2x'_3$. Then

$$\epsilon'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk}\epsilon_{ijk} \tag{1}$$

$$\frac{\partial A'_s}{\partial x'_r} = \ell_{sv}\ell_{ru}\frac{\partial A_v}{\partial x_u} \tag{2}$$

From equations (1) and (2), we get

$$\epsilon'_{mnp}\frac{\partial A'_s}{\partial x'_r} = \ell_{mi}\ell_{nj}\ell_{pk}\epsilon_{ijk}\ell_{sv}\ell_{ru}\frac{\partial A_v}{\partial x_u}$$

$$= \ell_{mi}\ell_{nj}\ell_{pk}\ell_{sv}\ell_{rv}\,\epsilon_{ijk}\frac{\partial A_v}{\partial x_u} \tag{3}$$

Let $p = s$ and $n = r$ in equation (3), we get

$$\epsilon'_{mnp}\frac{\partial A'_p}{\partial x'_n} = \ell_{mi}\ell_{nj}\ell_{pk}\ell_{pv}\ell_{nu}\epsilon_{ijk}\frac{\partial A_v}{\partial x_u}$$

$$= \ell_{mi}\,(\ell_{nj}\ell_{nu})(\ell_{pk}\ell_{pv})\epsilon_{ijk}\frac{\partial A_v}{\partial x_u}$$

$$= \ell_{mi}\delta_{ju}\delta_{kv}\epsilon_{ijk}\frac{\partial A_v}{\partial x_u}$$

$$\epsilon'_{mnp}\frac{\partial A'_p}{\partial x'_n} = \ell_{mi}\,\epsilon_{ijk}\frac{\partial A_k}{\partial x_j} \tag{4}$$

which shows that under a rotation of the coordinate axes, the components of $\nabla \times \overline{A}$ transform as the components of a vector.

(ii) Let $\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}$ be the components of $\nabla \times \overline{A}$ in the system $Ox_1x_2x_3$ and $\epsilon'_{mnp}\frac{\partial A'_p}{\partial x'_n}$ be its components in the system $Ox'_1x'_2x'_3$, then we know that

$$\epsilon'_{mnp}\frac{\partial A'_p}{\partial x'_n} = \ell_{mi}\epsilon_{ijk}\frac{\partial A_k}{\partial x_j} \tag{5}$$

Also $\hat{e}_m{}' = \ell_{mi}\,\hat{e}_i = \ell_{mr}\,\hat{e}_r$ (6)

From equations (1) and (2) we get

$$\epsilon'_{mnp}\frac{\partial A'_p}{\partial x'_n}\hat{e}'_m = \ell_{mi}\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}(\ell_{mr}\hat{e}_r)$$

$$= \ell_{mi}\ell_{mr}\,\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}\hat{e}_r$$

$$= \delta_{ir}\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}\hat{e}_r$$

$$= \epsilon_{ijk}\frac{\partial A_k}{\partial x_j}\hat{e}_i$$

or $\quad \nabla \times \vec{A}' = \nabla \times \vec{A}$

which shows that $\nabla \times \vec{A}$ is invariant vector field under the rotation of axes.

THEOREM (7.31): Prove the following formulas using tensor methods :

(i) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(ii) $\nabla.(\phi\vec{A}) = \phi(\nabla.\vec{A}) + (\nabla\phi).\vec{A}$

(iii) $\nabla \times (\phi\vec{A}) = \phi(\nabla \times \vec{A}) + (\nabla\phi) \times \vec{A}$

(iv) $\nabla.(\vec{A} \times \vec{B}) = \vec{B}.\nabla \times \vec{A} - \vec{A}.\nabla \times \vec{B}$

(v) $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla.\vec{B}) - \vec{B}(\nabla.\vec{A}) + (\vec{B}.\nabla)\vec{A} - (\vec{A}.\nabla)\vec{B}$

(vi) $\nabla(\vec{A}.\vec{B}) = (\vec{A}.\nabla)\vec{B} + (\vec{B}.\nabla)\vec{A} + \vec{A} \times \nabla \times \vec{B} + \vec{B} \times \nabla \times \vec{A}$

(vii) $\nabla.\nabla\phi = \frac{\partial^2\phi}{\partial x_i^2} \equiv \phi_{,ii}$

(viii) $\nabla.(\nabla \times \vec{A}) = 0$

(ix) $\nabla \times (\nabla\phi) = \vec{0}$

(x) $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A}$

PROOF: We know that

(i) $[\nabla(\phi\psi)]_i = \frac{\partial(\phi\psi)}{\partial x_i} = \phi\frac{\partial\psi}{\partial x_i} + \psi\frac{\partial\phi}{\partial x_i}$

$[\nabla(\phi\psi)]_i = \phi(\nabla\psi)_i + \psi(\nabla\phi)_i$

which gives the three components of the required formulas for $i = 1, 2, 3$. Hence

$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(ii) $\nabla.(\phi\vec{A}) = \frac{\partial(\phi A_i)}{\partial x_i} = \phi\frac{\partial A_i}{\partial x_i} + A_i\frac{\partial\phi}{\partial x_i}$

$= \phi(\nabla.\vec{A}) + \vec{A}.(\nabla\phi) \quad \left[\text{since } \vec{A}.\nabla = A_j\frac{\partial}{\partial x_j}\right]$

(iii) $[\nabla \times (\phi\vec{A})]_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\phi A)_k = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\phi A_k)$

$$= \epsilon_{ijk}\left[\phi\frac{\partial A_k}{\partial x_j}+\frac{\partial \phi}{\partial x_j}A_k\right]$$

$$= \phi\,\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}+\epsilon_{ijk}(\nabla\phi)_j A_k$$

$$= \phi(\nabla\times\vec{A})_i+(\nabla\phi\times\vec{A})_i$$

which shows that all the three components of the required formula are equal (for $i = 1, 2, 3$).

Hence $\nabla\times(\phi\vec{A}) = \phi(\nabla\times\vec{A})+(\nabla\phi)\times\vec{A}$

(iv) $\nabla\cdot(\vec{A}\times\vec{B}) = \frac{\partial}{\partial x_i}(\vec{A}\times\vec{B})_i$

$$= \frac{\partial}{\partial x_i}[\epsilon_{ijk}A_jB_k]$$

$$= \epsilon_{ijk}\left(A_j\frac{\partial B_k}{\partial x_i}+\frac{\partial A_j}{\partial x_i}B_k\right)$$

$$= A_j\left(\epsilon_{ijk}\frac{\partial B_k}{\partial x_i}\right)+\left(\epsilon_{ijk}\frac{\partial A_j}{\partial x_i}\right)B_k$$

$$= A_j\left(-\epsilon_{jik}\frac{\partial B_k}{\partial x_i}\right)+\left(\epsilon_{kij}\frac{\partial A_j}{\partial x_i}\right)B_k$$

$$= -A_j(\nabla\times\vec{B})_j+B_k(\nabla\times\vec{A})_k$$

$$= \vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$$

$\vec{A}\cdot\vec{B} = A_iB_i$

(v) $[\nabla\times(\vec{A}\times\vec{B})]_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{A}\times\vec{B})_k$

$$= \epsilon_{ijk}\frac{\partial}{\partial x_j}(\epsilon_{k\ell m}A_\ell B_m)$$

$$= \epsilon_{ijk}\epsilon_{\ell mk}\left(A_\ell\frac{\partial B_m}{\partial x_j}+B_m\frac{\partial A_\ell}{\partial x_j}\right)$$

$$= (\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell})\left[A_\ell\frac{\partial B_m}{\partial x_j}+B_m\frac{\partial A_\ell}{\partial x_j}\right]$$

$$= \delta_{i\ell}\delta_{jm}A_\ell\frac{\partial B_m}{\partial x_j}+\delta_{i\ell}\delta_{jm}B_m\frac{\partial A_\ell}{\partial x_j} - \delta_{im}\delta_{j\ell}A_\ell\frac{\partial B_m}{\partial x_j}-\delta_{im}\delta_{j\ell}B_m\frac{\partial A_\ell}{\partial x_j}$$

$$= \delta_{i\ell}A_\ell\frac{\partial B_j}{\partial x_j}+\delta_{jm}B_m\frac{\partial A_i}{\partial x_j}-\delta_{j\ell}A_\ell\frac{\partial B_i}{\partial x_j}-\delta_{im}B_m\frac{\partial A_j}{\partial x_j}$$

$$= A_i\frac{\partial B_j}{\partial x_j}+B_j\frac{\partial A_i}{\partial x_j}-A_j\frac{\partial B_i}{\partial x_j}-B_i\frac{\partial A_j}{\partial x_j}$$

$$= A_i\nabla\cdot\vec{B}+(\vec{B}\cdot\nabla)A_i-(\vec{A}\cdot\nabla)B_i-B_i\nabla\cdot\vec{A}$$

which shows that all the three components of the required formula are equal for $i = 1, 2, 3$.

Hence $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$

(vi)
$$[\vec{A} \times \nabla \times \vec{B}]_i = \epsilon_{ijk} A_j (\nabla \times \vec{B})_k$$
$$= \epsilon_{ijk} A_j \epsilon_{k\ell m} \frac{\partial B_m}{\partial x_\ell}$$
$$= \epsilon_{ijk} \epsilon_{\ell mk} A_j \frac{\partial B_m}{\partial x_\ell}$$
$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) A_j \frac{\partial B_m}{\partial x_\ell}$$
$$= \delta_{i\ell}\delta_{jm} A_j \frac{\partial B_m}{\partial x_\ell} - \delta_{im}\delta_{j\ell} A_j \frac{\partial B_m}{\partial x_\ell}$$
$$= A_j \frac{\partial B_j}{\partial x_i} - A_j \frac{\partial B_i}{\partial x_j}$$
$$= A_j \frac{\partial B_j}{\partial x_i} - (\vec{A} \cdot \nabla) B_i$$

Similarly $[\vec{B} \times \nabla \times \vec{A}]_i = B_j \frac{\partial A_j}{\partial x_i} - (\vec{B} \cdot \nabla) A_i$

Hence
$$[\vec{A} \times \nabla \times \vec{B}]_i + [\vec{B} \times \nabla \times \vec{A}]_i = A_j \frac{\partial B_j}{\partial x_i} + B_j \frac{\partial A_j}{\partial x_i} - (\vec{A} \cdot \nabla) B_i - (\vec{B} \cdot \nabla) A_i$$
$$= \frac{\partial}{\partial x_i}(A_j B_j) - (\vec{A} \cdot \nabla) B_i - (\vec{B} \cdot \nabla) A_i$$
$$= \frac{\partial}{\partial x_i}(\vec{A} \cdot \vec{B}) - (\vec{A} \cdot \nabla) B_i - (\vec{B} \cdot \nabla) A_i$$

or
$$[\nabla(\vec{A} \cdot \vec{B})]_i = (\vec{A} \cdot \nabla) B_i + (\vec{B} \cdot \nabla) A_i + [\vec{A} \times \nabla \times \vec{B}]_i + [\vec{B} \times \nabla \times \vec{A}]_i$$

which shows that all the three components of required formula are equal for $i = 1, 2, 3$. Hence

$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times \nabla \times \vec{B} + \vec{B} \times \nabla \times \vec{A}$$

(vii)
$$\nabla \cdot (\nabla \phi) = \frac{\partial}{\partial x_i} (\nabla \phi)_i$$
$$= \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_i}\right) = \frac{\partial^2 \phi}{\partial x_i \partial x_i} = \frac{\partial^2 \phi}{\partial x_i^2} = \phi_{,ii}$$

Thus we have proved that $\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_i^2}$

which implies that $\nabla^2 \equiv \frac{\partial^2}{\partial x_i^2}$ and is called the **Laplacian operator**.

(viii)
$$\nabla \cdot (\nabla \times \vec{A}) = \frac{\partial}{\partial x_i} (\nabla \times \vec{A})_i = \frac{\partial}{\partial x_i}\left[\epsilon_{ijk} \frac{\partial A_k}{\partial x_j}\right] = \epsilon_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j}$$

$3^3 = 27$

$$= \epsilon_{123} \frac{\partial^2 A_3}{\partial x_1 \partial x_2} + \epsilon_{231} \frac{\partial^2 A_1}{\partial x_2 \partial x_3} + \epsilon_{312} \frac{\partial^2 A_2}{\partial x_3 \partial x_1} + \epsilon_{132} \frac{\partial^2 A_2}{\partial x_1 \partial x_3} + \epsilon_{321} \frac{\partial^2 A_1}{\partial x_3 \partial x_2} + \epsilon_{213} \frac{\partial^2 A_3}{\partial x_2 \partial x_1} = 0,$$

where we have assumed that $\frac{\partial^2 A_k}{\partial x_i \partial x_j} = \frac{\partial^2 A_k}{\partial x_j \partial x_i}$

(ix) $$[\nabla \times (\nabla \phi)]_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \phi)_k = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial \phi}{\partial x_k} \right) = \epsilon_{ijk} \frac{\partial^2 \phi}{\partial x_j \partial x_k}$$

from which $$[\nabla \times (\nabla \phi)]_1 = \epsilon_{1jk} \frac{\partial^2 \phi}{\partial x_j \partial x_k}$$

$$= \epsilon_{123} \frac{\partial^2 \phi}{\partial x_2 \partial x_3} + \epsilon_{132} \frac{\partial^2 \phi}{\partial x_3 \partial x_2} = \frac{\partial^2 \phi}{\partial x_2 \partial x_3} - \frac{\partial^2 \phi}{\partial x_3 \partial x_2} = 0$$

Similarly $[\nabla \times (\nabla \phi)]_2 \equiv 0$ and $[\nabla \times (\nabla \phi)]_3 \equiv 0$

Hence $\nabla \times \nabla \phi \equiv 0$

(x) $$[\nabla \times (\nabla \times \vec{A})]_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \vec{A})_k$$

$$= \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\epsilon_{k\ell m} \frac{\partial A_m}{\partial x_\ell} \right) = \epsilon_{ijk} \, \epsilon_{\ell mk} \frac{\partial^2 A_m}{\partial x_j \partial x_\ell}$$

$$= (\delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}) \frac{\partial^2 A_m}{\partial x_j \partial x_\ell}$$

$$= \delta_{i\ell} \delta_{jm} \frac{\partial^2 A_m}{\partial x_j \partial x_\ell} - \delta_{im} \delta_{j\ell} \frac{\partial^2 A_m}{\partial x_j \partial x_\ell}$$

$$= \frac{\partial^2 A_j}{\partial x_j \partial x_i} - \frac{\partial^2 A_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_i} \left(\frac{\partial^2 A_j}{\partial x_j} \right) - \frac{\partial^2 A_i}{\partial x_j^2}$$

$$= \frac{\partial}{\partial x_i} (\nabla \cdot \vec{A}) - \nabla^2 A_i$$

$$= [\nabla (\nabla \cdot \vec{A})]_i - (\nabla^2 \vec{A})_i$$

which shows that all the three components of the required formula are equal for $i = 1, 2, 3$.

Hence $\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$.

7.28 INTEGRAL THEOREMS IN TENSOR FORM

GAUSS DIVERGENCE THEOREM

Let V be the volume of a region R bounded by a closed surface S and $\hat{n}$ the outward drawn unit normal to S. If $\vec{A}$ is any vector point function with continuous first partial derivatives, then

$$\iint_S \vec{A} \cdot \hat{n} \, dS = \iiint_R \nabla \cdot \vec{A} \, dV \qquad (1)$$

In tensor notation, $\vec{A} = A_i \hat{e}_i$ and $\hat{n} = n_i \hat{e}_i$ so that $\vec{A} \cdot \hat{n} = A_i n_i$ and $\nabla \cdot \vec{A} = \dfrac{\partial A_i}{\partial x_i}$ so that equation (1) becomes

$$\iint_S A_i n_i \, dS = \iiint_R \frac{\partial A_i}{\partial x_i} dV \qquad (2)$$

STOKES' THEOREM

Let S be an open two – sided surface bounded by a simple closed curve C. If $\vec{A}$ is any vector point function with continuous first partial derivatives, then

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} \, dS \qquad (3)$$

where $\hat{n}$ is the outward drawn unit normal to S.

In tensor notation, $\vec{A} = A_i \hat{e}_i$, $\hat{n} = n_i \hat{e}_i$, and $d\vec{r} = dx_i \hat{e}_i$ so that $\vec{A} \cdot d\vec{r} = A_i dx_i$, $\nabla \times \vec{A} = \epsilon_{ijk} \dfrac{\partial A_k}{\partial x_j} \hat{e}_i$, and $(\nabla \times \vec{A}) \cdot \hat{n} = \epsilon_{ijk} \dfrac{\partial A_k}{\partial x_j} n_i$. Thus equation (3) takes the form

$$\oint_C A_i dx_i = \iint_S \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} n_i \, dS \qquad (4)$$

7.29 EIGENVALUES AND EIGENVECTORS OF A SECOND ORDER TENSOR

Let A_{ij} $(i, j = 1, 2, 3)$ be a second order tensor and suppose x_i is a vector. Then the inner product $A_{ij} x_j$ is also a vector. Suppose the vector x_i is such that a scalar λ can be found so that $A_{ij} x_j$ can be written as a scalar multiple of x_i

i.e. $\quad A_{ij} x_j = \lambda x_i, \ (i = 1, 2, 3) \qquad (1)$

We can write equation (1) in the form

$$A_{ij} x_j = \lambda \delta_{ij} x_j$$

or $\quad (A_{ij} - \lambda \delta_{ij}) x_j = 0, \ \ i = 1, 2, 3 \qquad (2)$

which is equivalent to

$$\left.\begin{aligned} (A_{11} - \lambda) x_1 + A_{12} x_2 + A_{13} x_3 &= 0 \\ A_{21} x_1 + (A_{22} - \lambda) x_2 + A_{23} x_3 &= 0 \\ A_{31} x_1 + A_{23} x_2 + (A_{33} - \lambda) x_3 &= 0 \end{aligned}\right] \qquad (3)$$

The system of equations (2) or (3) has a non – trivial solution if and only if $|A_{ij} - \lambda \delta_{ij}| = 0$

or
$$\begin{vmatrix} A_{11} - \lambda & A_{12} & A_{13} \\ A_{21} & A_{22} - \lambda & A_{23} \\ A_{31} & A_{23} & A_{33} - \lambda \end{vmatrix} = 0 \qquad (4)$$

Equation (4) is called the **determinantal equation** or **characteristic equation** of the tensor A_{ij}. This equation being cubic in λ gives three values of λ denoted by $\lambda_1, \lambda_2, \lambda_3$ which are called the

eigenvalues (or characteristic values) of the tensor A_{ij}. Thus a tensor A_{ij} (of rank 2) has three eigenvalues in general (which may not be all distinct).

