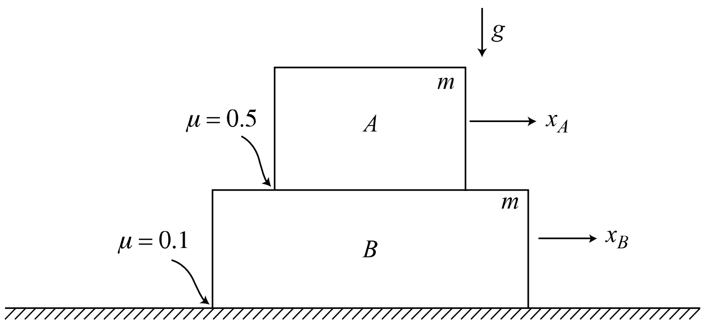


**Figure 1.18.**

Thus the relative acceleration between blocks $A$ and $B$ is $0.8g$ and the time to stop sliding is

$$t_1 = \frac{v_0}{0.8g} = 1.25\frac{v_0}{g} \tag{1.192}$$

At this time the common velocity of the two blocks is

$$v_1 = \ddot{x}_B t_1 = 0.375 v_0 \tag{1.193}$$

When $t > t_1$, $A$ and $B$ will slide as a single block of mass $2m$ with an acceleration equal to $-0.1g$. The stopping time for this combination is

$$t_2 = \frac{0.375 v_0}{0.1g} = 3.75\frac{v_0}{g} \tag{1.194}$$

Thus the time required for sliding to stop completely is

$$t = t_1 + t_2 = 5\frac{v_0}{g} \tag{1.195}$$

The final displacement of block $B$ is equal to its average velocity multiplied by the total time. We obtain

$$x_B = \frac{1}{2} v_1 t = 0.9375\frac{v_0^2}{g} \tag{1.196}$$

The final displacement of block $A$ is equal to $x_B$ plus the displacement of $A$ relative to $B$.

$$x_A = x_B + \frac{1}{2} v_0 t_1 = 1.5625\frac{v_0^2}{g} \tag{1.197}$$

## 1.3 Constraints and configuration space

### Generalized coordinates and configuration space

Consider a system of $N$ particles. The *configuration* of this system is specified by giving the locations of all the particles. For example, the inertial location of the first particle might be given by the Cartesian coordinates $(x_1, x_2, x_3)$, the location of the second particle by $(x_4, x_5, x_6)$, and so forth. Thus, the configuration of the system would be given by

$(x_1, x_2, \ldots, x_{3N})$. In the usual case, the particles cannot all move freely but are at least somewhat constrained kinematically in their differential motions, if not in their large motions as well. Under these conditions, it is usually possible to give the configuration of the system by specifying the values of fewer than $3N$ parameters. These $n \leq 3N$ parameters are called *generalized coordinates* ($q$s) and are related to the $x$s by the *transformation equations*

$$x_k = x_k(q_1, q_2, \ldots, q_n, t) \qquad (k = 1, \ldots, 3N) \tag{1.198}$$

The $q$s are not necessarily uniform in their dimensions. For example, the position of a particle in planar motion may be expressed by the polar coordinates $(r, \theta)$ which have differing dimensions. Thus, generalized coordinates may include common coordinate systems. However, a generalized coordinate may also be chosen such that it is not identified with any of the common coordinate systems, but represents a displacement form or shape involving several particles. In this case, the generalized coordinate is defined assuming certain displacement ratios and relative directions among the particles. For example, a generalized coordinate might consist of equal radial displacements of particles at the vertices of an equilateral triangle.

Frequently one attempts to find a set of independent generalized coordinates, but this is not always possible. So, in general, we assume that there are $m$ independent equations of constraint involving the $q$s and possibly the $\dot{q}$s. If, for the same system, there are $l$ independent equations of constraint involving the $3N$ $x$s (and possibly the corresponding $\dot{x}$s), then

$$3N - l = n - m \tag{1.199}$$

and this is equal to the number of *degrees of freedom*. The number of degrees of freedom is, in general, a property of the system and not of the choice of coordinates.

Since the configuration of a system is specified by the values of its $n$ generalized coordinates, one can represent any particular configuration by a point in $n$-dimensional *configuration space* (Fig. 1.19). If the values of all the $q$s and $\dot{q}$s are known at some initial time $t_0$, then, as time proceeds, the configuration point $C$ will trace a solution path in configuration space in accordance with the dynamical equations of motion and any constraint equations. For the case of independent $q$s, the curve will be continuous but otherwise not constrained. If, however, there are holonomic constraints expressed as functions of the $q$s and possibly time, then the solution point must remain on a hypersurface having fewer than $n$ dimensions, and which may be moving and possibly changing shape. In general, then, one can represent an evolving mechanical system by an $n$-dimensional vector $\mathbf{q}$, drawn from the origin to the configuration point $C$, tracing a path in configuration space as time proceeds. This will be discussed further in Chapter 2.

