Differentiation

Introduction to derivative

Def:- Suppose $y = f(x)$ be a function then rate of change in 'y' w.r.t 'x' is called differentiation of 'y' w.r.t 'x'.

The result of diff. is denoted by $\frac{d}{dx} f(x) = \frac{dy}{dx}$ and it called derivative of y.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, slope of curve at a point.

Question:

$$f(x) = 3x^2 - 2x$$

Find: (1) $f'(x)$

(2) Eq. of tangent line at point $P(2,3)$.

1): Sol

$$f(x) = 3x^2 - 2x$$

$$f(x+h) = 3(x+h)^2 - 2(x+h)$$

$$= 3x^2 + 3h^2 + 6xh - 2x - 2h$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{3x^2} + 3h^2 + 6xh - \cancel{2x} - 2h - \cancel{3x^2} + \cancel{2x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(3h + 6x - 2)}{\cancel{h}}$$

$$f'(x) = 6x - 2$$

2): Sol

$f'(x) = 6x - 2$, Slope $= m = f'(x)$

$$m|_{P(2,3)} = f'(2) = 6(2) - 2 = 10$$

$m = 10 \;,\; x_1 = 2 \;,\; y_1 = 3$

For tangent line, use Point slope Formula.

$$y - y_1 = m(x - x_1)$$

$$y - 3 = 10(x - 2)$$

$$\frac{d}{dx}(x^n) = ?$$

Let $f(x) = x^n$

$$F(x+h) = (x+h)^n$$

Now $\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$

$$\frac{d}{dx}(x^n) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\left(\cancel{x^n} + {}^nC_1 x^{n-1} h + {}^nC_2 x^{n-2} h^2 + {}^nC_3 x^{n-3} h^3 + \cdots + h^n\right) - \cancel{x^n}\right]$$

$$= \lim_{h \to 0} \left[{}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} h + {}^nC_3 x^{n-3} h^2 + \cdots + h^{n-1}\right]$$

$$= {}^nC_1 x^{n-1}$$

$$= n x^{n-1}$$

Hence; $\frac{d}{dx}(x^n) = n x^{n-1}$

$$\frac{d}{dx}(\sin x) = ?$$

Let $f(x) = \sin x$

$$F(x+h) = \sin(x+h)$$

Now;

$$\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$\therefore$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$= \lim_{h \to 0} \cos\left(x + \frac{h}{2}\right) \frac{\sin(h/2)}{h/2} \qquad \because \lim_{x \to 0}\left(\frac{\sin x}{x}\right) = 1$$

$\therefore$

$$= \cos x \cdot 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

Techniques of Derivative with Examples

$$\frac{d}{dx}(C) = 0, \quad C = \text{constant}$$

$$\frac{d}{dx}(C \cdot F(x)) = C \frac{d}{dx}(F(x))$$

$$\frac{d}{dx}(F(x) \pm g(x)) = \frac{d}{dx}(F(x)) \pm \frac{d}{dx}(g(x))$$

$$\frac{d}{dx}(F(x) \cdot g(x)) = F(x) \cdot \frac{d}{dx}(g(x)) + g(x) \frac{d}{dx}(F(x))$$

$$\frac{d}{dx}(F(x) / g(x)) = \frac{g(x) \frac{d}{dx}(F(x)) - F(x) \frac{d}{dx} g(x)}{(g(x))^2}$$

Different Notations for Derivative

Dependent Variable $\leftarrow y = F(x) \rightarrow$ Independent Variable

$$\frac{d}{dx} y = \frac{d}{dx} F(x) \Rightarrow \frac{dy}{dx} = \frac{dF}{dx}$$

y' ,	y'' ,	y''' ,	$y^{(iv)}$
F' ,	F'' ,	F''' ,	$F^{(iv)}$
$\frac{dy}{dx}$,	$\frac{d^2y}{dx^2}$,	$\frac{d^3y}{dx^3}$,	$\frac{d^4y}{dx^4}$
$D_x y$,	$D_x^2 y$,	$D_x^3 y$,	$D_x^4 y$