The vectors corresponding to the values $\lambda_1, \lambda_2, \lambda_3$ of λ are called the **eigenvectors** of A_{ij}.

Since the system is homogeneous , it is clear that if $\vec{X}$ is an eigenvector of A_{ij}, then $K\vec{X}$, where K is any non – zero constant , is also an eigenvector of A_{ij} corresponding to the same eigenvalue .

REPEATED ROOT OF THE CHARACTERISTIC EQUATION

In general , the characteristic equation will have three distinct roots for λ , each leading to a single eigenvector which is linearly independent from the other two . That is , the eigenvectors corresponding to distinct eigenvalues are linearly independent . If , however , the characteristic equation has a repeated root , λ is said to be an eigenvalue of multiplicity 2 . We shall show that corresponding to such an eigenvalue , there may be one or two linearly independent eigenvectors . A tensor with only one eigenvector corresponding to the repeated eigenvalue is said to **defective** .

INVARIANCE OF EIGENVALUES

THEOREM (7.32): The eigenvalues of a second order tensor A_{ij} are independent of the coordinate system .

PROOF: Let A_{ij}, x_i, and x_j, be the components of a second order tensor and two vectors in the system K . Let A'_{mn}, x'_m, and x'_n be the components of the second order tensor and two vectors in the system K', and let λ be an eigenvalue in the system K', then we have

$$A'_{mn} x'_n = \lambda x'_m \qquad (1)$$

Now

$$A'_{mn} = \ell_{mi} \ell_{nj} A_{ij} \qquad (2)$$

$$x'_n = \ell_{np} x_p \qquad (3)$$

$$x'_m = \ell_{mq} x_q \qquad (4)$$

Substituting from equations (2) , (3) , and (4) in equation (1) , we get

$$\ell_{mi} \ell_{nj} A_{ij} \ell_{np} x_p = \lambda \ell_{mq} x_q$$

Multiplying both sides by ℓ_{mr}, we get

$$\ell_{mr} \ell_{mi} \ell_{nj} \ell_{np} A_{ij} x_p = \lambda \ell_{mr} \ell_{mq} x_q$$

$$\delta_{ri} \delta_{jp} A_{ij} x_p = \lambda \delta_{rq} x_q$$

$$\delta_{ri} A_{ij} x_j = \lambda x_r$$

or $\quad A_{rj} x_j = \lambda x_r \quad$ or $\quad A_{ij} x_j = \lambda x_i$

which shows that λ is also an eigenvalue in the coordinate system K .

EXAMPLE (20): Find the eigenvalues and eigenvectors of the second order tensor

$$\begin{bmatrix} 13 & -3 & 5 \\ 0 & 4 & 0 \\ -15 & 9 & -7 \end{bmatrix}$$

SOLUTION: In this case, the characteristic equation is $\begin{vmatrix} 13-\lambda & -3 & 5 \\ 0 & 4-\lambda & 0 \\ -15 & 9 & -7-\lambda \end{vmatrix} = 0$

Expanding with the help of second row, we obtain

$$(4-\lambda)[(13-\lambda)(-7-\lambda)+75] = 0$$

or $\quad (4-\lambda)(\lambda^2-6\lambda-16) = 0$

or $\quad (4-\lambda)(\lambda+2)(\lambda-8) = 0$. Therefore, $\lambda = -2, 4, 8$.

Thus the eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 4$, $\lambda_3 = 8$.

Corresponding to the first eigenvalue $\lambda_1 = -2$, the above equations (3) take the form

$$15x_1 - 3x_2 + 5x_3 = 0 \quad (1)$$
$$0x_1 + 6x_2 + 0x_3 = 0 \quad (2)$$
$$-15x_1 + 9x_2 - 5x_3 = 0 \quad (3)$$

From equation (2) we have $x_2 = 0$, therefore equations (1) and (3) become

$$15x_1 + 5x_3 = 0$$
$$-15x_1 - 5x_3 = 0$$

which are practically the same equations. These can be written as $\frac{x_1}{x_3} = \frac{-1}{3}$

So if $x_3 = 3$ then $x_1 = -1$

and corresponding to $\lambda_1 = -2$, the eigenvector $\vec{X} = (-1, 0, 3)$.

Corresponding to the second eigenvalue $\lambda_2 = 4$, the above equations (3) become

$$9x_1 - 3x_2 + 5x_3 = 0 \quad (4)$$
$$0x_1 + 0x_2 + 0x_3 = 0 \quad (5)$$
$$-15x_1 + 9x_2 - 11x_3 = 0 \quad (6)$$

Equation (5) does not help us to find the values of x_1, x_2, and x_3. So practically we have two equations to find x_1, x_2, x_3. Multiplying equation (4) by 3 and adding it to (6) (to eliminate x_2) we obtain

$$12x_1 + 4x_3 = 0 \quad \text{or} \quad \frac{x_1}{x_3} = \frac{-1}{3}$$

So if we take $x_3 = 3$, then $x_1 = -1$ and hence from equations (4) or (6) $x_2 = 2$.

Thus corresponding to $\lambda_2 = 4$, the eigenvector is $\vec{X} = (-1, 2, 3)$.

Similarly corresponding to the third eigenvalue $\lambda_3 = 8$, the above equations (3) are

$$5x_1 - 3x_2 + 5x_3 = 0 \quad (7)$$
$$0x_1 - 4x_2 + 0x_3 = 0 \quad (8)$$
$$-15x_1 + 9x_2 - 15x_3 = 0 \quad (9)$$

From equation (8) we have $x_2 = 0$, therefore equations (7) and (9) become

$$5x_1 + 5x_3 = 0$$
$$-15x_1 - 15x_3 = 0$$

which are practically the same equations.

These can be written as $\frac{x_1}{x_3} = -\frac{1}{1}$

So if $x_1 = 1$ then $x_3 = -1$

Thus corresponding to $\lambda_3 = 8$, the eigenvector is $\vec{X} = (1, 0, -1)$.

7.30 EIGENVALUES AND EIGENVECTORS OF A SECOND ORDER REAL SYMMETRIC TENSOR

REALITY OF EIGENVALUES

THEOREM (7.33): Prove that the eigenvalues of a real symmetric tensor A_{ij} are all real.

PROOF: Let λ be an eigenvalue of a real symmetric tensor A_{ij} so that there exists a non zero vector x_i such that $A_{ij} x_j = \lambda x_i$ (1)

Inner multiplication of both sides with $\bar{x}_i$ gives

$$A_{ij} x_j \bar{x}_i = \lambda x_i \bar{x}_i \quad \text{(bar stands for complex conjugate)}$$

or $$\lambda = \frac{A_{ij} x_j \bar{x}_i}{x_i \bar{x}_i} \qquad (2)$$

Now $x_i \bar{x}_i = x_1 \bar{x}_1 + x_2 \bar{x}_2 + x_3 \bar{x}_3 = |x_1|^2 + |x_2|^2 + |x_3|^2$ is real

Also $$\overline{A_{ij} x_j \bar{x}_i} = \bar{A}_{ij} \bar{x}_j x_i$$

$$= A_{ij} \bar{x}_j x_i \quad \text{(since } A_{ij} \text{ is real)}$$

$$= A_{ji} \bar{x}_i x_j \quad \text{(interchanging the dummies } i \text{ and } j \text{)}$$

$$= A_{ij} \bar{x}_i x_j \quad \text{(since } A_{ij} \text{ is symmetic)}$$

which shows that $A_{ij} x_j \bar{x}_i$ is real. Now since the numerator and denominator of equation (2) are real, therefore λ is real.

ORTHOGONALITY OF EIGENVECTORS

THEOREM (7.34): Prove that eigenvectors of a real symmetric tensor A_{ij} corresponding to two distinct eigenvalues are orthogonal (i.e. perpendicular to each other).

PROOF: Let λ_1 and λ_2 be two distinct eigenvalues of A_{ij}. Then there exist non – zero vectors x_i and y_i such that

$$A_{ij} x_j = \lambda_1 x_i \qquad (1)$$

$$A_{ij} y_j = \lambda_1 y_i \qquad (2)$$

Forming the inner product of equations (1) and (2) with y_i and x_i respectively, we get

$$A_{ij} x_j y_i = \lambda x_i y_i \qquad (3)$$

$$A_{ij} y_j x_i = \lambda y_i x_i \qquad (4)$$

Now $A_{ij} x_j y_i = A_{ji} x_i y_j$ (interchanging the dummies i and j)

$= A_{ij} x_i y_j$ (since A_{ij} is symmetric)

Thus from equations (3) and (4) , we get

$$\lambda_1 x_i y_i = \lambda_2 x_i y_i \quad \text{or} \quad (\lambda_1 - \lambda_2) x_i y_i = 0$$

But since $\lambda_1 - \lambda_2 \neq 0$ (By hypothesis) , therefore $x_i y_i = 0$ or $\vec{X} \cdot \vec{Y} = 0$

i..e $\vec{X}$ and $\vec{Y}$ are orthogonal to each other.

NOTE: (i) A real symmetric tensor has three real eigenvalues of which two or three may be repeated .

(ii) A real symmetric tensor with distinct eigenvalues has a unique set of three mutually orthogonal eigenvectors .

(iii) A real symmetric tensor with one distinct and one repeated eigenvalue has a unique eigenvector corresponding to the distinct eigenvalue . Two eigenvectors may be found corresponding to the repeated eigenvalue such that the three eigenvectors form a mutually orthogonal set .

7.31 PRINCIPAL AXES AND PRINCIPAL DIRECTIONS

A second order symmetric tensor A_{ij} has the property , that by a suitable transformation of the coordinate axes , it can be reduced to diagonal form , i.e. the components A_{ij} are only non zero if $i = j$. The coordinate axes of the new coordinate system (which are mutually perpendicular) are known as the **principal axes** of the tensor and their directions are called the **principal directions** of the tensor . Furthermore , the three diagonal elements A_{11} , A_{22} , A_{33} are the eigenvalues of the tensor A_{ij} .

The following classical theorem is of great importance .

DIAGONALISATION OF A SECOND ORDER SYMMETRIC TENSOR OR REDUCTION TO PRINCIPAL AXIS FORM

THEOREM (7.35): If A is a real second order symmetric tensor , it is always possible to choose a set of principal axes , say $Ox'_1 x'_2 x'_3$ relative to which the components of A are $A'_{1} = \lambda_1$, $A'_{22} = \lambda_2$, $A'_{33} = \lambda_3$ and $A'_{ij} = 0$ for $i \neq j$, where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A .

PROOF: We know that in case of a real symmetric tensor , the eigenvalues are all real and the eigenvectors corresponding to these eigenvalues are mutually orthogonal .

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues and $\vec{X}^{(1)}, \vec{X}^{(2)}, \vec{X}^{(3)}$ the corresponding eigenvectors of the second order symmetric tensor A_{ij} . Then in the system $Ox_1 x_2 x_3$, $A_{ij} x_j^{(m)} = \lambda_m x_i^{(m)}$, (i , m = 1 , 2 , 3) where m is not a summation index . Rotate the system $Ox_1 x_2 x_3$ about the origin to form another orthogonal system $Ox'_1 x'_2 x'_3$ such that the eigenvectors $\vec{X}^{(1)}, \vec{X}^{(2)}, \vec{X}^{(3)}$ are in the directions of Ox'_1, Ox'_2, Ox'_3 axis , respectively . The unit vectors in the directions of $\vec{X}^{(1)}, \vec{X}^{(2)}, \vec{X}^{(3)}$ are given by

$$\hat{e}'_m = \frac{\vec{X}^{(m)}}{|\vec{X}^{(m)}|}, \quad m = 1, 2, 3 .$$

Since $\vec{X}^{(1)}, \vec{X}^{(2)}, \vec{X}^{(3)}$ are eigenvectors, therefore $\hat{e}_1', \hat{e}_2', \hat{e}_3'$ are also eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$, respectively. If A'_{mn} are the components of the tensor in the system $Ox_1' x_2' x_3'$, the eigenvector equation (1) now becomes $A'_{mn}\hat{e}_n' = \lambda_m \hat{e}_m'$

where m is not a summation index.

Taking the dot product of both sides of this equation with $\hat{e}_n'$, we get

$$A'_{mn}\,\hat{e}_n' \cdot \hat{e}_n' = \lambda_m \hat{e}_m' \cdot \hat{e}_n'$$

or $$A'_{mn} = \lambda_m \delta_{mn}$$

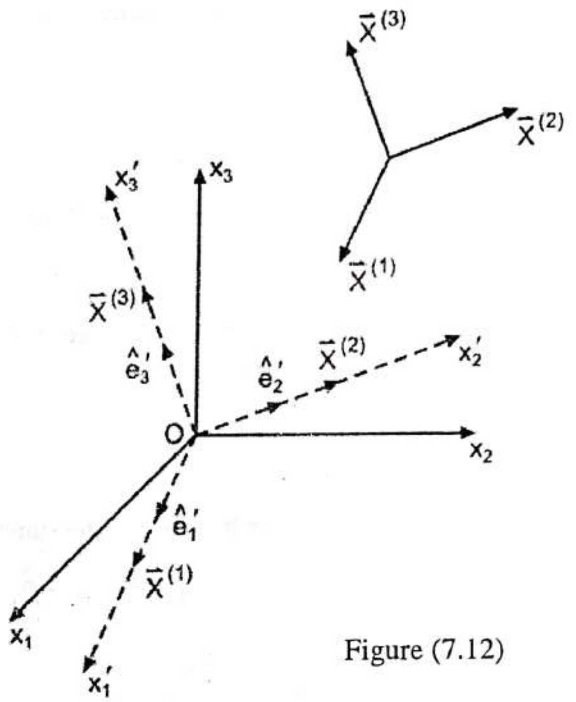

Figure (7.12)

which implies $A'_{11} = \lambda_1,\; A'_{22} = \lambda_2,$

$$A'_{33} = \lambda_3,\; A'_{12} = 0,\; A'_{23} = 0,\; A'_{31} = 0$$

Thus the matrix of the components of the tensor A_{ij} in the system $Ox_1' x_2' x_3'$ takes the form.

$$[A'_{mn}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \qquad (2)$$

From equation (2), we see that the off – diagonal elements are all zero and the elements on the principal diagonal are the eignvalues $\lambda_1, \lambda_2, \lambda_3$ of the tensor A_{ij}. The form of the tensor A_{ij} given by equation (2) is called the principal axis form or diagonal form.

EXAMPLE (21): **Find the eighenvalues and eigenvectors of the second order symmetric tensor**

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & -2 & 0 \\ 6 & 0 & 6 \end{bmatrix}$$

Also, reduce it to the principal axis (i.e. diagonal) form.

SOLUTION: In this case, the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 0 & 6 \\ 0 & -2-\lambda & 0 \\ 6 & 0 & 6-\lambda \end{vmatrix} = 0$$

Expanding the determinant, we get

$$-(1-\lambda)(2+\lambda)(6-\lambda) + 36(2+\lambda) = 0$$

or $$\lambda^3 - 5\lambda^2 - 44\lambda - 60 = 0$$

or $$(\lambda+2)(\lambda^2 - 7\lambda - 30) = 0$$

$$(\lambda+2)(\lambda+3)(\lambda-10) = 0 \quad \text{which gives } \lambda = -2, -3, 10$$

Thus the eigenvalues are $\lambda_1 = -2,\; \lambda_2 = -3,\; \lambda_3 = 10$.

Corresponding to the first eigenvalue $\lambda_1 = -2$, equations (3) on page (432) take the form

$$3x_1 + 6x_3 = 0 \quad (1)$$

$$6x_1 + 8x_3 = 0 \quad (2)$$

which give $x_1 = x_3 = 0$. Since x_2 is undetermined, we take it 1.