## Holonomic constraints

Suppose that the configuration of a system is specified by $n$ generalized coordinates $(q_1, \ldots, q_n)$ and assume that there are $m$ independent equations of constraint of the form

$$\phi_j(q_1, \ldots, q_n, t) = 0 \qquad (j = 1, \ldots, m) \tag{1.200}$$

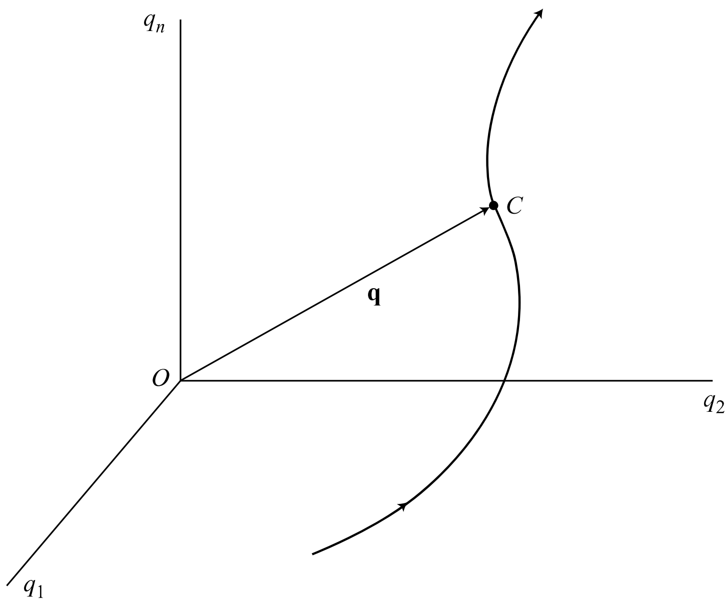


**Figure 1.19.**

A constraint of this form is called a *holonomic constraint.* A dynamical system whose constraint equations, if any, are all of the holonomic form is called a *holonomic system.*

An example of a holonomic constraint is provided by a particle which is forced to move on a sphere of radius $R$ centered at the origin of a Cartesian frame. In this case the equation of constraint is

$$\phi_j = x^2 + y^2 + z^2 - R^2 = 0 \tag{1.201}$$

where $(x, y, z)$ is the location of the particle. The sphere is a two-dimensional constraint surface which is embedded in a three-dimensional Cartesian space.

The configuration of a holonomic system can always be specified using a minimal set of generalized coordinates equal in number to the degrees of freedom. This is also the number of dimensions of the constraint hypersurface, that is, $n - m$. Hence it is always possible in theory to find a set of *independent* $q$s describing a holonomic system. For the case of a spherical constraint surface, one could use angles of latitude and longitude to describe the position of a particle. Another possibility might be to use the cylindrical coordinates $\phi$ and $z$ as $q$s, where $\phi$ effectively gives the longitude and $z$ the latitude.

## Nonholonomic constraints

Nonholonomic constraints may have the general form

$$f_j(q, \dot{q}, t) = 0 \qquad (j = 1, \ldots, m) \tag{1.202}$$

but usually they have a simpler form which is *linear* in the velocities. Thus, we nearly always assume that nonholonomic constraints have the form

$$f_j = \sum_{i=1}^{n} a_{ji}(q, t)\, \dot{q}_i + a_{jt}(q, t) = 0 \qquad (j = 1, \ldots, m) \tag{1.203}$$

or the alternate differential form

$$\sum_{i=1}^{n} a_{ji}(q,t)\,dq_i + a_{jt}(q,t)\,dt = 0 \qquad (j = 1,\ldots,m) \tag{1.204}$$

where, in either case, these expressions are *not integrable*. If either expression were integrable, then a function $\phi_j(q,t)$ would exist and (1.200) would apply, indicating that the constraint is actually *holonomic*. In this case, we would have

$$\dot{\phi}_j(q,\dot{q},t) = \sum_{i=1}^{n} \frac{\partial \phi_j}{\partial q_i}\,\dot{q}_i + \frac{\partial \phi_j}{\partial t} = 0 \tag{1.205}$$

and, comparing (1.203) and (1.205), we find that

$$a_{ji} \equiv \frac{\partial \phi_j}{\partial q_i}, \qquad a_{jt} \equiv \frac{\partial \phi_j}{\partial t} \qquad (i = 1,\ldots,n;\ j = 1,\ldots,m) \tag{1.206}$$

for this holonomic constraint. We conclude that holonomic and "linear" nonholonomic constraints can be expressed in the forms of (1.203) and (1.204). The holonomic case is distinguished by its integrability. Note that the coefficients $a_{ji}(q,t)$ and $a_{jt}(q,t)$ are generally *nonlinear* in the $q$s and $t$.