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

$$\frac{d}{dx}[F(x)]^n = n.[F(x)]^{n-1} \frac{d}{dx}(f(x))$$

Q#1 Find $f'(x) = ?$

Sol

$$f(x) = -5$$

$$\frac{d}{dx}(F(x)) = \frac{d}{dx}(-5)$$

$$F'(x) = 0$$

Q#2 If $y = 2x$; $y' = ?$

Sol

$$\frac{dy}{dx} = \frac{d}{dx}(2x)$$

$$= 2\frac{d}{dx}(x')$$

$$= 2(1.x^{1-1})$$

$$= 2$$

Q#3

$F'(x) = ?$ if $f(x) = x^3 + 4x + 5$

Sol

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x^3 + 4x + 5)$$

$$F'(x) = \frac{d}{dx}(x^3) + 4\frac{d}{dx}(x) + \frac{d}{dx}(5).$$

$$= 3x^{3-1}\frac{d}{dx}(x) + 4(1) + 0$$

$$F'(x) = 3x^2 + 4$$

Q#4 $F'(x) = ?$ if $F(x) = (x+1)(x+2)$

Sol $F'(x) = \frac{d}{dx}[(x+1)(x+2)]$

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$= (x+1)\frac{d}{dx}(x+2) \quad + (x+2)\frac{d}{dx}(x+1)$

$= \quad (x+1)(1+0) \quad + (x+2)(1+0)$

$= \quad x+1 + x + 2$

$= \quad 2x+3$

$f'(x) = 2x+3$

Q#5

$f'(x) = ? \quad , \quad f(x) = \frac{2x+5}{x^2+1}$

Sol

$$F'(x) = \frac{(x^2+1)\frac{d}{dx}(2x+5) - (2x+5)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(2) - (2x+5)(2x)}{(x^2+1)^2} = \frac{2x^2+2-4x^2-10x}{(x^2+1)^2}$$

$$F'(x) = \frac{-2x^2-10x+2}{(x^2+1)^2}$$

Q#6

$F'(t) = ? \quad , \quad f(t) = \sqrt[3]{t^2}$

Sol

$F(t) = (t^2)^{1/3}$

$F(t) = t^{2/3}$

$\frac{d}{dt}(F(t)) = \frac{d}{dt}(t^{2/3})$

$F'(t) = 2/3 \, t^{2/3-1} \quad \frac{d}{dt}(t)$

$F'(t) = \frac{2}{3} t^{-1/3}$

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Implicit Differentiation

Q#1

$$y^2 + x^2y + xy = 3$$

Sol

$$\frac{dy}{dx} = ?$$

$$\frac{d}{dx}(y^2 + x^2y + xy) = \frac{d}{dx}(3)$$

$$= \frac{d}{dx}(y^2) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy) = 0$$

$$= 2y \cdot \frac{dy}{dx} + \left[y\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(y)\right] + \left[y\frac{d}{dx}(x) + x\frac{d}{dx}(y)\right] = 0$$

$$= 2y\frac{dy}{dx} + y(2x) + x^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$

$$= \frac{dy}{dx}(2y + x^2 + x) = -2xy - y$$

$$\frac{dy}{dx} = \frac{-2xy - y}{x^2 + x + 2y}$$

Q#2

$$\sqrt{y^3} + \sqrt{x} + \sqrt{xy} = 1, \quad y = f(x), \quad y' = ?$$

Sol

$$= y^{3/2} + x^{1/2} + (xy)^{1/2} = 1$$

$$\frac{d}{dx}(y^{3/2}) + \frac{d}{dx}(x^{1/2}) + \frac{d}{dx}(xy)^{1/2} = \frac{d}{dx}(1)$$

$$\frac{3}{2}y^{3/2-1}\frac{dy}{dx} + \frac{1}{2}x^{1/2-1} \cdot 1 + \frac{1}{2}[xy]^{1/2-1}\frac{d}{dx}(xy) = 0$$

$$\frac{3}{2}y^{1/2}y' + \frac{1}{2}x^{-1/2} + \frac{1}{2}(xy)^{-1/2}[1 \cdot y + xy'] = 0$$

$$\frac{3}{2}\sqrt{y}\,y' + \frac{1}{2\sqrt{x}} + \frac{y}{2\sqrt{xy}} + \frac{xy'}{2\sqrt{xy}} = 0$$

$$y'\left[3\sqrt{y}+\frac{x}{2\sqrt{xy}}\right]=-\frac{1}{2\sqrt{x}}-\frac{y}{2\sqrt{xy}}$$

$$y'=\frac{\left(-\frac{1}{2\sqrt{x}}-\frac{y}{2\sqrt{xy}}\right)}{\left(3\sqrt{y}+\frac{x}{2\sqrt{xy}}\right)}$$