Thus corresponding to $\lambda = -2$, the eigenvector is

$$\vec{X}^{(1)} = (0, 1, 0) \quad (3)$$

Corresponding to the second eigenvalue $\lambda_2 = -3$, the equations (3) on page (432) take the form

$$4x_1 + 6x_3 = 0 \quad (4)$$

$$x_2 = 0 \quad (5)$$

$$6x_1 + 9x_3 = 0 \quad (6)$$

Equations (4) and (6) are practically the same equation. These can be written as

$$2x_1 + 3x_3 = 0 \quad \text{or} \quad \frac{x_1}{x_3} = \frac{-3}{2}$$

So if $x_3 = 2$, $x_1 = -3$

Thus corresponding to $\lambda = -3$, the eigenvector is

$$\vec{X}^{(2)} = (-3, 0, 2) \quad (7)$$

Corresponding to the third eigenvalue $\lambda = 10$, equations (3) on page (432) become

$$-9x_1 + 6x_3 = 0 \quad (8)$$

$$-12x_2 = 0 \quad (9)$$

$$6x_1 - 4x_3 = 0 \quad (10)$$

From equation (9) we have $x_2 = 0$. Equations (8) and (10) are practically the same equation. These can be written as

$$3x_1 - 2x_3 = 0 \quad \text{or} \quad \frac{x_1}{x_3} = \frac{2}{3}$$

So if $x_3 = 3$, we get $x_1 = 2$

Thus corresponding to $\lambda = 10$, the eigenvector is

$$\vec{X}^{(3)} = (2, 0, 3) \quad (11)$$

Now the unit vectors in the new system $Ox'_1 x'_2 x'_3$ are given by

$$\hat{e}_1' = \frac{\vec{X}^{(1)}}{|\vec{X}^{(1)}|} = (0, 1, 0) \quad (12)$$

$$\hat{e}_2' = \frac{\vec{X}^{(2)}}{|\vec{X}^{(2)}|} = \left(-\frac{3}{\sqrt{13}}, 0, \frac{2}{\sqrt{13}}\right) \quad (13)$$

$$\hat{e}_3' = \frac{\vec{X}^{(3)}}{|\vec{X}^{(3)}|} = \left(\frac{2}{\sqrt{13}}, 0, \frac{3}{\sqrt{13}}\right) \quad (14)$$

From equations (12), (13), and (14), we get

$$\left.\begin{aligned} \hat{e}_1' \cdot \hat{e}_1' = \hat{e}_2' \cdot \hat{e}_2' = \hat{e}_3' \cdot \hat{e}_3' = 1 \\ \hat{e}_1' \cdot \hat{e}_2' = \hat{e}_2' \cdot \hat{e}_3' = \hat{e}_3' \cdot \hat{e}_1' = 0 \end{aligned}\right] \quad (15)$$

Now we know that

$$A'_{mn}\,\hat{e}_n' = \lambda_m \hat{e}_m' \quad (16)$$

For $m = 1$, equation (16) becomes

$$A'_{1n}\,\hat{e}_n' = \lambda_1 \hat{e}_1'$$

or $\quad A'_{1n}(\hat{e}_n' \cdot \hat{e}_n') = \lambda_1(\hat{e}_1' \cdot \hat{e}_n')$

or $\quad A'_{1n} = \lambda_1(\hat{e}_1' \cdot \hat{e}_n')$

which implies on using equation (15), $A'_{11} = \lambda_1 = -2, \quad A'_{12} = 0, \quad A'_{13} = 0$

For $m = 2$, equation (16) becomes

$$A'_{2n}\,\hat{e}_n' = \lambda_2 \hat{e}_2'$$

or $\quad A'_{2n}(\hat{e}_n' \cdot \hat{e}_n') = \lambda_2(\hat{e}_2' \cdot \hat{e}_n')$

or $\quad A'_{2n} = \lambda_2(\hat{e}_2' \cdot \hat{e}_n')$

which implies on using equation (15), $A'_{21} = 0, \quad A'_{22} = \lambda_2 = -3, \quad A'_{23} = 0$

For $m = 3$, equation (16) becomes

$$A'_{3n}\,\hat{e}_n' = \lambda_3 \hat{e}_3'$$

or $\quad A'_{3n}(\hat{e}_n' \cdot \hat{e}_n') = \lambda_3(\hat{e}_3' \cdot \hat{e}_n')$

or $\quad A'_{3n} = \lambda_3(\hat{e}_3' \cdot \hat{e}_n')$

which implies on using equation (15), $A'_{31} = 0, \quad A'_{32} = 0, \quad A'_{33} = \lambda_3 = 10$

Thus the required diagonal form of the given tensor is $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

7.32 INVARIANTS OF A TENSOR

An expression which does not change under transformations from one coordinate system to another is called an **invariant**.

There is only one independent invariant that can be constructed from a first order tensor i.e. vector say A_i. Denoting the invariant by I, we have

$$I = A_i A_i = A_1^2 + A_2^2 + A_3^2$$

The invariants of a second order tensor A_{ij} are obtained from the characteristic equation

$$\begin{vmatrix} A_{11} - \lambda & A_{12} & A_{13} \\ A_{21} & A_{22} - \lambda & A_{23} \\ A_{31} & A_{32} & A_{33} - \lambda \end{vmatrix} = 0 \quad (1)$$

The equation (1) can be written as (after expanding the determinant) .

$$\lambda^3 - \lambda^2 (A_{11} + A_{22} + A_{33}) + \lambda [A_{11}A_{22} + A_{22}A_{33} + A_{33}A_{11} - A_{12}A_{21} - A_{23}A_{32} - A_{31}A_{13}] - \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = 0 \qquad (2)$$

The numbers $\lambda, \lambda^2, \lambda^3$ being scalars , are independent of the choice of the coordinate system , and hence so are the coefficients in equation (2) . Therefore the quantities

$$I_1 = A_{11} + A_{22} + A_{33} = A_{ii}$$

$$I_2 = A_{11}A_{22} + A_{22}A_{33} + A_{33}A_{11} - A_{12}A_{21} - A_{23}A_{32} - A_{31}A_{13} = \frac{1}{2}(A_{ii}A_{jj} - A_{ij}A_{ji})$$

$$I_3 = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = |A_{ij}|$$

are all the invariants of the tensor A_{ij}. Using equation (3) , we can form infinitely many other invariants, for example ,

$$I_1^2 = (A_{ii})^2,$$

$$I_1^2 - 2I_2 = A_{ij}A_{ji} \text{ and so on .}$$

We now establish the invariance of the quantities I_1, I_2, I_3 under right – handed orthogonal transformation of axes . Consider $A_{ij} - \lambda\delta_{ij}$ where λ is a scalar constant . Now A_{ij} and $\lambda\delta_{ij}$ are second order tensors , therefore $A_{ij} - \lambda\delta_{ij}$ is also a second order tensor . Let $A'_{mn} - \lambda\delta'_{mn}$ be the components of this tensor in the system K' , then the transformation law for this tensor is

$$A'_{mn} - \lambda\delta'_{mn} = \ell_{mi}\ell_{nj}(A_{ij} - \lambda\delta_{ij})$$

Now we know that $\delta'_{mn} = \delta_{mn}$ (the unit matrix) and also that $|\ell_{mi}| = |\ell_{nj}| = 1$, so that

$$|A'_{mn} - \lambda\delta_{mn}| = |A_{ij} - \lambda\delta_{ij}|$$

or

$$\begin{vmatrix} A'_{11}-\lambda & A'_{12} & A'_{13} \\ A'_{21} & A'_{22}-\lambda & A'_{23} \\ A'_{31} & A'_{32} & A'_{33}-\lambda \end{vmatrix} = \begin{vmatrix} A_{11}-\lambda & A_{12} & A_{13} \\ A_{21} & A_{22}-\lambda & A_{23} \\ A_{31} & A_{32} & A_{33}-\lambda \end{vmatrix}$$

Each side of this equation is a cubic in λ and since it must hold for any value of λ , then the coefficients of each cubic must be the same . The coefficient of λ^2 gives

$$A'_{11} + A'_{22} + A'_{33} = A_{11} + A_{22} + A_{33}$$

i.e. $A_{ii} = A_{11} + A_{22} + A_{33}$ called the **trace** (i.e. the sum of elements on the principal diagonal) is invariant under transformation of axes .

Next , coefficient of λ gives

$$\begin{vmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{vmatrix} + \begin{vmatrix} A'_{22} & A'_{23} \\ A'_{32} & A'_{33} \end{vmatrix} + \begin{vmatrix} A'_{33} & A'_{31} \\ A'_{13} & A'_{11} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{33} & A_{31} \\ A_{13} & A_{11} \end{vmatrix}$$

which shows that $\frac{1}{2}(A_{ii} - A_{ij}A_{ji})$ is invariant under the transformation of axes . Finally , from the constant term , we find

$$\begin{vmatrix} A'_{11} & A'_{12} & A'_{13} \\ A'_{21} & A'_{22} & A'_{23} \\ A'_{31} & A'_{32} & A'_{33} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

which show that the determinant of the tensor i.e. $|A_{ij}|$ is invariant under transformation .

INVARIANTS OF A SECOND ORDER SYMMETRIC TENSOR

We know that if $\lambda_1 , \lambda_2 , \lambda_3$ are the eigenvalues of a second order symmetric tensor A_{ij} , then in the principal axes system , the tensor A_{ij} has a diagonal matrix of the form

$$[A'_{mn}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Therefore , $I_1 = A'_{mm} = A'_{11} + A'_{22} + A'_{33} = \lambda_1 + \lambda_2 + \lambda_3$

$$I_2 = \begin{vmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{vmatrix} + \begin{vmatrix} A'_{22} & A'_{23} \\ A'_{32} & A'_{33} \end{vmatrix} + \begin{vmatrix} A'_{33} & A'_{31} \\ A'_{13} & A'_{11} \end{vmatrix}$$

$$= \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{vmatrix} + \begin{vmatrix} \lambda_3 & 0 \\ 0 & \lambda_1 \end{vmatrix} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$I_3 = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3 .$$

EXAMPLE (22): **Find the invariants of the following second order tensors :**

(i) $\begin{bmatrix} 2 & 4 & -1 \\ 6 & -7 & 10 \\ 3 & -4 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & -1 & -2 \\ -1 & 7 & 2 \\ -2 & 2 & 4 \end{bmatrix}$

SOLUTION: (i) The invariants of a second order tensor are given by

$$I_1 = A_{11} + A_{22} + A_{33} = 2 - 7 + 6 = 1$$

$$I_2 = A_{11}A_{22} + A_{22}A_{33} + A_{33}A_{11} - A_{12}A_{21} - A_{23}A_{32} - A_{31}A_{13}$$

$$= (2)(-7) + (-7)(6) + (6)(2) - (4)(6) - (10)(-4) - (3)(-1)$$

$$= -14 - 42 + 12 - 24 + 40 + 3 = -80 + 55 = -25$$

$$I_3 = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} 2 & 4 & -1 \\ 6 & -7 & 10 \\ 3 & -4 & 6 \end{vmatrix}$$

$$= 2(-2) - 4(6) - 1(-3) = -4 - 24 + 3 = -25$$

(ii) In this case , the characteristic equation is $\begin{vmatrix} 7-\lambda & -1 & -2 \\ -1 & 7-\lambda & 2 \\ -2 & 2 & 4-\lambda \end{vmatrix} = 0$

or $(7-\lambda)(24 - 11\lambda + \lambda^2) + 1(\lambda) - 2(12 - 2\lambda) = 0$

$$168 - 101\lambda + 18\lambda^2 - \lambda^3 + \lambda - 24 + 4\lambda = 0$$

or $$-\lambda^3 + 18\lambda^2 - 96\lambda + 144 = 0$$

or $$\lambda^3 - 18\lambda^2 + 96\lambda - 144 = 0$$

or $$(\lambda - 6)(\lambda^2 - 12\lambda + 24) = 0$$

This implies that $\lambda = 6,\ 6 + 2\sqrt{3},\ 6 - 2\sqrt{3}$. Thus the three eigenvalues are

$$\lambda_1 = 6, \quad \lambda_2 = 6 + 2\sqrt{3}, \quad \lambda_3 = 6 - 2\sqrt{3}$$

Since the given tensor is symmetric, therefore the invariants of this tensor are given by

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3 = 6 + 6 + 2\sqrt{3} + 6 - 2\sqrt{3} = 18$$

$$I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 6(6 + 2\sqrt{3}) + (6 + 2\sqrt{3})(6 - 2\sqrt{3}) + (6 - 2\sqrt{3})(6)$$

$$= 36 + 12\sqrt{3} + 36 - 12 + 36 - 12\sqrt{3} = 96$$

$$I_3 = \lambda_1\lambda_2\lambda_3 = 6(6 + 2\sqrt{3})(6 - 2\sqrt{3}) = 6(36 - 12) = 144$$

7.33 DEVIATORS

Tensors for which the invariant I_1 vanishes are called deviators.

THEOREM (7.36): Prove that any second order tensor A_{ij} can be written as the sum of a deviator and an isotropic tensor.

PROOF:

$$A_{ij} = A_{ij} - \frac{1}{3} A_{kk}\delta_{ij} + \frac{1}{3} A_{kk}\delta_{ij}$$

$$= D_{ij} + \frac{1}{3} A_{kk}\delta_{ij} \qquad (1)$$

where $$D_{ij} = A_{ij} - \frac{1}{3} A_{kk}\delta_{ij} \qquad (2)$$

Now we shall prove that D_{ij} is a deviator and $\frac{1}{3} A_{kk}\delta_{ij}$ is isotropic.

From equation (2) we have

$$D_{ii} = A_{ii} - \frac{1}{3} A_{kk}\delta_{ii}$$

which is equivalent to the following three equations

$$D_{11} = A_{11} - \frac{1}{3}(A_{11} + A_{22} + A_{33})$$

$$D_{22} = A_{22} - \frac{1}{3}(A_{11} + A_{22} + A_{33})$$

$$D_{33} = A_{33} - \frac{1}{3}(A_{11} + A_{22} + A_{33})$$

Adding we have $D_{11} + D_{22} + D_{33} = 0$ i.e. $D_{ii} = 0$

which shows that D_{ij} is a deviator. Again, we know that A_{ik} is a second order tensor. If we apply contraction to it, we get A_{kk} as scalar. Thus $\frac{1}{3} A_{kk}\delta_{ij}$ is nothing but δ_{ij} multiply by a scalar. Again we know that δ_{ij} is isotropic. Therefore $\frac{1}{3} A_{kk}\delta_{ij}$ must be isotropic.

7.34 SOLVED PROBLEMS

PROBLEM (1): **Write the following system of equations using summation convention .**

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

SOLUTION: The given system can be written as

$$\left.\begin{aligned} a_{1j}x_j &= b_1 \\ a_{2j}x_j &= b_2 \\ a_{3j}x_j &= b_3 \end{aligned}\right\} \qquad (1)$$

where the summation is on the repeated index j from 1 to 3 . The system (1) can further be written as

$$a_{ij}x_j = b_i, \quad i = 1, 2, 3.$$

Note that assigning three values to the index i in the above equation generates three such equations of the system (1) .

PROBLEM (2): **Write out explicitly the following equation in full :**

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = F_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i}, \quad i = 1, 2, 3.$$

SOLUTION: Assigning three values to the free index i , the given equation generates three equations as follows :

$$\left.\begin{aligned} \frac{\partial u_1}{\partial t} + u_j \frac{\partial u_1}{\partial x_j} &= F_1 - \frac{1}{\rho}\frac{\partial p}{\partial x_1} \\ \frac{\partial u_2}{\partial t} + u_j \frac{\partial u_2}{\partial x_j} &= F_2 - \frac{1}{\rho}\frac{\partial p}{\partial x_2} \\ \frac{\partial u_3}{\partial t} + u_j \frac{\partial u_3}{\partial x_j} &= F_3 - \frac{1}{\rho}\frac{\partial p}{\partial x_3} \end{aligned}\right\} \qquad (1)$$

Now the summation is on the repeated index j from 1 to 3 , therefore the system (1) can be written as

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2} + u_3\frac{\partial u_1}{\partial x_3} = F_1 - \frac{1}{\rho}\frac{\partial p}{\partial x_1}$$

$$\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2} + u_3\frac{\partial u_2}{\partial x_3} = F_2 - \frac{1}{\rho}\frac{\partial p}{\partial x_2}$$

$$\frac{\partial u_3}{\partial t} + u_1\frac{\partial u_3}{\partial x_1} + u_2\frac{\partial u_3}{\partial x_2} + u_3\frac{\partial u_3}{\partial x_3} = F_3 - \frac{1}{\rho}\frac{\partial p}{\partial x_3}$$

In Fluid mechanics , these equations are called the Euler's equations of motion for an inviscid incompressible flow .

PROBLEM (3): Write out explicitly the terms in the expression $a_{ij}\, a_{km}\, x_{jm}$; $j, m = 1, 2, 3.$

SOLUTION:
$$a_{ij}\, a_{km}\, x_{jm} = a_{i1}\, a_{km}\, x_{1m} + a_{i2}\, a_{km}\, x_{2m} + a_{i3}\, a_{km}\, x_{3m}$$
$$= a_{i1}\, a_{k1}\, x_{11} + a_{i1}\, a_{k2}\, x_{12} + a_{i1}\, a_{k3}\, x_{13}$$
$$+ a_{i2}\, a_{k1}\, x_{21} + a_{i2}\, a_{k2}\, x_{22} + a_{i3}\, a_{k3}\, x_{23}$$
$$+ a_{i3}\, a_{k1}\, x_{31} + a_{i3}\, a_{k2}\, x_{32} + a_{i3}\, a_{k3}\, x_{33}$$

PROBLEM (4): Write the following equation using summation convention :

$$y_{11} = a_{11}\, a_{11}\, x_{11} + a_{11}\, a_{12}\, x_{12} + a_{11}\, a_{13}\, x_{13} + a_{12}\, a_{11}\, x_{21} + a_{12}\, a_{12}\, x_{22} + a_{12}\, a_{13}\, x_{23}$$
$$+ a_{13}\, a_{11}\, x_{31} + a_{13}\, a_{12}\, x_{32} + a_{13}\, a_{13}\, x_{33}.$$

SOLUTION: Write the given equation as

$$y_{11} = a_{11}\,(a_{11}\, x_{11} + a_{12}\, x_{12} + a_{13}\, x_{13}) + a_{12}\,(a_{11}\, x_{21} + a_{12}\, x_{22} + a_{13}\, x_{23})$$
$$+ a_{13}\,(a_{11}\, x_{31} + a_{12}\, x_{32} + a_{13}\, x_{33})$$
$$= a_{11}\, a_{1j}\, x_{1j} + a_{12}\, a_{1j}\, x_{2j} + a_{13}\, a_{1j}\, x_{3j}$$
$$= a_{1i}\, a_{1j}\, x_{ij}$$

PROBLEM (5): Suppose that $Q = g_{ir}\, a_{rs}\, y_s$ and $y_i = b_{ir}\, x_r$. If further, $a_{ir}\, b_{rj} = \delta_{ij}$, find Q in terms of x_r.