## Other constraint classifications

A constraint is classed as *scleronomic* if the time $t$ does not appear explicitly in the equation of constraint. Otherwise, it is *rheonomic*. Thus, a scleronomic holonomic constraint has the form

$$\phi_j(q) = 0 \tag{1.207}$$

A scleronomic nonholonomic constraint has the form

$$\sum_{i=1}^{n} a_{ji}(q)\dot{q}_i = 0 \tag{1.208}$$

where we note that $a_{jt} \equiv 0$.

Constraints having $a_{jt} \neq 0$, or $a_{ji} = a_{ji}(q,t)$, or $\phi_j = \phi_j(q,t)$ are classed as *rheonomic* constraints. Typical examples of rheonomic constraints are a rod of varying length $l(t)$ connecting two particles in the holonomic case, or a knife edge whose orientation angle is an explicit function of time in the nonholonomic case.

A *catastatic* constraint has $\partial\phi_j/\partial t \equiv 0$ if holonomic, or $a_{jt} \equiv 0$ if nonholonomic. Catastatic constraints have an important place in dynamical theory. Note that all scleronomic constraints are also catastatic, but not necessarily *vice versa*. For example, a nonholonomic constraint having the form

$$\sum_{i=1}^{n} a_{ji}(q,t)\,\dot{q}_i = 0 \tag{1.209}$$

is catastatic but is not scleronomic.

Dynamical *systems* can also be classified as scleronomic or rheonomic. A *scleronomic system* satisfies the conditions that: (1) all constraints, if any, are scleronomic; and (2) the transformation equations, given by (1.198), which relate inertial Cartesian coordinates and generalized coordinates do not contain time explicitly. As an example illustrating the importance of the second condition, consider a particle moving on a spherical surface, centered at the origin, whose radius $R(t)$ is a known function of time. We can use the spherical coordinates $\theta$ and $\phi$ as *independent generalized coordinates*, that is, there are no constraints on the $q$s. Hence, the first condition is satisfied. The transformation equations, however, are

$$\begin{aligned} x &= R(t)\sin\theta\cos\phi \\ y &= R(t)\sin\theta\sin\phi \\ z &= R(t)\cos\theta \end{aligned} \tag{1.210}$$

These transformation equations do not satisfy the second condition, so the system is rheonomic.

Another classification of dynamical systems involves the categories *catastatic* or *acatastatic*. A catastatic system satisfies the conditions that: (1) all constraints, if any, are catastatic; and (2) the system is capable of being continuously at rest by setting all the $\dot{q}$s equal to zero. This implies that $\partial x_k/\partial t \equiv 0$ for $k = 1, \ldots, 3N$; that is, for all the transformation equations. In other words, $x_k = x_k(q)$ in agreement with the second condition for a scleronomic system. Acatastatic means not catastatic.

## Accessibility

Constraint equations, either holonomic or nonholonomic, are *kinematic* in nature; that is, they put restrictions on the possible motions of a system, irrespective of the dynamical equations. There is an important difference in these possible motions that distinguishes holonomic from nonholonomic systems. It lies in the *accessibility* of points in configuration space. For the case of *holonomic constraints*, the configuration point moves in a reduced space of $(n - m)$ dimensions since it must remain on each of $m$ constraint surfaces, that is, on their common intersection. Thus, certain regions of $n$-dimensional configuration space are no longer accessible.

By contrast, for *nonholonomic constraints*, it is the differential motions which are constrained. Since the differential equations representing these nonholonomic constraints are not integrable, there are no finite constraint surfaces in configuration space and there is no reduction of the accessible region. In other words, by properly choosing the path, it is possible to reach any point in $n$-dimensional $q$-space from any other point. As an example, a scleronomic nonholonomic constraint, as given in (1.208), can be represented by an $(n - m)$-dimensional planar differential surface element. Any differential displacement $d\mathbf{q}$ must lie within that plane but is otherwise unconstrained. It is possible, however, to steer the configuration point $C$ and its differential element to any point of configuration space, provided that more than one degree of freedom exists. A scleronomic system with only one degree of freedom is not steerable, and therefore any such system must be integrable and holonomic, and not fully accessible.