Q#3 Find the eq. of tangent line of curve $2y^2-4x^2=2$ at point $(-2,1)$.

Point slope formula

$$y-y_1=m(x-x_1)$$

Sol $\because$ $m=y'$

$$2y^2-4x^2=2$$

$$\frac{d}{dx}(2y^2)-\frac{d}{dx}(4x^2)=\frac{d}{dx}(2)$$

$$4y.y'-8x=0$$

$$y'=\frac{8x}{4y}=\frac{2x}{y}$$

$$m=y'=\frac{2x}{y}$$

$$m|_{P(-2,1)}=\frac{2(-2)}{1}=-4$$

$$\Rightarrow \quad y-1=-4(x+2)$$

$$y+4x=-7$$

Derivative of Trigonometric Function

Q#1 $f'(x)=?$

$$f(x)=\sqrt{1+\tan 5x}$$

$$\frac{d}{dx}(f(x))=\frac{d}{dx}(1+\tan 5x)^{\frac{1}{2}}$$

$$f'(x)=\frac{1}{2}(1+\tan 5x)^{\frac{1}{2}-1}\frac{d}{dx}(1+\tan 5x)$$

$$=\frac{1}{2}(1+\tan 5x)^{-\frac{1}{2}}\left[0+\sec^2(5x)\frac{d}{dx}(5x)\right]$$

$$=\frac{1}{2\sqrt{1+\tan 5x}}\cdot 5\sec^2 5x$$

i) $\frac{d}{dx}(\sin x)=\cos x$

ii) $\frac{d}{dx}(\cos x)=-\sin x$

iii) $\frac{d}{dx}(\tan x)=\sec^2 x$

iv) $\frac{d}{dx}(\cot x)=-\csc^2 x$

v) $\frac{d}{dx}(\sec x)=\sec x\tan x$

vi) $\frac{d}{dx}(\csc x)=-\cot x\csc x$

Q#2 Find $F'(x)$ if $F(x) = (\cos\sqrt[3]{x} - \sin\sqrt[3]{x})^4$

Sol $F(x) = (\cos x^{1/3} - \sin x^{1/3})^4$

$$F'(x) = 4[\cos x^{1/3} - \sin x^{1/3}]^{4-1} \frac{d}{dx}[\cos x^{1/3} - \sin x^{1/3}]$$

$$= 4(\cos x^{1/3} - \sin x^{1/3})^3 \left[-\sin x^{1/3} \frac{d}{dx}(x)^{1/3} - \cos x^{1/3} \frac{d}{dx}(x^{1/3})\right]$$

$$= 4(\cos x^{1/3} - \sin x^{1/3})^3 \left[-\sin x^{1/3}\left(\frac{1}{3}x^{-2/3}\right) - \cos x^{1/3}\left(\frac{1}{3}x^{-2/3}\right)\right]$$

$$= 4(\cos x^{1/3} - \sin x^{1/3})^3 \left[-\frac{1}{3}x^{-2/3}\sin x^{1/3} - \frac{1}{3}x^{-2/3}\cos x^{1/3}\right]$$

Q#3 Find $\frac{dy}{dx} = ?$ if $y = \frac{\sin 2x}{\cos 4x^2}$

Sol $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin 2x}{\cos 4x^2}\right)$$

$$= \frac{(\cos 4x^2)\frac{d}{dx}(\sin 2x) - (\sin 2x)\frac{d}{dx}(\cos 4x^2)}{(\cos 4x^2)^2}$$

$$= \frac{(\cos 4x^2)(2\cos 2x) - (\sin 2x)(-8x\sin 4x^2)}{(\cos 4x^2)^2}$$