SOLUTION: First write $y_s = b_{st}\, x_t$. Then by substitution,

$$Q = g_{ir}\, a_{rs}\, b_{st}\, x_t$$
$$= g_{ir}\, \delta_{rt}\, x_t = g_{ir}\, x_r$$

PROBLEM (6): Consider a system of linear equations of the form $y_i = A_{ij}\, x_j$ and suppose that $[B_{ij}]$ is a matrix of numbers such that for all i and j, $B_{ir}\, A_{rj} = \delta_{ij}$ (i.e. the matrix $[B_{ij}]$ is the inverse of the matrix $[A_{ij}]$). Solve the system for x_i in terms of the y_j.

SOLUTION: We have $y_i = A_{ij}\, x_j$

Multiply both sides of the this equation by B_{ki} and sum over i :

$$B_{ki}\, y_i = B_{ki}\, A_{ij}\, x_j = \delta_{kj}\, x_j = x_k$$

or $x_i = B_{ij}\, y_j$.

PROBLEM (7): If $r^2 = x_1^2 + x_2^2 + x_3^2$, show that $\dfrac{\partial r}{\partial x_k} = \dfrac{x_k}{r}$.

SOLUTION: We can write $r^2 = x_i\, x_i$

Differentiating both sides w.r.t. x_k, we get

$$2r\frac{\partial r}{\partial x_k} = x_i\frac{\partial x_i}{\partial x_k} + x_i\frac{\partial x_i}{\partial x_k} = 2x_i\frac{\partial x_i}{\partial x_k} = 2x_i\,\delta_{ik} = 2x_k$$

or $$\frac{\partial r}{\partial x_k} = \frac{x_k}{r}$$

PROBLEM (8): **If a_{ij} are constants, calculate the partial derivative $\frac{\partial}{\partial x_k}(a_{ij}x_ix_j)$**

SOLUTION: We have

$$\frac{\partial}{\partial x_k}(a_{ij}x_ix_j) = a_{ij}\frac{\partial}{\partial x_k}(x_ix_j)$$

$$= a_{ij}\left(x_i\frac{\partial x_j}{\partial x_k} + x_j\frac{\partial x_i}{\partial x_k}\right)$$

$$= a_{ij}(x_i\delta_{jk} + x_j\delta_{ik}) = a_{ik}x_i + a_{kj}x_j$$

$$= a_{ik}x_i + a_{ki}x_i = (a_{ik} + a_{ki})x_i$$

$a_{ij}\delta_{jk} = a_{ik}$

PROBLEM (9): **If $a_{ij} = a_{ji}$ are constants, calculate $\frac{\partial^2}{\partial x_k \partial x_m}(a_{ij}x_ix_j)$**

SOLUTION: Using problem (8), we have

$$\frac{\partial^2}{\partial x_k \partial x_m}(a_{ij}x_ix_j) = \frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_m}(a_{ij}x_ix_j)\right]$$

$$= \frac{\partial}{\partial x_k}\left[(a_{im} + a_{mi})x_i\right] = \frac{\partial}{\partial x_k}(2a_{im}x_i)$$

$$= 2a_{im}\frac{\partial x_i}{\partial x_k} = 2a_{im}\delta_{ik} = 2a_{km}$$

PROBLEM (10): **Write the transformation matrix for a rotation of angle π in the positive sense about (i) the x_1 – axis (ii) the x_2 – axis (iii) the x_3 – axis.**

SOLUTION: (i) The new system $Ox'_1x'_2x'_3$ formed by rotation of the system $Ox_1x_2x_3$ through an angle of π in the positive sense about the x_1 – axis is shown in figure (7.13). Then by definition,

$$\ell_{11} = \cos(x'_1 O x_1) = \cos 0 = 1$$

$$\ell_{12} = \cos(x'_1 O x_2) = \cos \pi/2 = 0$$

$$\ell_{13} = \cos(x'_1 O x_3) = \cos \pi/2 = 0$$

$$\ell_{21} = \cos(x'_2 O x_1) = \cos \pi/2 = 0$$

$$\ell_{22} = \cos(x'_2 O x_2) = \cos \pi = -1$$

$$\ell_{23} = \cos(x'_2 O x_3) = \cos \pi/2 = 0$$

$$\ell_{31} = \cos(x'_3 O x_1) = \cos \pi/2 = 0$$

$$\ell_{32} = \cos(x'_3 O x_2) = \cos \pi/2 = 0$$

$$\ell_{33} = \cos(x'_3 O x_3) = \cos \pi = -1$$

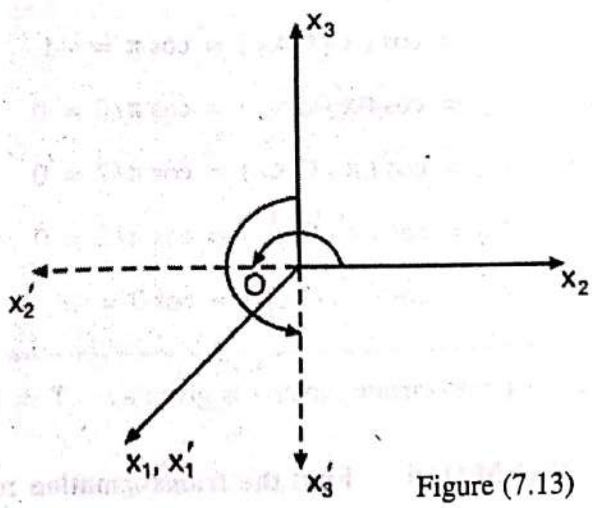

Figure (7.13)

Thus the transformation matrix is given by $T = [\ell_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(ii) The new system $Ox'_1 x'_2 x'_3$ formed by rotating the system $Ox_1 x_2 x_3$ through an angle of π in the positive sense about the x_2 – axis is shown in figure (7.14). Then by definition,

$\ell_{11} = \cos(x'_1 O x_1) = \cos\pi = -1$

$\ell_{12} = \cos(x'_1 O x_2) = \cos\pi/2 = 0$

$\ell_{13} = \cos(x'_1 O x_3) = \cos\pi/2 = 0$

$\ell_{21} = \cos(x'_2 O x_1) = \cos\pi/2 = 0$

$\ell_{22} = \cos(x'_2 O x_2) = \cos 0 = 1$

$\ell_{23} = \cos(x'_2 O x_3) = \cos\pi/2 = 0$

$\ell_{31} = \cos(x'_3 O x_1) = \cos\pi/2 = 0$

$\ell_{32} = \cos(x'_3 O x_2) = \cos\pi/2 = 0$

$\ell_{33} = \cos(x'_3 O x_3) = \cos\pi = -1$

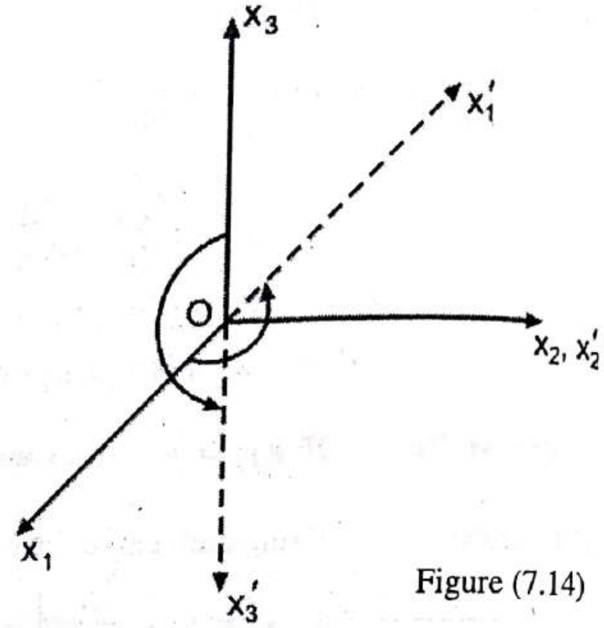

Figure (7.14)

Thus the transformation matrix is given by $T = [\ell_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(iii) The new system $Ox'_1 x'_2 x'_3$ formed by rotating the system $Ox_1 x_2 x_3$ through an angle of π in the positive sense about the x_3 – axis is shown in figure (7.15). Then by definition,

$\ell_{11} = \cos(x'_1 O x_1) = \cos\pi = -1$

$\ell_{12} = \cos(x'_1 O x_2) = \cos\pi/2 = 0$

$\ell_{13} = \cos(x'_1 O x_3) = \cos\pi/2 = 0$

$\ell_{21} = \cos(x'_2 O x_1) = \cos\pi/2 = 0$

$\ell_{22} = \cos(x'_2 O x_2) = \cos\pi = -1$

$\ell_{23} = \cos(x'_2 O x_3) = \cos\pi/2 = 0$

$\ell_{31} = \cos(x'_3 O x_1) = \cos\pi/2 = 0$

$\ell_{32} = \cos(x'_3 O x_2) = \cos\pi/2 = 0$

$\ell_{33} = \cos(x'_3 O x_3) = \cos 0 = 1$

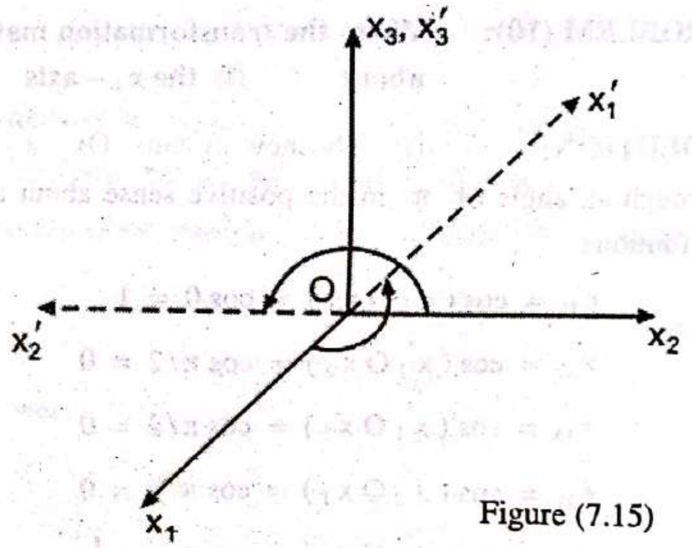

Figure (7.15)

Thus the transformation matrix is given by $T = [\ell_{ij}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

PROBLEM (11): Find the transformation matrix for a rotation of angle $\pi/4$ in the negative sense about the x_1 – axis.

SOLUTION: The new system $Ox'_1 x'_2 x'_3$ formed by rotation of the system $Ox_1 x_2 x_3$ through an angle of $\pi/4$ in the negative sense about the x_1 – axis is shown in figure (7.16). Then by definition,

$\ell_{11} = \cos(x'_1 O x_1) = \cos 0 = 1$

$\ell_{12} = \cos(x'_1 O x_2) = \cos \pi/2 = 0$

$\ell_{13} = \cos(x'_1 O x_3) = \cos \pi/2 = 0$

$\ell_{21} = \cos(x'_2 O x_1) = \cos \pi/2 = 0$

$\ell_{22} = \cos(x'_2 O x_2) = \cos(-\pi/4) = \frac{1}{\sqrt{2}}$

$\ell_{23} = \cos(x'_2 O x_3) = \cos(-3\pi/4) = -\frac{1}{\sqrt{2}}$

$\ell_{31} = \cos(x'_3 O x_1) = \cos \pi/2 = 0$

$\ell_{32} = \cos(x'_3 O x_2) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$\ell_{33} = \cos(x'_3 O x_3) = \cos(-\pi/4) = \frac{1}{\sqrt{2}}$

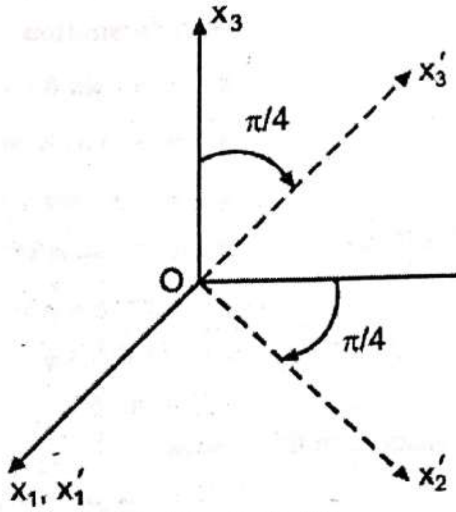

Figure (7.16)

Thus the transformation matrix is given by $T = [\ell_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

PROBLEM (12): Find the transformation matrix for a rotation of angle $\pi/2$ about the x_1–axis, followed by a rotation of angle $\pi/2$ about the x'_2 – axis, both in the positive sense.

SOLUTION: The system $Ox''_1 x''_2 x''_3$ obtained after the combined rotation is shown in figure (7.17). Then by definition,

$\ell_{11} = \cos(x''_1 O x_1) = \cos \pi/2 = 0$

$\ell_{12} = \cos(x''_1 O x_2) = \cos 0 = 1$

$\ell_{13} = \cos(x''_1 O x_3) = \cos \pi/2 = 0$

$\ell_{21} = \cos(x''_2 O x_1) = \cos \pi/2 = 0$

$\ell_{22} = \cos(x''_2 O x_2) = \cos \pi/2 = 0$

$\ell_{23} = \cos(x''_2 O x_3) = \cos 0 = 1$

$\ell_{31} = \cos(x''_3 O x_1) = \cos 0 = 1$

$\ell_{32} = \cos(x''_3 O x_2) = \cos \pi/2 = 0$

$\ell_{33} = \cos(x''_3 O x_3) = \cos \pi/2 = 0$

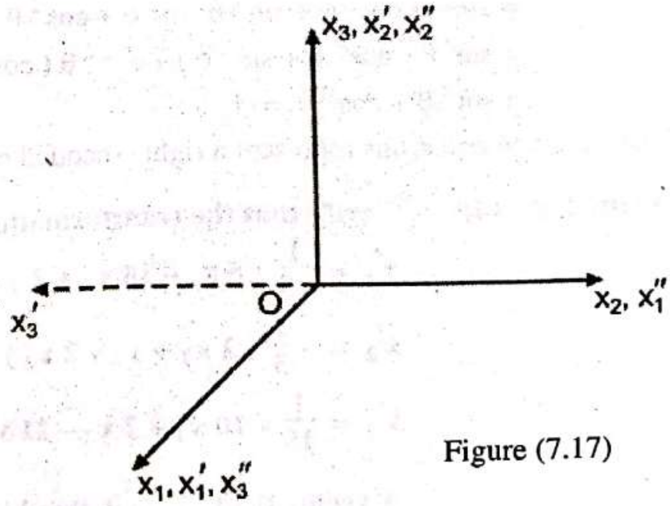

Figure (7.17)

Thus the transformation matrix is given by $T = [\ell_{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

PROBLEM (13): **Show that the following equations represent a right – handed orthogonal transformation :**

$$x'_1 = x_1 \sin\theta\cos\phi + x_2 \sin\theta \sin\phi + x_3 \cos\theta$$

$$x'_2 = x_1 \cos\theta\cos\phi + x_2 \cos\theta\sin\phi - x_3 \sin\theta$$

$$x'_3 = -x_1 \sin\phi + x_2 \cos\phi$$

SOLUTION: From the transformation equations , we can write the transformation matrix T as

$$T = [\ell_{ij}] = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}$$

The transpose of this matrix is

$$T' = [\ell_{ji}] = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$$

and so $$TT' = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}\begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

which shows that the transformation is orthogonal .

Also $$\det T = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,(0 + \sin\theta\cos\phi) + \sin\theta\sin\phi\,(\sin\theta\sin\phi - 0) + \cos\theta\,(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi)$$

$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi$$

$$= \sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta\,(\cos^2\phi + \sin^2\phi)$$

$$= \sin^2\theta + \cos^2\theta = 1$$

Thus the given equations represent a right – handed orthogonal transformation .

PROBLEM (14): **Verify that the transformation**

$$x'_1 = \frac{1}{15}(5x_1 - 14x_2 + 2x_3)$$

$$x'_2 = -\frac{1}{3}(2x_1 + x_2 + 2x_3)$$

$$x'_3 = \frac{1}{15}(10x_1 + 2x_2 - 11x_3)$$ **is orthogonal and right – handed .**

A vector field $\vec{A}$ is defined in the system $Ox_1x_2x_3$ by

$$A_1 = x_1^2,\ A_2 = x_2^2,\ A_3 = x_3^2 .$$

Evaluate the components A'_j of the vector field in the new system $Ox'_1x'_2x'_3$.