## Exactness and integrability

Now let us consider the conditions under which a constraint function $f_j(q, \dot{q}, t)$ is integrable. If it is integrable, then a function $\phi_j(q, t)$ exists whose total time derivative is equal to $f_j(q, \dot{q}, t)$, that is,

$$f_j = \dot{\phi}_j(q, \dot{q}, t) = \sum_{i=1}^{n} \frac{\partial \phi_j}{\partial q_i}\, \dot{q}_i + \frac{\partial \phi_j}{\partial t} \tag{1.211}$$

We see immediately that $f_j$ must be linear in the $\dot{q}$s, so let us consider the linear form

$$f_j = \sum_{i=1}^{n} a_{ji}(q, t)\, \dot{q}_i + a_{jt}(q, t) \tag{1.212}$$

as in (1.203). By comparing (1.211) and (1.212) we can equate coefficients as follows:

$$a_{ji}(q, t) = \frac{\partial \phi_j}{\partial q_i}, \qquad a_{jt}(q, t) = \frac{\partial \phi_j}{\partial t} \tag{1.213}$$

for all $i$ and $j$. If $f_j(q, \dot{q}, t)$ is integrable, we know that a function $\phi_j(q, t)$ exists and that

$$\frac{\partial^2 \phi_j}{\partial q_k \partial q_i} = \frac{\partial^2 \phi_j}{\partial q_i \partial q_k}, \qquad \frac{\partial^2 \phi_j}{\partial q_i \partial t} = \frac{\partial^2 \phi_j}{\partial t\, \partial q_i} \tag{1.214}$$

that is, the order of partial differentiation is immaterial. In terms of $a_{ji}$s we have

$$\begin{aligned} \frac{\partial a_{ji}}{\partial q_k} &= \frac{\partial a_{jk}}{\partial q_i} \qquad (i, k = 1, \ldots, n) \\ \frac{\partial a_{jt}}{\partial q_i} &= \frac{\partial a_{ji}}{\partial t} \qquad (i = 1, \ldots, n) \end{aligned} \tag{1.215}$$

These are the *exactness conditions* for integrability. In general, a function $f_j(q, \dot{q}, t)$ is integrable if it has the linear form of (1.212) and if it is either (1) exact as it stands or (2) can be made exact through multiplication by an *integrating factor* of the form $M_j(q, t)$.

There is an alternative form of the exactness conditions for the case in which $f_j(q, \dot{q}, t)$ has the linear form of (1.212). First, we see that

$$\frac{d}{dt}\left(\frac{\partial f_j}{\partial \dot{q}_i}\right) = \dot{a}_{ji} = \sum_{k=1}^{n} \frac{\partial a_{ji}}{\partial q_k}\, \dot{q}_k + \frac{\partial a_{ji}}{\partial t} \tag{1.216}$$

Also, changing the summing index from $i$ to $k$ in (1.212), we obtain

$$\frac{\partial f_j}{\partial q_i} = \sum_{k=1}^{n} \frac{\partial a_{jk}}{\partial q_i} \dot{q}_k + \frac{\partial a_{jt}}{\partial q_i} \tag{1.217}$$

Hence we find that

$$\frac{d}{dt}\left(\frac{\partial f_j}{\partial \dot{q}_i}\right) - \frac{\partial f_j}{\partial q_i} = \sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k} - \frac{\partial a_{jk}}{\partial q_i}\right)\dot{q}_k + \frac{\partial a_{ji}}{\partial t} - \frac{\partial a_{jt}}{\partial q_i} \tag{1.218}$$

If the exactness conditions of (1.215) are satisfied, it follows that

$$\frac{d}{dt}\left(\frac{\partial f_j}{\partial \dot{q}_i}\right) - \frac{\partial f_j}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{1.219}$$

Conversely, if (1.219) is satisfied for all $\dot{q}$s satisfying the constraints, then, from (1.218), we see that the exactness conditions must apply. Hence, (1.215) and (1.219) are equivalent statements of the exactness conditions.