This equation implies the following three equations :

$$A'_1 = \ell_{11} A_1 + \ell_{12} A_2 + \ell_{13} A_3$$

$$A'_2 = \ell_{21} A_1 + \ell_{22} A_2 + \ell_{23} A_3$$

$$A'_3 = \ell_{31} A_1 + \ell_{32} A_2 + \ell_{33} A_3$$

where $A_1 = x_1^2 = \frac{1}{9}(x'_1 - 2x'_2 + 2x'_3)^2$

$$A_2 = x_2^2 = \frac{1}{225}(14x'_1 + 5x'_2 - 2x'_3)^2$$

$$A_3 = x_3^2 = \frac{1}{225}(2x'_1 - 10x'_2 - 11x'_3)^2$$

PROBLEM (15): **The vector $\vec{A}$ has components $(4, 5, 6)$ in the system $Ox_1x_2x_3$. A new coordinate system $Ox'_1x'_2x'_3$ is formed by rotating the original system through an angle $\pi/6$ in the positive sense about the x_3 – axis. The system $Ox'_1x'_2x'_3$ is further rotated through an angle of $\pi/2$ in the positive sense about the x'_1 – axis to obtain another coordinate system $Ox''_1x''_2x''_3$. Find the transformation matrix for this combined rotation. Hence determine the components of $\vec{A}$ in the new system $Ox''_1x''_2x''_3$.**

SOLUTION: The new system $Ox''_1x''_2x''_3$ formed by the combined rotation is shown in figure (7.18). Then by definition,

$$\ell_{11} = \cos(x''_1 O x_1) = \cos\pi/6 = \frac{\sqrt{3}}{2}$$

$$\ell_{12} = \cos(x''_1 O x_2) = \cos\pi/3 = \frac{1}{2}$$

$$\ell_{13} = \cos(x''_1 O x_3) = \cos\pi/2 = 0$$

$$\ell_{21} = \cos(x''_2 O x_1) = \cos\pi/2 = 0$$

$$\ell_{22} = \cos(x''_2 O x_2) = \cos\pi/2 = 0$$

$$\ell_{23} = \cos(x''_2 O x_3) = \cos 0 = 1$$

$$\ell_{31} = \cos(x''_3 O x_1) = \cos\pi/3 = \frac{1}{2}$$

$$\ell_{32} = \cos(x''_3 O x_2) = \cos 5\pi/6 = -\frac{\sqrt{3}}{2}$$

$$\ell_{33} = \cos(x''_3 O x_3) = \cos\pi/2 = 0$$

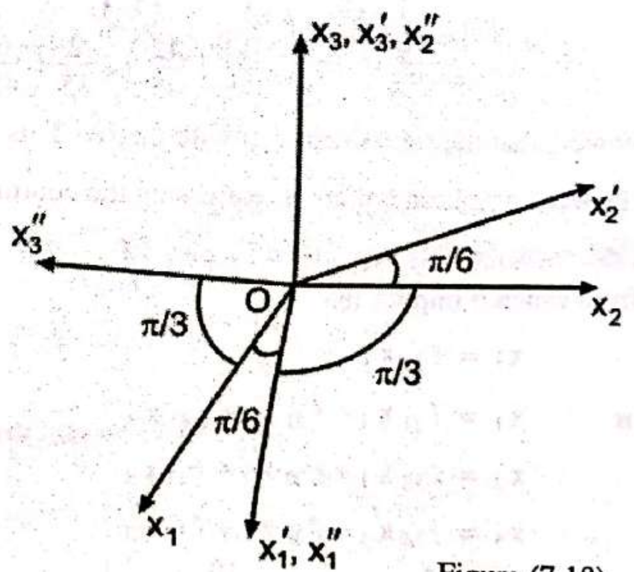

Figure (7.18)

Thus the transformation matrix for the combined rotation is

$$T = [\ell_{ij}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{bmatrix}$$

Now $A''_1 = \ell_{11} A_1 + \ell_{12} A_2 + \ell_{13} A_3 = \frac{\sqrt{3}}{2}(4) + \frac{1}{2}(5) + (0)6 = 2\sqrt{3} + \frac{5}{2} = \frac{1}{2}(5 + 4\sqrt{3})$

$A''_2 = \ell_{21} A_1 + \ell_{22} A_2 + \ell_{23} A_3 = (0)(4) + (0)(5) + 1(6) = 6$

$A''_3 = \ell_{31} A_1 + \ell_{32} A_2 + \ell_{33} A_3 = \frac{1}{2}(4) + \left(\frac{-\sqrt{3}}{2}\right)(5) + (0)(6) = \frac{1}{2}(4 - 5\sqrt{3})$

The components of $\vec{A}$ in the new system are $\left[\frac{1}{2}(5 + 4\sqrt{3}) \,,\ 6 \,,\ \frac{1}{2}(4 - 5\sqrt{3})\right]$

It can be seen that $A_1^2 + A_2^2 + A_3^2 = A''^2_1 + A''^2_2 + A A''^2_3 = 77$.

PROBLEM (16): **The vector $\vec{A}$ has components (A_1, A_2, A_3) in the system $O x_1 x_2 x_3$. A new coordinate system $Ox'_1 x'_2 x'_3$ is formed by rotating the original system through an angle α in the positive sense about the x_3 – axis. The system $Ox'_1 x'_2 x'_3$ is further rotated through an angle of β in the positive sense about the x'_1 – axis to obtain another coordinate system $Ox''_1 x''_2 x''_3$. Find the transformation matrix for this combined rotation. Hence determine the components of $\vec{A}$ in the new system $Ox''_1 x''_2 x''_3$.**

SOLUTION: The new system $Ox''_1 x''_2 x''_3$ formed by the combined rotation is shown in figure (7.19). For such a rotation the concept of projection of a vector will be used. Let $\hat{e}''_1, \hat{e}''_2, \hat{e}''_3$ be the unit vectors in the directions of x''_1, x''_2, x''_3 respectively.

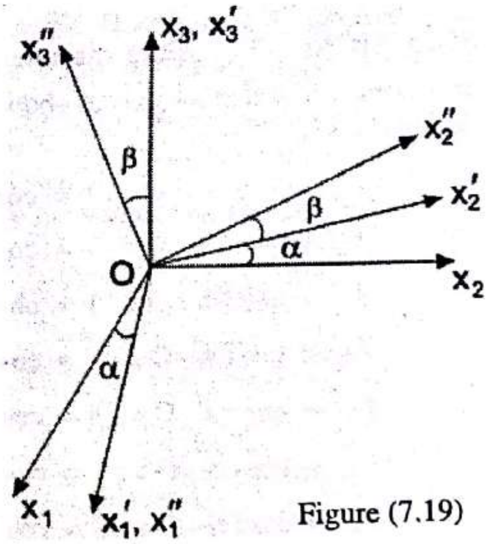

Figure (7.19)

Then, for example, to calculate ℓ_{22}, we have

$$\text{Proj}_{\hat{e}'_2}\, \hat{e}''_2 = |\hat{e}''_2| \cos\beta\ \hat{e}'_2 = \cos\beta\, \hat{e}'_2$$

and $$\text{Proj}_{\hat{e}_2} (\cos\beta\, \hat{e}'_2) = |\cos\beta\, \hat{e}'_2| \cos\alpha = \cos\beta\cos\alpha$$

Hence, by definition

$$\ell_{11} = \cos(x''_1 O x_1) = \cos\alpha$$

$$\ell_{12} = \cos(x''_1 O x_2) = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$$

$$\ell_{13} = \cos(x''_1 O x_3) = \cos\frac{\pi}{2} = 0$$

$$\ell_{21} = \cos(x''_2 O x_1) = \cos\beta\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha\cos\beta$$

$$\ell_{22} = \cos(x''_2 O x_2) = \cos\beta\cos\alpha = \cos\alpha\cos\beta$$

$$\ell_{23} = \cos(x''_2 O x_3) = \cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta$$

$$\ell_{31} = \cos(x''_3 O x_1) = \cos\left(\frac{\pi}{2} + \beta\right)\cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\beta\,(-\sin\alpha) = \sin\alpha\sin\beta$$

$$\ell_{32} = \cos(x''_3 O x_2) = \cos\left(\frac{\pi}{2} + \beta\right)\cos\alpha = -\cos\alpha\sin\beta$$

$$\ell_{33} = \cos(x''_3 O x_3) = \cos\beta$$

Thus the transformation matrix for this combined rotation becomes

$$T = [\ell_{ij}] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha\cos\beta & \cos\alpha\cos\beta & \sin\beta \\ \sin\alpha\sin\beta & -\cos\alpha\sin\beta & \cos\beta \end{bmatrix}$$

The components in the new system $Ox''_1 x''_2 x''_3$ are given by

$$A''_1 = \ell_{11} A_1 + \ell_{12} A_2 + \ell_{13} A_3 = (\cos\alpha) A_1 + (\sin\alpha) A_2$$

$$A''_2 = \ell_{21} A_1 + \ell_{22} A_2 + \ell_{23} A_3 = (-\sin\alpha\cos\beta) A_1 + (\cos\alpha\cos\beta) A_2 + (\sin\beta) A_3$$

$$A''_3 = \ell_{31} A_1 + \ell_{32} A_2 + \ell_{33} A_3 = (\sin\alpha\sin\beta) A_1 - (\cos\alpha\sin\beta) A_2 + (\cos\beta) A_3$$

NOTE: Problem (15) is a special case of this problem when

$$A_1 = 4, \quad A_2 = 5, \quad A_3 = 6 \quad \text{and} \quad \alpha = \frac{\pi}{6} \quad \text{and} \quad \beta = \frac{\pi}{2}.$$

PROBLEM (17): **A second order tensor has components** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **in the system $Ox_1 x_2 x_3$. The axes of this system are rotated by an angle θ in the positive sense about the x_1 – axis. Find the components of this tensor in the new system $Ox'_1 x'_2 x'_3$ and evaluate these components when $\theta = \pi/2$ and π.**

SOLUTION: Since the rotation is by an angle θ about the x_1 – axis, therefore x'_1 – axis coincides with the x_1 – axis as shown in figure (7.20). If the angle $x'_2 O x_2 = x'_3 O x_3 = \theta$, then by definition

$$\ell_{11} = \cos(x'_1 O x_1) = \cos 0 = 1$$

$$\ell_{12} = \cos(x'_1 O x_2) = \cos 90 = 0$$

$$\ell_{13} = \cos(x'_1 O x_3) = \cos 90 = 0$$

$$\ell_{21} = \cos(x'_2 O x_1) = \cos 90 = 0$$

$$\ell_{22} = \cos(x'_2 O x_2) = \cos\theta$$

$$\ell_{23} = \cos(x'_2 O x_3) = \cos(90 - \theta) = \sin\theta$$

$$\ell_{31} = \cos(x'_3 O x_1) = \cos 90 = 0$$

$$\ell_{32} = \cos(x'_3 O x_2) = \cos(90 + \theta) = -\sin\theta$$

$$\ell_{33} = \cos(x'_3 O x_3) = \cos\theta$$

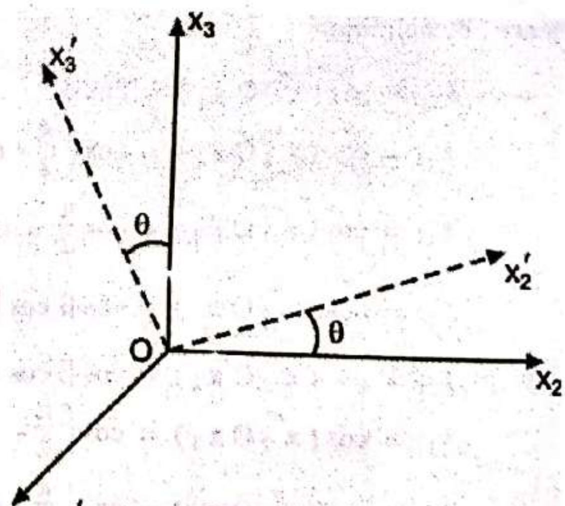

Figure (7.20)

Thus the transformation matrix is given by

$$T = [\ell_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, $T' = [\ell_{ji}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

The components of the tensor in the new system $Ox'_1 x'_2 x'_3$ are given by $= TAT'$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \sin\theta & \cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \sin\theta & \cos^2\theta & -\sin\theta\cos\theta \\ \cos\theta & -\sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

At $\theta = \frac{\pi}{2}$, the components of the tensor are given by $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

At $\theta = \pi$, the components of the tensor are given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

PROBLEM (18): Show that the transformation

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -3 & -6 & -2 \\ -2 & 3 & -6 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is orthogonal and right–handed. A second order tensor A_{ij} is defined in the system $Ox_1 x_2 x_3$ by $A_{ij} = x_i x_j$, $i, j = 1, 2, 3$. Evaluate its components at the point P where $x_1 = 0, x_2 = x_3 = 1$. Also evaluate the component A'_{11} of the tensor at P.

SOLUTION: From the transformation equations, we can write the transformation matrix T as

$$T = \frac{1}{7} \begin{bmatrix} -3 & -6 & -2 \\ -2 & 3 & -6 \\ 6 & -2 & -3 \end{bmatrix}$$

The transpose of this matrix is $T' = \frac{1}{7} \begin{bmatrix} -3 & -2 & 6 \\ -6 & 3 & -2 \\ -2 & -6 & -3 \end{bmatrix}$

and so $$TT' = \frac{1}{49} \begin{bmatrix} -3 & -6 & -2 \\ -2 & 3 & -6 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -3 & -2 & 6 \\ -6 & 3 & -2 \\ -2 & -6 & -3 \end{bmatrix}$$

$$= \frac{1}{49} \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Also $$\det T = \begin{vmatrix} -\frac{3}{7} & -\frac{6}{7} & -\frac{2}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{6}{7} \\ \frac{6}{7} & -\frac{2}{7} & -\frac{3}{7} \end{vmatrix} = \frac{9}{49} + \frac{36}{49} + \frac{4}{49} = \frac{49}{49} = 1$$

Thus the transformation is orthogonal and right – handed. We are given that

$$A_{ij} = x_i x_j \quad (i, j = 1, 2, 3).$$

The nine components A_{ij} at the point P where $x_1 = 0, x_2 = x_3 = 1$ are given by

$$A_{11} = x_1 x_1 = 0, \quad A_{12} = x_1 x_2 = 0, \quad A_{13} = x_1 x_3 = 0$$
$$A_{21} = x_2 x_1 = 0, \quad A_{22} = x_2 x_2 = 1, \quad A_{23} = x_2 x_3 = 1$$
$$A_{31} = x_3 x_1 = 0, \quad A_{32} = x_3 x_2 = 1, \quad A_{33} = x_3 x_3 = 1$$

Now we know that

$$A'_{11} = \ell_{1i}\ell_{1j}A_{ij}$$
$$= \ell_{11}\ell_{1j}A_{1j} + \ell_{12}\ell_{1j}A_{2j} + \ell_{13}\ell_{1j}A_{3j}$$
$$= \ell_{11}(\ell_{11}A_{11} + \ell_{12}A_{12} + \ell_{13}A_{13}) + \ell_{12}(\ell_{11}A_{21} + \ell_{12}A_{22} + \ell_{13}A_{23}) + \ell_{13}(\ell_{11}A_{31} + \ell_{12}A_{32} + \ell_{13}A_{33})$$
$$= \ell_{11}(0+0+0) + \ell_{12}(0 + \ell_{12}1 + \ell_{13}1) + \ell_{13}(0 + \ell_{12}1 + \ell_{13}1)$$
$$= \ell_{12}^2 + \ell_{12}\ell_{13} + \ell_{13}\ell_{12} + \ell_{13}^2 = \ell_{12}^2 + 2\ell_{12}\ell_{13} + \ell_{13}^2$$
$$= (\ell_{12} + \ell_{13})^2 = \left(-\frac{6}{7} - \frac{2}{7}\right)^2 = \left(-\frac{8}{7}\right)^2 = \frac{64}{49}$$

PROBLEM (19): **A_{ijk} is a third – order tensor, such that $A_{111} = A_{222} = 1, A_{212} = -2$, all other components being zero. Evaluate the components of the vector A_{iji}. The same transformation is made from $Ox_1x_2x_3$ to $Ox'_1x'_2x'_3$ as in problem (18). Evaluate the component A'_{123} of the tensor in the system $Ox'_1x'_2x'_3$.**

SOLUTION: Since A_{ijk} is a third order tensor, therefore contracting w.r.t. i and k, i.e. A_{iji} is a first order tensor i.e. vector. We are given that $A_{111} = A_{222} = 1$, $A_{212} = -2$, while all other components are zero.

Let $B_j = A_{iji} \quad (j = 1, 2, 3)$

then $B_1 = A_{i1i} = A_{111} + A_{212} + A_{313} = 1 - 2 + 0 = -1$

$B_2 = A_{i2i} = A_{121} + A_{222} + A_{323} = 0 + 1 + 0 = 1$

$B_3 = A_{i3i} = A_{131} + A_{232} + A_{333} = 0 + 0 + 0 = 0$

Thus the components of $B_j = A_{iji}$ are $(-1, 1, 0)$. Now we know that

$$A'_{123} = \ell_{1i}\ell_{2j}\ell_{3k}A_{ijk}$$
$$= \ell_{11}\ell_{21}\ell_{31}A_{111} + \ell_{12}\ell_{22}\ell_{32}A_{222} + \ell_{12}\ell_{21}\ell_{32}A_{212}$$
$$= \left(-\frac{3}{7}\right)\left(-\frac{2}{7}\right)\left(\frac{6}{7}\right) + \left(-\frac{6}{7}\right)\left(\frac{3}{7}\right)\left(-\frac{2}{7}\right) + \left(-\frac{6}{7}\right)\left(-\frac{2}{7}\right)\left(-\frac{2}{7}\right)(-2)$$
$$= \frac{36}{343} + \frac{36}{343} + \frac{48}{343} = \frac{120}{343}$$

PROBLEM (20): **For the second – order tensor A_{ij}, show that the quantities**

(i) A_{ii} (ii) $A_{ij}A_{ji}$ (iii) $A_{ij}A_{jk}A_{ki}$

are invariant under an orthogonal transformation.

SOLUTION: From the transformation equations , we can write the transformation matrix T as

$$T = [\ell_{ij}] = \begin{bmatrix} \frac{5}{15} & -\frac{14}{15} & \frac{2}{15} \\ -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{10}{15} & \frac{2}{15} & -\frac{11}{15} \end{bmatrix}$$

The transpose of this matrix is $T' = [\ell_{ji}] = \begin{bmatrix} \frac{5}{15} & -\frac{2}{3} & \frac{10}{15} \\ -\frac{14}{15} & -\frac{1}{3} & \frac{2}{15} \\ \frac{2}{15} & -\frac{2}{3} & -\frac{11}{15} \end{bmatrix}$

and so $T T' = \begin{bmatrix} \frac{5}{15} & -\frac{14}{15} & \frac{2}{15} \\ -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{10}{15} & \frac{2}{15} & -\frac{11}{15} \end{bmatrix} \begin{bmatrix} \frac{5}{15} & -\frac{2}{3} & \frac{10}{15} \\ -\frac{14}{15} & -\frac{1}{3} & \frac{2}{15} \\ \frac{2}{15} & -\frac{2}{3} & -\frac{11}{15} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Also , det $T = \begin{vmatrix} \frac{5}{15} & -\frac{14}{15} & \frac{2}{15} \\ -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{10}{15} & \frac{2}{15} & -\frac{11}{15} \end{vmatrix}$

$$= \frac{5}{15}\left(\frac{15}{45}\right) + \frac{14}{15}\left(\frac{42}{45}\right) + \frac{2}{15}\left(\frac{6}{45}\right) = \frac{75 + 588 + 12}{675} = \frac{675}{675} = 1$$

showing that the transformation with matrix T is orthogonal and right – handed .

The transformation equation expressing the coordinates of a point from the system K to K' is

$$x'_j = \ell_{ji}\, x_i \quad (j = 1, 2, 3)$$

from which it implies that

$$x_i = \ell_{ji}\, x'_j$$

or
$$x_1 = \ell_{11} x'_1 + \ell_{21} x'_2 + \ell_{31} x'_3$$
$$x_2 = \ell_{12} x'_1 + \ell_{22} x'_2 + \ell_{32} x'_3$$
$$x_3 = \ell_{13} x'_1 + \ell_{23} x'_2 + \ell_{33} x'_3$$

or
$$x_1 = \frac{5}{15} x'_1 - \frac{2}{3} x'_2 + \frac{10}{15} x'_3 = \frac{1}{3}\,(x'_1 - 2 x'_2 + 2 x'_3)$$
$$x_2 = -\frac{14}{15} x'_1 - \frac{1}{3} x'_2 + \frac{2}{15} x'_3 = -\frac{1}{15}\,(14 x'_1 + 5 x'_2 - 2 x'_3)$$
$$x_3 = \frac{2}{15} x'_1 - \frac{2}{3} x'_2 - \frac{11}{15} x'_3 = \frac{1}{15}\,(2 x'_1 - 10 x'_2 - 11 x'_3)$$

The transformation equation for the first order tensor (i.e. vector) from the system K to K' is

$$A'_j = \ell_{ji}\, A_i \quad (j = 1, 2, 3)$$

SOLUTION: (i) Since A_{ij} is a second order tensor, therefore $A'_{mn} = \ell_{mi}\ell_{nj}A_{ij}$

Let $m = n$, then

$$A'_{mm} = \ell_{mi}\ell_{mj}A_{ij} = \delta_{ij}A_{ij} = A_{ii}$$

or $$A'_{11} + A'_{22} + A'_{33} = A_{11} + A_{22} + A_{33}$$

showing that A_{ii} is invariant. This result says that the trace of the matrix $[A_{ij}]$ i.e. the sum of its diagonal elements remains invariant under orthogonal transformation. In the mathematical theory of elasticity, the stress tensor σ_{ij} arises and the above result means that the sum of the direct stresses $\sigma_{11} + \sigma_{22} + \sigma_{33}$ is invariant under an orthogonal transformation of axes.

(ii) Since $A'_{mn} = \ell_{mi}\ell_{nj}A_{ij}$ and $A'_{nm} = \ell_{nr}\ell_{ms}A_{rs}$, we have

$$A'_{mn}A'_{nm} = (\ell_{mi}\ell_{nj}A_{ij})(\ell_{nr}\ell_{ms}A_{rs})$$
$$= (\ell_{mi}\ell_{ms})(\ell_{nj}\ell_{nr})A_{ij}A_{rs}$$
$$= \delta_{is}\delta_{jr}A_{ij}A_{rs} = A_{ij}\delta_{is}(\delta_{jr}A_{rs}) = A_{ij}\delta_{is}A_{js} = A_{ij}A_{ji}$$

showing that $A_{ij}A_{ji}$ is invariant.

(iii) $$A'_{mn}A'_{np}A'_{pm} = (\ell_{mi}\ell_{nj}A_{ij})(\ell_{nr}\ell_{ps}A_{rs})(\ell_{pk}\ell_{mv}A_{kv})$$
$$= (\ell_{mi}\ell_{mv})(\ell_{nj}\ell_{nr})(\ell_{ps}\ell_{pk})A_{ij}A_{rs}A_{kv}$$
$$= \delta_{iv}\delta_{jr}\delta_{sk}A_{ij}A_{rs}A_{kv}$$
$$= A_{ij}(\delta_{sk}\delta_{jr}A_{rs})(\delta_{iv}A_{kv})$$
$$= A_{ij}(\delta_{sk}A_{js})(A_{ki}) = A_{ij}A_{jk}A_{ki}$$

showing that $A_{ij}A_{jk}A_{ki}$ is invariant under orthogonal transformation.

PROBLEM (21): **Prove that**

(i) $$\Delta = \det(A_{ij}) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk}A_{1i}A_{2j}A_{3k} = \epsilon_{ijk}A_{i1}A_{j2}A_{k3}$$

(ii) $$\epsilon_{rst}\Delta = \begin{vmatrix} A_{r1} & A_{r2} & A_{r3} \\ A_{s1} & A_{s2} & A_{s3} \\ A_{t1} & A_{t2} & A_{t3} \end{vmatrix} = \epsilon_{ijk}A_{ri}A_{sj}A_{tk} = \epsilon_{ijk}A_{ir}A_{js}A_{kt}$$

SOLUTION: (i) Using the definition of ϵ_{ijk}, we can write

$$\epsilon_{ijk}A_{1i}A_{2j}A_{3k} = \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32} + \epsilon_{231}A_{12}A_{23}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{312}A_{13}A_{21}A_{32} + \epsilon_{321}A_{13}A_{22}A_{31}$$
$$= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$
$$= A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}$$
$$= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \Delta$$

Similarly, we can show that $\Delta = \epsilon_{ijk}A_{i1}A_{j2}A_{k3}$.

(ii) $$\begin{aligned}\epsilon_{ijk}A_{ri}A_{sj}A_{tk} &= \epsilon_{123}A_{r1}A_{s2}A_{t3}+\epsilon_{132}A_{r1}A_{s3}A_{t2}+\epsilon_{231}A_{r2}A_{s3}A_{t1}\\ &\quad+\epsilon_{213}A_{r2}A_{s1}A_{t3}+\epsilon_{312}A_{r3}A_{s1}A_{t2}+\epsilon_{321}A_{r3}A_{s2}A_{t1}\\ &= A_{r1}(A_{s2}A_{t3}-A_{s3}A_{t2})+A_{r2}(A_{s3}A_{t1}-A_{s1}A_{t3})\\ &\quad+A_{r3}(A_{s1}A_{t2}-A_{s2}A_{t1})\\ &= A_{r1}\begin{vmatrix}A_{s2} & A_{s3}\\ A_{t2} & A_{t3}\end{vmatrix} - A_{r2}\begin{vmatrix}A_{s1} & A_{s3}\\ A_{t1} & A_{t3}\end{vmatrix} + A_{r3}\begin{vmatrix}A_{s1} & A_{s2}\\ A_{t1} & A_{t2}\end{vmatrix}\\ &= \begin{vmatrix}A_{r1} & A_{r2} & A_{r3}\\ A_{s1} & A_{s2} & A_{s3}\\ A_{t1} & A_{t2} & A_{t3}\end{vmatrix} = \Delta\end{aligned}$$

Similarly, we can show that $\epsilon_{rst}\Delta = \epsilon_{ijk}A_{ir}A_{js}A_{kt}$.

PROBLEM (22): **Prove that** $\Delta = \begin{vmatrix}\delta_{m1} & \delta_{m2} & \delta_{m3}\\ \delta_{n1} & \delta_{n2} & \delta_{n3}\\ \delta_{p1} & \delta_{p2} & \delta_{p3}\end{vmatrix} = \epsilon_{mnp}$

and $\epsilon_{ijk}\,\epsilon_{mnp} = \begin{vmatrix}\delta_{mi} & \delta_{mj} & \delta_{mk}\\ \delta_{ni} & \delta_{nj} & \delta_{nk}\\ \delta_{pi} & \delta_{pj} & \delta_{pk}\end{vmatrix}$

Hence prove that

(i) $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}$

(ii) $\epsilon_{ijk}\epsilon_{mjk} = 2\delta_{im}$ **(iii)** $\epsilon_{ijk}\epsilon_{ijk} = 6$

SOLUTION: $\epsilon_{mnp} = \begin{vmatrix}\delta_{m1} & \delta_{m2} & \delta_{m3}\\ \delta_{n1} & \delta_{n2} & \delta_{n3}\\ \delta_{p1} & \delta_{p2} & \delta_{p3}\end{vmatrix}$

$$\epsilon_{123} = \begin{vmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{vmatrix} = \begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = 1$$

$$\epsilon_{231} = \begin{vmatrix}\delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\\ \delta_{11} & \delta_{12} & \delta_{13}\end{vmatrix} = \begin{vmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{vmatrix} = 1$$

$$\epsilon_{312} = \begin{vmatrix}\delta_{31} & \delta_{32} & \delta_{33}\\ \delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\end{vmatrix} = \begin{vmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{vmatrix} = 1$$

$$\epsilon_{132} = \begin{vmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{31} & \delta_{32} & \delta_{33}\\ \delta_{21} & \delta_{22} & \delta_{23}\end{vmatrix} = -\begin{vmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{vmatrix} = -1$$

$$\epsilon_{213} = \begin{vmatrix}\delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{vmatrix} = -\begin{vmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{vmatrix} = -1$$

$$\epsilon_{321} = \begin{vmatrix}\delta_{31} & \delta_{32} & \delta_{33}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{11} & \delta_{12} & \delta_{13}\end{vmatrix} = -\begin{vmatrix}\delta_{11} & \delta_{12} & \delta_{13}\\ \delta_{21} & \delta_{22} & \delta_{23}\\ \delta_{31} & \delta_{32} & \delta_{33}\end{vmatrix} = -1$$

If i, j, k is not a permutation of $(1, 2, 3)$, so that at least two of i, j, k are equal, the determinant has two equal rows and so $\Delta = 0$.

Alternatively, we can prove this result following the procedure of problem (21). We can write

$$\Delta = \epsilon_{ijk}\delta_{mi}\delta_{nj}\delta_{pk}$$

$$= \epsilon_{mjk}\delta_{nj}\delta_{pk} = \epsilon_{mnk}\delta_{pk} = \epsilon_{mnp}$$

Hence $\epsilon_{ijk}\epsilon_{mnp} = \epsilon_{ijk}\Delta = \begin{vmatrix} \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \\ \delta_{pi} & \delta_{pj} & \delta_{pk} \end{vmatrix}$

since multiplication of Δ by ϵ_{ijk} is equivalent to interchanging the columns $1, i; 2, j; 3, k$.

(i) Taking $p = k$ in the last result, we get

$$\epsilon_{ijk}\epsilon_{mnk} = \begin{vmatrix} \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \\ \delta_{pi} & \delta_{pj} & \delta_{kk} \end{vmatrix} = \begin{vmatrix} \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \\ \delta_{ki} & \delta_{kj} & 3 \end{vmatrix}$$

$$= \delta_{mi}(3\delta_{nj} - \delta_{nk}\delta_{kj}) - \delta_{mj}(3\delta_{ni} - \delta_{nk}\delta_{ki}) + \delta_{mk}(\delta_{ni}\delta_{kj} - \delta_{nj}\delta_{ki})$$

$$= \delta_{mi}(3\delta_{nj} - \delta_{nj}) - \delta_{mj}(3\delta_{ni} - \delta_{ni}) + \delta_{mk}(\delta_{ni}\delta_{kj} - \delta_{nj}\delta_{ki})$$

$$= 3\delta_{mi}\delta_{nj} - \delta_{mi}\delta_{nj} - 3\delta_{ni}\delta_{mj} + \delta_{ni}\delta_{mj} + \delta_{ni}\delta_{mj} - \delta_{nj}\delta_{mi}$$

$$= \delta_{mi}\delta_{nj} - \delta_{ni}\delta_{mj} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

(ii) Taking $n = j$ in the result of part (i), we get

$$\epsilon_{ijk}\epsilon_{mjk} = \delta_{im}\delta_{jj} - \delta_{ij}\delta_{jm} = 3\delta_{im} - \delta_{im} = 2\delta_{im}$$

(iii) Taking $m = i$ in the result of part (ii), we get $\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{ii} = 6$

PROBLEM (23): **If $A_i B_i$ is a scalar or invariant, where A_i is an arbitrary vector, then prove that B_i is a vector.**

SOLUTION: Since $A_i B_i$ is a scalar, then by definition

$$A'_j B'_j = A_i B_i \qquad (1)$$

where the undashed and dashed symbols have the usual meanings.

Now since A_i is a vector, therefore

$$A'_j = \ell_{ji} A_i$$

or $\quad A_i = \ell_{ji} A'_j \qquad (2)$

From equations (1) and (2), we have

$$A'_j B'_j = \ell_{ji} A'_j B_i$$

or $\quad A'_j (B'_j - \ell_{ji} B_i) = 0$

Now A'_j being an arbitrary vector $A'_j \neq 0$ and the above relation will be true only when

$$B'_j = \ell_{ji} B_i$$

which shows that B_i is a vector.

PROBLEM (24): **If $A_{ij} B_k$ is a tensor of rank 3, where B_k is an arbitrary vectors, then prove that the 2 – suffix set A_{ij} is also a tensor of rank 2.**

SOLUTION: Let $C_{ijk} = A_{ij} B_k$, $C'_{pqr} = A'_{pq} B'_r$ where A_{ij}, B_k, C_{ijk} and A'_{pq}, B'_r, C'_{pqr} are the components of the 2 – suffix set, the vector and the tensor relatively to the two systems K and K' respectively. Now since C_{ijk} is a tensor of rank 3, therefore

$$C'_{pqr} = \ell_{pi} \ell_{qj} \ell_{rk} C_{ijk}$$

or $$A'_{pq} B'_r = \ell_{pi} \ell_{qj} \ell_{rk} A_{ij} B_k \tag{1}$$

Also B_k being an arbitrary vector,

$$B'_r = \ell_{rk} B_k \tag{2}$$

Writing B_r' for $\ell_{rk} B_k$ in the R.H.S. of equation (1), we get

$$A'_{pq} B'_r = \ell_{pi} \ell_{qj} A_{ij} B'_r$$

or $$(A'_{pq} - \ell_{pi} \ell_{qj} A_{ij}) B'_r = 0$$

Since the vector B_k is arbitrary, B'_r is also arbitrary and the above equation will be true only when

$$A'_{pq} - \ell_{pi} \ell_{qj} A_{ij} = 0$$

or $$A'_{pq} = \ell_{pi} \ell_{qj} A_{ij}$$

which shows that the 2 – suffix set A_{ij} is a tensor of rank 2.

PROBLEM (25): **If $A_{ijkm} B_{km}$ is a second order tensor where B_{km} is an arbitrary second order tensor, then prove that the 4–suffix set A_{ijkm} is a fourth–order tensor.**

SOLUTION: Let $C_{ij} = A_{ijkm} B_{km}$ and $C'_{pq} = A'_{pqrs} B'_{rs}$

where A_{ijkm}, B_{km}, C_{ij} and $A'_{pqrs}, \beta'_{rs}, C'_{pq}$ are the components of the 4 – suffix set and the two second order tensors in the systems K and K', respectively. Now since C_{ij} is a second order tensor, therefore $C'_{pq} = \ell_{pi} \ell_{qj} C_{ij}$

or $$A'_{pqrs} B'_{rs} = \ell_{pi} \ell_{qj} A_{ijkm} B_k \tag{1}$$

Also B_{km} being an arbitrary second order tensor, we have

$$B'_{rs} = \ell_{rk} \ell_{sm} B_{km}$$

This implies $B_{km} = \ell_{rk} \ell_{sm} B'_{rs}$ (2)

From equations (1) and (2), we get

$$A'_{pqrs} \beta'_{rs} = \ell_{pi} \ell_{qj} \ell_{rk} \ell_{sm} A_{ijkm} B'_{rs}$$

or $$(A'_{pqrs} - \ell_{pi} \ell_{qj} \ell_{rk} \ell_{sm} A_{ijkm}) B'_{rs} = 0$$

Since the tensor B_{km} is arbitrary, B'_{rs} is also arbitrary, therefore the above relation will be true only when $A'_{pqrs} - \ell_{pi} \ell_{qj} \ell_{rk} \ell_{sm} A_{ijkm} = 0$

or $$A'_{pqrs} = \ell_{pi} \ell_{qj} \ell_{rk} \ell_{sm} A_{ijkm}$$

which shows that the 4 – suffix set A_{ijkm} is a tensor of order 4.

PROBLEM (26): **If A_{ij} and A_{ji} are two second order tensors, then show that $A_{ij} + A_{ji}$ is a second order symmetric tensor, while $A_{ij} - A_{ji}$ is a second order anti – symmetric tensor.**

SOLUTION: We have

$$A_{ij}+A_{ji} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} + \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 2A_{11} & A_{12}+A_{21} & A_{13}+A_{31} \\ A_{21}+A_{12} & 2A_{22} & A_{23}+A_{32} \\ A_{31}+A_{13} & A_{32}+A_{23} & 2A_{33} \end{bmatrix}$$

which is a symmetric tensor.

Next
$$A_{ij}-A_{ji} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} - \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & A_{12}-A_{21} & A_{13}-A_{31} \\ -(A_{12}-A_{21}) & 0 & A_{23}-A_{32} \\ -(A_{13}-A_{31}) & -(A_{23}-A_{32}) & 0 \end{bmatrix}$$

which is an anti – symmetric tensor.

PROBLEM (27): **Find the symmetric and anti–symmetric parts of the tensor $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.**

SOLUTION: We know that any second order tensor A_{ij} can be written as the sum of a symmetric and an anti – symmetric tensor

i.e. $$A_{ij} = \frac{1}{2}(A_{ij}+A_{ji}) + \frac{1}{2}(A_{ij}-A_{ji})$$

$$= B_{ij} + C_{ij}$$

where $B_{ij} = \frac{1}{2}(A_{ij}+A_{ji})$ is symmetric

and $C_{ij} = \frac{1}{2}(A_{ij}-A_{ji})$ an anti – symmetric parts of the tensor A_{ij}.

Now $A_{ij} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ therefore, $A_{ji} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

$$A_{ij}+A_{ji} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix}$$

$$A_{ij}-A_{ji}=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}-\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}=\begin{bmatrix}0&-2&-4\\2&0&-2\\4&2&0\end{bmatrix}$$

Thus $$B_{ij}=\frac{1}{2}(A_{ij}+A_{ji})=\begin{bmatrix}1&3&5\\3&5&7\\5&7&9\end{bmatrix}$$

and $$C_{ij}=\frac{1}{2}(A_{ij}-A_{ji})=\begin{bmatrix}0&-1&-2\\1&0&-1\\2&1&0\end{bmatrix}$$

are the symmetric and anti – symmetric parts respectively, of the given tensor.

PROBLEM (28): Show that under orthogonal transformation

(i) a second – order symmetric tensor remains a second order symmetric tensor.

(ii) a second order anti – symmetric tensor remains a second order anti – symmetric tensor.

SOLUTION: (i) Let A_{ij} be a second order symmetric tensor so that

$$A_{ij} = A_{ji} \qquad (1)$$

Since A_{ij} is a second order tensor, its equation of transformation is

$$A'_{mn} = \ell_{mi}\ell_{nj}A_{ij}$$

$$= \ell_{mi}\ell_{nj}A_{ji} \qquad [\text{using equation (1)}]$$

$$= \ell_{nj}\ell_{mi}A_{ji} = A'_{nm}$$

Thus $A'_{mn} = A'_{nm}$ showing that the tensor is symmetric in the new coordinate system as well.

(ii) Let A_{ij} be a second order anti – symmetric tensor so that

$$A_{ij} = -A_{ji} \qquad (2)$$

Since A_{ij} is a second order tensor its equation of transformation is

$$A'_{mn} = \ell_{mi}\ell_{nj}A_{ij}$$

$$= \ell_{mi}\ell_{nj}(-A_{ji}) \qquad [\text{using equation (2)}]$$

$$= -\ell_{mi}\ell_{nj}A_{ji} = -\ell_{nj}\ell_{mi}A_{ji} = -A'_{nm}$$

or $A'_{mn} = -A'_{nm}$ showing that the tensor is anti–symmetric in the new coordinate system as well.

Thus the properties of symmetry and anti – symmetry of second order tensors do not change under orthogonal transformation, i.e. they are independent of the coordinate system.

PROBLEM (29): Prove that if A_{ij} is a second order symmetric tensor and B_{ij} is a second order anti – symmetric tensor, then $A_{ij}B_{ij} = 0$.

SOLUTION: We can write

$$A_{ij}B_{ij} = \frac{1}{2}(A_{ij}B_{ij}+A_{ij}B_{ij}) \qquad (1)$$

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Interchanging the dummies i and j in the second term on the R.H.S. of equation (1), we get

$$A_{ij}B_{ij} = \frac{1}{2}(A_{ij}B_{ij} + A_{ji}B_{ji})$$

$$= \frac{1}{2}(A_{ij}B_{ij} - A_{ij}B_{ij}) \quad \text{since } A_{ji} = A_{ij} \text{ and } B_{ji} = -B_{ij}$$

or $\quad A_{ij}B_{ij} = 0$.

PROBLEM (30): **Write $A_{ij}\, x_i\, x_j$ in full and prove that it equals zero if A_{ij} be a anti symmetric tensor.**

SOLUTION:

$$A_{ij}x_ix_j = A_{11}x_1^2 + A_{22}x_2^2 + A_{33}x_3^2 + (A_{12} + A_{21})x_1x_2 + (A_{13} + A_{31})x_1x_3 + (A_{23} + A_{32})x_2x_3$$

In case A_{ij} is antisymmetric,

$$A_{11} = 0 = A_{22} = A_{33}, \quad A_{21} = -A_{12}, \quad A_{13} = -A_{31}, \quad A_{32} = -A_{23}$$

therefore, $A_{ij}x_ix_j = 0$

NOTE: In case A_{ij} is symmetric,

$$A_{ij}x_ix_j = A_{11}x_1^2 + A_{22}x_2^2 + A_{33}x_3^2 + 2A_{12}x_1x_2 + 2A_{13}x_1x_3 + 2A_{23}x_2x_3.$$

PROBLEM (31): **If A_{jk} are the components of a second order tensor, show that**

$$B_i = \frac{1}{2}\epsilon_{ijk}A_{jk} \quad (i = 1, 2, 3)$$

are the components of a first order tensor i.e. vector $\vec{B}$. If A_{jk} are the components of an anti–symmetric tensor, verify that $\vec{B} = (A_{23}, A_{31}, A_{12})$.

SOLUTION: Since A_{jk} and ϵ_{ijk} are second order and third order tensors, their equations of transformations from the system K to K′, are

$$A'_{mn} = \ell_{mj}\ell_{nk}A_{jk}$$

$$\epsilon'_{pqr} = \ell_{pi}\ell_{qs}\ell_{rt}\epsilon_{ist}$$

$$\epsilon'_{pqr}A'_{mn} = \ell_{pi}\ell_{qs}\ell_{rt}\epsilon_{ist}\ell_{mj}\ell_{nk}A_{jk}$$

$$= \ell_{pi}\ell_{qs}\ell_{rt}\ell_{mj}\ell_{nk}\epsilon_{ist}A_{jk}$$

Let $q = m$ and $r = n$, we get

$$\epsilon'_{pqr}A'_{mn} = \ell_{pi}\ell_{ms}\ell_{nt}\ell_{mj}\ell_{nk}\epsilon_{ist}A_{jk}$$

$$= (\ell_{ms}\ell_{mj})(\ell_{nt}\ell_{nk})\ell_{pi}\epsilon_{ist}A_{jk}$$

$$= \delta_{sj}\delta_{tk}\ell_{pi}\epsilon_{ist}A_{jk}$$

$$= \ell_{pi}(\delta_{sj}\epsilon_{ist})\delta_{tk}A_{jk} = \ell_{pi}\epsilon_{ijt}\delta_{tk}A_{jk}$$

$$= \ell_{pi}\epsilon_{ijk}A_{jk}$$

or $\quad \frac{1}{2}\epsilon'_{pmn}A'_{mn} = \ell_{pi}\left(\frac{1}{2}\epsilon_{ijk}A_{jk}\right)$

(1)

or $B'_p = \ell_{pi} B_i$

where $B'_p = \frac{1}{2} \epsilon'_{pmn} A'_{mn}$

Equation (1) shows that

$$B_i = \frac{1}{2} \epsilon_{ijk} A_{jk} \qquad (2)$$

are the components of a first order tensor (i.e. vector) . B_i is called the **vector** associated with the given second order tensor A_{jk}.

Since A_{jk} are the components of an antisymmetric tensor , therefore ,

$$A_{jk} = 0 \text{ for } j = k \text{ and } A_{jk} = -A_{kj} \text{ for } j \neq k$$

Then $B_1 = \frac{1}{2}\epsilon_{1jk} A_{jk} = \frac{1}{2}\epsilon_{123} A_{23} + \frac{1}{2}\epsilon_{132} A_{32}$

$$= \frac{1}{2}A_{23} - \frac{1}{2}A_{32} = \frac{1}{2}A_{23} + \frac{1}{2}A_{23} = A_{23}$$

$B_2 = \frac{1}{2}\epsilon_{2jk} A_{jk} = \frac{1}{2}\epsilon_{231} A_{31} + \frac{1}{2}\epsilon_{213} A_{13}$

$$= \frac{1}{2}A_{31} - \frac{1}{2}A_{13} = \frac{1}{2}A_{31} + \frac{1}{2}A_{31} = A_{31}$$

Then $B_3 = \frac{1}{2}\epsilon_{3jk} A_{jk} = \frac{1}{2}\epsilon_{312} A_{12} + \frac{1}{2}\epsilon_{321} A_{21}$

$$= \frac{1}{2}A_{12} - \frac{1}{2}A_{21} = \frac{1}{2}A_{12} + \frac{1}{2}A_{12} = A_{12}$$

Thus $\vec{B} = (A_{23}, A_{31}, A_{12})$ (3)

Thus a second order anti – symmetric tensor has three independent quantities associated with it which themselves form the vector of the tensor .

If A_{jk} are the components of a symmetric tensor , equation (2) evidently gives zero .

PROBLEM (32): If A_{jk} are the components of a second order anti – symmetric tensor and the vector $\vec{B}$ has components given by $B_i = \frac{1}{2} \epsilon_{ijk} A_{jk}$ $(i = 1, 2, 3)$, show that $A_{rs} = \epsilon_{irs} B_i$. Write down the matrix of components of A in terms of components of $\vec{B}$.

SOLUTION: Since A_{jk} is an anti – symmetric tensor , therefore

$$A_{jk} = -A_{kj} \qquad (1)$$

$$\epsilon_{irs} B_i = \epsilon_{irs}\left(\frac{1}{2}\epsilon_{ijk} A_{jk}\right) = \frac{1}{2}(\epsilon_{irs}\epsilon_{ijk}) A_{jk}$$

$$= \frac{1}{2}(\delta_{rj}\delta_{sk} - \delta_{rk}\delta_{sj}) A_{jk}$$

$$= \frac{1}{2}(\delta_{rj}\delta_{sk} A_{jk} - \delta_{rk}\delta_{sj} A_{jk})$$

$$= \frac{1}{2}(\delta_{rj} A_{js} - \delta_{rk} A_{sk}) = \frac{1}{2}(A_{rs} - A_{sr})$$

$$= \frac{1}{2}(A_{rs} + A_{rs}) \text{ [since using equation (1) } A_{rs} = -A_{sr} \text{]}$$

$$= A_{rs}$$

Thus with any vector we can associate an anti – symmetric tensor of the second order . Conversely , with any anti – symmetric tensor of the second order we can associate a vector . This result is true only in Cartesian tensors . Furthermore , since

$$A_{rs} = \epsilon_{irs} B_i$$

$$A_{11} = \epsilon_{i11} B_i = 0 , \quad A_{12} = \epsilon_{i12} B_i = \epsilon_{312} B_3 = B_3 , \quad A_{13} = \epsilon_{i13} B_i = \epsilon_{213} B_2 = -B_2$$

$$A_{21} = \epsilon_{i21} B_i = \epsilon_{321} B_3 = -B_3 , \quad A_{22} = \epsilon_{i22} B_i = 0 , \quad A_{23} = \epsilon_{i23} B_i = \epsilon_{123} B_1 = B_1$$

$$A_{31} = \epsilon_{i31} B_i = \epsilon_{231} B_2 = B_2 , \quad A_{32} = \epsilon_{i32} B_i = \epsilon_{132} B_1 = -B_1 , \quad A_{33} = \epsilon_{i33} B_i = 0$$

Thus the required matrix of components is given by

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{bmatrix}$$

PROBLEM (33) : Find the eigenvalues and eigenvectors of $\begin{bmatrix} -2 & 2 & 3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

SOLUTION: In this case , the characteristic equation is

$$\begin{vmatrix} -2-\lambda & 2 & 3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

Expanding the determinant and solving the cubic in λ , we obtain $\lambda = 5 , -3 , -3$.

Thus $\quad \lambda_1 = 5 , \lambda_2 = -3 , \lambda_3 = -3$.

Corresponding to $\lambda = \lambda_1 = 5$, equations (3) on page (432) become

$$-7x_1 + 2x_2 - 3x_3 = 0 \qquad (1)$$

$$2x_1 - 4x_2 - 6x_3 = 0 \qquad (2)$$

$$-x_1 - 2x_2 - 5x_3 = 0 \qquad (3)$$

Adding (1) and (2) to eliminate x_2 , we obtain

$$-8x_1 - 8x_3 = 0 \text{ or } x_1 + x_3 = 0$$

So if we take $x_3 = -1$, then $x_1 = 1$ and therefore , $x_2 = 2$

So $(1 , 2 , -1)$ is an eigenvector of A_{ij} corresponding to the eigenvalue 5 .

Corresponding to $\lambda = \lambda_2 = -3$, equation (3) on page (432) become

$$\left. \begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + 4x_2 - 6x_3 &= 0 \\ -x_1 - 2x_2 + 3x_3 &= 0 \end{aligned} \right\}$$

Each of these equations reduces to

$$x_1 + 2x_2 - 3x_3 = 0 \qquad (4)$$

So to get the non–zero values of x_1, x_2, x_3, we give arbitrary values to x_2 and x_3 and obtain the value of x_1, from equation (4). If we choose $x_2 = a$, $x_3 = b$, then from equation (4) $x_1 = -2a + 3b$.

Thus $(-2a + 3b, a, b)$ is the general form of the eigenvectors.

Now if $a = 1$, $b = 2$, then $(4, 1, 2)$ is an eigenvector corresponding to $\lambda = \lambda_2 = -3$.

The third eigenvalue is also -3 so corresponding to $\lambda = \lambda_3 = -3$, the equation (3) will also reduce to the equation (4). So to get the eigenvector corresponding to this eighenvalue, we give to a and b some other arbitrary values, say $a = 2$, $b = 1$ and therefore $(-1, 2, 1)$ is an eigenvector which may be taken as the eigenvector corresponding to the eigenvalue $\lambda = \lambda_3 = -3$.

So $(4, 1, 2)$ and $(-1, 2, 1)$ are two linearly independent eigenvectors corresponding to the eigenvalues -3. This example shows that to an eigenvalue there may correspond several linearly independent eigenvectors.

PROBLEM (34): Find the eigenvalues and eigenvectors of the second order real symmetric tensor

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 2 & 2 \\ -2 & 2 & 5 \end{bmatrix}$$

Also, reduce it to the principal axis (i.e. diagonal) form.

SOLUTION: In this case, the characteristic equation is

$$\begin{vmatrix} 2-\lambda & 4 & -2 \\ 4 & 2-\lambda & 2 \\ -2 & 2 & 5-\lambda \end{vmatrix} = 0$$

Expanding the dterminant, we get

$$(2-\lambda)(6-7\lambda+\lambda^2) - 4(24-4\lambda) - 2(12-2\lambda) = 0$$

$$12 - 20\lambda + 9\lambda^2 - \lambda^3 - 96 + 16\lambda - 24 + 4\lambda = 0$$

or $\quad \lambda^3 - 9\lambda^2 + 108 = 0 \quad$ or $\quad (\lambda+3)(6-\lambda)^2 = 0$,

which gives $\lambda = -3, 6, 6$

Thus the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 6$, $\lambda_3 = 6$

Corresponding to the first eigenvalue $\lambda_1 = -3$, equation (3) on page (432) take the form

$$5x_1 + 4x_2 - 2x_3 = 0 \qquad (1)$$

$$4x_1 + 5x_2 + 2x_3 = 0 \qquad (2)$$

$$-2x_1 + 2x_2 + 8x_3 = 0 \qquad (3)$$

Adding equations (1) and (2), we get $\quad 9x_1 + 9x_2 = 0$

which can be written as $\quad x_2 = -x_1$

So if we take $x_1 = 2$, then $x_2 = -2$ and from either equation we get $x_3 = 1$

Thus corresponding to $\lambda_1 = -3$, the eigenvector is

$$\vec{X}^{(1)} = (2, -2, 1) \tag{4}$$

Corresponding to the second eigenvalue $\lambda = 6$, the equations (3) on page (432) take the form

$$-4x_1 + 4x_2 - 2x_3 = 0 \tag{5}$$

$$4x_1 - 4x_2 + 2x_3 = 0 \tag{6}$$

$$-2x_1 + 2x_2 - x_3 = 0 \tag{7}$$

Equations (5), (6) and (7) represent the same equation given by

$$2x_1 - 2x_2 + x_3 = 0$$

or $\quad x_3 = -2x_1 + 2x_2$

If we choose $x_1 = a$ and $x_2 = b$, then $x_3 = -2a + 2b$

and so the eigenvectors have the general form $(a, b, -2a + 2b)$. This is the most general form for a vector perpendicular to $\vec{X}^{(1)}$.

We may arbitrarily choose $a = b = 1$ to get

$$\vec{X}^{(2)} = (1, 1, 0) \tag{8}$$

The third eigenvector perpendicular to both $\vec{X}^{(1)}$ and $\vec{X}^{(2)}$ is given by

$$\vec{X}^{(3)} = \vec{X}^{(1)} \times \vec{X}^{(2)} = (2, -2, 1) \times (1, 1, 0) = (-1, 1, 4) \tag{9}$$

It can be seen that the three eigenvectors are mutually orthogonal.

Now the unit vectors in the new system $Ox'_1 x'_2 x'_3$ are given by

$$\hat{e}_1' = \frac{\vec{X}^{(1)}}{|\vec{X}^{(1)}|} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \tag{10}$$

$$\hat{e}_2' = \frac{\vec{X}^{(2)}}{|\vec{X}^{(2)}|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \tag{11}$$

$$\hat{e}_3' = \frac{\vec{X}^{(3)}}{|\vec{X}^{(3)}|} = \left(-\frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{4}{\sqrt{18}}\right) \tag{12}$$

From equations (10), (11), and (12), we get

$$\left.\begin{aligned} \hat{e}_1' \cdot \hat{e}_1' &= \hat{e}_2' \cdot \hat{e}_2' = \hat{e}_3' \cdot \hat{e}_3' = 1 \\ \hat{e}_1' \cdot \hat{e}_2' &= \hat{e}_2' \cdot \hat{e}_3' = \hat{e}_3' \cdot \hat{e}_1' = 0 \end{aligned}\right] \tag{13}$$

Now we know that

$$A'_{mn} \hat{e}_n' = \lambda_m \hat{e}_m' \tag{14}$$

For $m = 1$, equation (14) becomes

$$A'_{1n}\hat{e}_n{}' = \lambda_1 \hat{e}_1{}'$$

or $$A'_{1n}(\hat{e}_n{}'.\hat{e}_n{}') = \lambda_1(\hat{e}_1{}'.\hat{e}_n{}')$$

or $$A'_{1n} = \lambda_1(\hat{e}_1{}'.\hat{e}_n{}')$$

which implies on using equation (13),

$$A'_{11} = \lambda_1 = -3, \quad A'_{12} = 0, \quad A'_{13} = 0$$

For $m = 2$, equation (14) becomes

$$A'_{2n}\hat{e}_n{}' = \lambda_2 \hat{e}_2{}'$$

or $$A'_{2n}(\hat{e}_n{}'.\hat{e}_n{}') = \lambda_2(\hat{e}_2{}'.\hat{e}_n{}')$$

or $$A'_{2n} = \lambda_2(\hat{e}_2{}'.\hat{e}_n{}')$$

which implies on using equation (13),

$$A'_{21} = 0, \quad A'_{22} = \lambda_2 = 6, \quad A'_{23} = 0$$

For $m = 3$, equation (14) becomes

$$A'_{3n}\hat{e}_n{}' = \lambda_3 \hat{e}_3{}'$$

or $$A'_{3n}(\hat{e}_n{}'.\hat{e}_n{}') = \lambda_3(\hat{e}_3{}'.\hat{e}_n{}')$$

or $$A'_{3n} = \lambda_3(\hat{e}_3{}'.\hat{e}_n{}')$$

which implies on using equation (13)

$$A'_{31} = 0, \quad A'_{32} = 0, \quad A'_{33} = \lambda_3 = 6$$

Thus the required diagonal form of the given tensor is $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

7.35 EXERCISE

PROBLEM (1): Write the following expressions using summation convention :

(i) $a_{11} b_{11} + a_{21} b_{12} + a_{31} b_{13}$

(ii) $a_{13} b_{13} + a_{23} b_{23} + a_{33} b_{33}$

(iii) $a_1 x_1 x_3 + a_2 x_2 x_3 + a_3 x_3 x_3$

(iv) $a_{1j} x_1 + a_{2j} x_2 + a_{3j} x_3$

PROBLEM (2): Write the following equations using summation convention :

$$\frac{\partial e_{11}}{\partial x_1} + \frac{\partial e_{12}}{\partial x_2} + \frac{\partial e_{13}}{\partial x_3} = 0$$

$$\frac{\partial e_{21}}{\partial x_1} + \frac{\partial e_{22}}{\partial x_2} + \frac{\partial e_{23}}{\partial x_3} = 0$$

$$\frac{\partial e_{31}}{\partial x_1} + \frac{\partial e_{32}}{\partial x_2} + \frac{\partial e_{33}}{\partial x_3} = 0$$

PROBLEM (3): Write the following expression using summation convention :

$$a_{11}(x_1)^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3 + a_{21} x_2 x_1 + a_{22}(x_2)^2 + a_{23} x_2 x_3 + a_{31} x_3 x_1 + a_{32} x_3 x_2 + a_{33}(x_3)^2$$

PROBLEM (4): Write the following summations in full :

(i) $a_i x_i x_j$ **(ii)** $a_{ij} x_j$ **(iii)** $a_{ij} x_{jk}$ **(iv)** a_{ijki}

PROBLEM (5): Write the following equation in full : $\ell_{ij} \ell_{jk} = \delta_{ik}$.

PROBLEM (6): Write the terms in the sum A_{ijij}.

PROBLEM (7): Evaluate (i) $Q = a_{ij} y_j$ if $y_i = b_{ij} x_j$ (ii) $Q = a_{ijk} y_i y_j y_k$ if $y_i = b_{ij} x_j$.

PROBLEM (8): Express $Q = g_{ij} y_i y_j$ in terms of x–variables, if $y_i = a_{ij} x_j$ and $g_{ij} a_{ik} = \delta_{jk}$

PROBLEM (9): If the a_{ij} are constants, calculate the following partial derivatives :

(i) $\frac{\partial}{\partial x_k}(a_{11} x_1 + a_{12} x_2 + a_{13} x_3), k = 1, 2, 3$ **(ii)** $\frac{\partial}{\partial x_k}(a_{ij} x_j)$

PROBLEM (10): Calculate the partial derivative $\frac{\partial}{\partial x_k}[a_{ij} x_i (x_j)^2]$

where a_{ij} are constants such that $a_{ij} = a_{ji}$.

PROBLEM (11): Calculate the partial derivative $\frac{\partial}{\partial x_\ell}(a_{ijk} x_i x_j x_k)$ where a_{ijk} are constants.

PROBLEM (12): If $a_{11} = 1, a_{12} = -1, a_{13} = 0, \ a_{21} = -2, a_{22} = 3, a_{23} = 1$

$a_{31} = 2, a_{32} = 0, a_{33} = 4, \ b_1 = 1, b_2 = -1, b_3 = 4$ then show that

(i) $a_{ii} = 8$ **(ii)** $a_{i1} a_{i2} = -7$ **(iii)** $a_{i2} a_{i3} = 3$

(iv) $a_{1i} a_{2i} = -5$ **(v)** $a_{2i} a_{3i} = 0$ **(vi)** $a_{i1} a_{2i} = -6$.

(vii) $a_{1i} b_i = 2$ **(viii)** $a_{j1} b_j = 11$ **(ix)** $a_{ji} a_{i1} b_j = 49$

(x) $a_{1j} \delta_{1j} = 1$ **(xi)** $a_{12} \delta_{ii} = -3$ **(xii)** $a_{1i} a_{2k} \delta_{ik} = -5$

PROBLEM (13): Write down the transformation matrix for a rotation of angle $\pi/4$ in the positive sense about **(i)** x_1 – axis **(ii)** x_2 – axis **(iii)** x_3 – axis .

PROBLEM (14): Write down the transformation matrix for a rotation of angle $\pi/2$ in the positive sense about **(i)** x_1 – axis **(ii)** x_2 – axis **(iii)** x_3 – axis .

PROBLEM (15): Show that the transformation given by each of the following matrix is orthogonal and right – handed .

(i) $T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **(ii)** $T = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ 0 & 0 & 1 \\ -\frac{4}{5} & -\frac{3}{5} & 0 \end{bmatrix}$

(iii) $T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

PROBLEM (16): A set of axes $Ox'_1 x'_2 x'_3$ is initially coincident with a set $Ox_1 x_2 x_3$. The set $Ox'_1 x'_2 x'_3$ is rotated so as to bring x'_1 along the old x_2 , x'_2 along the old x_3 , and x'_3 along the old x_1 . Find the transformation matrix T for this rotation and show that this transformation is orthogonal and right–handed . Also , find the equations of transformation expressing the coordinates x'_j in the system $Ox'_1 x'_2 x'_3$ in terms of coordinates x_i of the system $Ox_1 x_2 x_3$.

PROBLEM (17): The system $Ox'_1 x'_2 x'_3$ is initially coincident with the system $Ox_1 x_2 x_3$. The system $Ox'_1 x'_2 x'_3$ is then rotated through an angle π in the positive sense about the x_1 – axis . Write down the transformation matrix corresponding to this rotation . If a point has coordinates (1 , 1 , 1) relative to the coordinate system $Ox_1 x_2 x_3$, find its coordinates w.r.t. the system $Ox'_1 x'_2 x'_3$.

PROBLEM (18): The transformation matrix T for a rotation of the system $Ox_1 x_2 x_3$ to the system $Ox'_1 x'_2 x'_3$ is given by $T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Show that this transformation is othogonal and right–handed . Find the transformation equations between the coordinates x'_j of the system $Ox'_1 x'_2 x'_3$ in terms of coordinates x_i of the system $Ox_1 x_2 x_3$. A vector $\overrightarrow{A}$ has components (2 , 1 , 2) relative to the coordinate system $Ox_1 x_2 x_3$. Find the components of the same vector relative to the system $Ox'_1 x'_2 x'_3$.

PROBLEM (19): Verify that the transformation

$$x'_1 = \frac{1}{13}(5x_1 + 12x_2)$$

$$x'_2 = \frac{1}{13}(12x_1 - 5x_2)$$

$$x'_3 = x_3$$

is orthogonal and left–handed . A vector field $\vec{A}$ is defined in the system $Ox_1x_2x_3$ by $A_1 = x_2x_3$, $A_2 = x_3x_1$, $A_3 = x_1x_2$.

Evaluate the components A'_j of the vector field in the new system $Ox'_1x'_2x'_3$ in terms of the coordinates x_i. Verify that $\nabla \cdot \vec{A}$ is invariant under the transformation.

PROBLEM (20): Verify that the transformation between the coordinates x_1, x_2, x_3 and x'_1, x'_2, x'_3 given by

$$x'_1 = \frac{1}{3}(2x_1 + 2x_2 - x_3)$$

$$x'_2 = \frac{1}{3}(2x_1 - x_2 + 2x_3)$$

$$x'_3 = \frac{1}{3}(-x_1 + 2x_2 + 2x_3)$$

is orthogonal and left–handed . A vector $\vec{A}$ referred to axes $Ox_1x_2x_3$ has components $(2, 1, -2)$. Find its components when referred to axes $Ox'_1x'_2x'_3$.

Also , write the transformation equations for a second order tensor A_{ij} w.r.t. this transformation and verify that $A'_{ii} = A_{ii}$.

PROBLEM (21): For the following transformation , $\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -3 & -6 & -2 \\ -2 & 3 & -6 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

write down the equations of the inverse transformation . If B_{ij} is a second order tensor whose components in the system $Ox'_1x'_2x'_3$ all vanish except that $B'_{13} = 1$, evaluate B_{11} and B_{12}.

PROBLEM (22): A second order tensor has components $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ in the system $Ox_1x_2x_3$.

The axes of this system are rotated about the origin to form the new system $Ox'_1x'_2x'_3$ whose transformation matrix is $T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$

Find the components of the tensor in the new system $Ox'_1x'_2x'_3$.

PROBLEM (23): A second order tensor has components $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ in the system $Ox_1x_2x_3$. The axes of this system are rotated by an angle θ in the positive sense about the x_3 – axis. Write the transformation matrix for this rotation. Also, find the components of this tensor in the new system $Ox'_1x'_2x'_3$.

PROBLEM (24): Write the transformation matrix for a rotation of angle α about x_2 – axis, followed by a rotation of angle β about the x'_3 – axis, both in the positive sense.

PROBLEM (25): The vector $\vec{A}$ has components (A_1, A_2, A_3) in the system $Ox_1x_2x_3$. A new coordinate system $Ox'_1x'_2x'_3$ is formed by rotating the original system through an angle α in the positive sense about the x_1 – axis. The system $Ox'_1x'_2x'_3$ is further rotated through an angle of β in the positive sense about the x'_2 – axis to obtain another coordinate system $Ox''_1x''_2x''_3$. Find the transformation matrix for this combined rotation. Hence determine the components of $\vec{A}$ in the new system when $\alpha = \beta = \frac{\pi}{6}$.

PROBLEM (26): If A_i and B_i are the components of two first order tensors, prove that their sum $(A_i + B_i)$ is also a first order tensor.

PROBLEM (27): If A_{ij} is a second order tensor, then show that A_{ji} is also a second order tensor.

PROBLEM (28): If A_i is a vector and δ_{ij} is a second order tensor, then prove that their product $\delta_{ij}A_k$ is a tensor of order 3.

PROBLEM (29): If A_{ij} is a second order tensor, prove that $\delta_{ij}A_{ij} = A_{ii}$ is a tensor of order zero i.e. a scalar.

PROBLEM (30): If A_{ijk} is a third order tensor, show that $\delta_{ij}A_{ijk} = A_{iik}$ are the components of a first order tensor. Write out the components of this tensor.

PROBLEM (31): Prove that

(i) $\delta_{ik}\delta_{mp}A_{ik} = \delta_{mp}A_{ii}$ (ii) $(\delta_{im}\delta_{kp} \pm \delta_{ip}\delta_{km})A_{ik} = A_{mp} \pm A_{pm}$

PROBLEM (32): If $A_{ij}B_i$ is a vector, and A_{ij} is an arbitrary tensor of rank 2, then prove that B_i is a vector.

PROBLEM (33): If A_iB_{ijk} is a tensor of rank 2, where A_i is an arbitrary vector, then prove that B_{ijk} is a tensor of rank 3.

PROBLEM (34): State whether the following tensors are symmetric, anti – symmetric or neither.

(i) $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & -2 \\ 1 & -2 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 2 & -7 \\ -2 & 1 & 4 \\ 7 & -4 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$

PROBLEM (35): Find the symmetric and anti – symmetric parts of the following second order tensors:

(i) $\begin{bmatrix} 10 & 8 & -1 \\ 2 & 7 & 5 \\ 6 & -9 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 2 & -7 \\ -4 & 1 & 4 \\ 8 & 2 & 5 \end{bmatrix}$

PROBLEM (36): Show that the following second – order tensors can be expressed as the sum of a symmetric and an anti – symmetric tensor :

(i) $A_i B_j$ (ii) $\dfrac{\partial u_i}{\partial x_j}$

PROBLEM (37): Prove from first principle that $\delta_{ij}\delta_{pq}$ are the components of a fourth order isotropic tensor .

PROBLEM (38): Prove that $\delta_{ij}\epsilon_{mnk}$ are the components of a fifth – order isotropic tensor . Furthermore , show that $\delta_{ij}\epsilon_{ijk}$ is the zero – vector .

PROBLEM (39): If A_{ij} is a second order symmetric tensor , show that $\epsilon_{ijk} A_{ij} = 0$, for all values of k . Prove the converse , that is if A_{ij} is a second order tensor such that $\epsilon_{ijk} A_{ij} = 0$, then A_{ij} is symmetric .

PROBLEM (40): If A_{ij} is a second order tensor , then $A_{ij,k}$ is a third order tensor . Furthermore , show that $A_{ij,j}$ is a first order tensor .

PROBLEM (41): If $A_{ij} = x_i x_j \quad (i, j = 1, 2, 3)$, evaluate $A_{ij,j}$; $A_{ij,ij}$.

PROBLEM (42): A third – order tensor field in the system $Ox_1x_2x_3$ is defined by

$A_{ijk} = x_i^2 + 2x_j^2 + x_k^2$. Find

(i) the divergence of the vector field A_{ijj} (ii) the curl of the vector field A_{ijj}

PROBLEM (43): Find the order or rank of tensors having the components

(i) A_{ijkmn} (ii) $A_{ijk} B_{km}$ (iii) $(B_{ijk} + C_{ijk})$

(iv) $A_{ij} D_{imn} E_n$ (v) $\dfrac{\partial^2 A_{ij}}{\partial x_i \partial x_j}$ (vi) $\iint A_{ijk}\, dx_i\, dx_j$

PROBLEM (44): Find the eigenvalues and eigenvectors of the following tensors :

(i) $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 4 & -1 \\ 6 & -7 & 10 \\ 3 & -4 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & -1 \\ -6 & -3 & 1 \end{bmatrix}$

EXERCISE (45): Find the eigenvalues and eigenvectors of the real symmetric tensors .

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 0 & 4 \\ 0 & 7 & 3 \\ 4 & 3 & 7 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

PROBLEM (46): Prove that the eigenvectors of each of the following tensors are mutually perpendicular :

(i) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

PROBLEM (47): Given a tensor A_{ij} whose components relative to a coordinate system $Ox_1x_2x_3$ are the following:

(i) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

Find the coordinate system relative to which the given tensors are diagonal .

PROBLEM (48): Find the invariants of the following second order tensors .

(i) $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 6 & 3 \\ 2 & 4 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 & 1 \\ 3 & 6 & 3 \\ 4 & 5 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 5 & 3 \\ 4 & 4 & 4 \\ 3 & 2 & 6 \end{bmatrix}$

PROBLEM (49): If λ_1, λ_2, and λ_3 are the eigenvalues , express

(i) $I = \lambda_1^2 \lambda_2 \lambda_3 + \lambda_1 \lambda_2^2 \lambda_3 + \lambda_1 \lambda_2 \lambda_3^2$ and

(ii) $I = \lambda_1^3 \lambda_2 \lambda_3 + \lambda_1 \lambda_2^3 \lambda_3 + \lambda_1 \lambda_2 \lambda_3^3$ as functions of principal invariants I_1, I_2, and I_3 .

PROBLEM (50): Let A_{ij} be the components of a second order symmetric tensor A . By referring A to its principal axes , express the following invariants

$$T_1 = A_{ii}, \quad T_2 = A_{ij} A_{ji}, \quad T_3 = A_{ij} A_{jk} A_{ki}$$

in terms of the eigenvalues λ_1, λ_2, and λ_3 of A . The principal invariants of A are I_1, I_2, and I_3 , verify that

$$I_1 = T_1, \quad I_2 = \frac{1}{2} \left(T_1^2 - T_2 \right)$$

$$I_3 = \frac{1}{6} \left(2T_3 + T_1^3 - 3T_1 T_2 \right)$$