Another approach to the question of integrability lies in the use of *Pfaffian differential forms*. A Pfaffian differential form $\Omega$ in the $r$ variables $x_1, \ldots, x_r$ can be written as

$$\Omega = X_1(x)dx_1 + \cdots + X_r(x)dx_r \tag{1.220}$$

where the coefficients are functions of the $x$s, in general. If the differential form is exact it is equal to the total differential $d\Phi$ of a function $\Phi(x)$. The exactness conditions are

$$\frac{\partial X_i}{\partial x_j} = \frac{\partial X_j}{\partial x_i} \qquad (i, j = 1, \ldots, r) \tag{1.221}$$

that is, for all $i$ and $j$. The differential form $\Omega$ is *integrable* if it is exact, or if it can be made exact through multiplication by an *integrating factor* of the form $M(x)$.

Returning now to a consideration of nonholonomic constraints, we can write the Pfaffian differential form

$$\Omega_j = \sum_{i=1}^{n} a_{ji}(q,t)dq_i + a_{jt}(q,t)dt = 0 \qquad (j = 1, \ldots, m) \tag{1.222}$$

which we recognize as having been presented previously in (1.204). Of course, the exactness conditions are, as before, those given in (1.215).

Differential forms have wide application in the study of dynamics. A common example is the differential expression

$$dW = \sum_{k=1}^{3N} F_k(x)dx_k \tag{1.223}$$

for the work done in an inertial Cartesian frame on a system of $N$ particles by the Cartesian force components $F_k(x)$. This differential form may or may not be integrable. The question of integrability is important since it relates to the existence of a potential energy function. If integrable, there exists a potential energy function of the form $V(x)$.

## 1.4 Work, energy and momentum

With the introduction of generalized coordinates and their use in specifying the kinematic constraints on dynamical systems, we need to consider an expanded, generalized view of work, energy, and momentum.

## Virtual displacements

Suppose that the vector $\mathbf{r}_i(q, t)$ gives the location of some point in a mechanical system; for example, it might be the position vector of the $i$th particle written in terms of the $n$ $q$s and time. Now consider an *actual* differential displacement

$$d\mathbf{r}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_j} dq_j + \frac{\partial \mathbf{r}_i}{\partial t} dt \tag{1.224}$$

which occurs during an infinitesimal time interval $dt$. If there are $m$ holonomic constraint equations, the $dq$s must satisfy

$$d\phi_j = \sum_{i=1}^{n} \frac{\partial \phi_j}{\partial q_i} dq_i + \frac{\partial \phi_j}{\partial t} dt = 0 \qquad (j = 1, \ldots, m) \tag{1.225}$$

On the other hand, if there are $m$ nonholonomic constraints, the $dq$s satisfy

$$\sum_{i=1}^{n} a_{ji}(q, t)\, dq_i + a_{jt}(q, t)\, dt = 0 \qquad (j = 1, \ldots, m) \tag{1.226}$$

as given previously in (1.204).

Now let us hold time fixed by setting $dt = 0$ and imagine a *virtual displacement* $\delta\mathbf{r}_i$ in ordinary three-dimensional space for each of $N$ particles.

$$\delta\mathbf{r}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_i} \delta q_i \qquad (i = 1, \ldots, N) \tag{1.227}$$

The virtual displacement of the system can also be described by the $n$-dimensional vector $\delta\mathbf{q}$ in configuration space. If there are constraints acting on the system, the $\delta q$s must satisfy *instantaneous* or *virtual constraint equations* of the form

$$\sum_{i=1}^{n} \frac{\partial \phi_j}{\partial q_i} \delta q_i = 0 \qquad (j = 1, \ldots, m) \tag{1.228}$$

for the holonomic case, or

$$\sum_{i=1}^{n} a_{ji}\, \delta q_i = 0 \qquad (j = 1, \ldots, m) \tag{1.229}$$

for nonholonomic constraints.

A comparison of (1.228) and (1.229) with (1.225) and (1.226) shows that virtual displacements and actual displacements are different, in general, since they satisfy different constraint equations. If the constraints are *catastatic*, however; that is, if any holonomic constraints are of the form $\phi_j(q) = 0$, and if all nonholonomic constraints have $a_{jt} = 0$, then the virtual and actual small displacements satisfy the same set of constraint equations. Of course, the actual motion also satisfies the equations of motion.

The forms of (1.228) and (1.229), which are linear in the $\delta q$s, indicate that the virtual displacements lie in an $(n - m)$-dimensional hyperplane at the operating point in $n$-dimensional configuration space, in accordance with the constraints. For holonomic constraints, the plane is tangent to the constraint surface at the operating point.

## Virtual work

The concept of virtual work is fundamental to a proper understanding of dynamical theory. First, it must be emphasized that there is a distinction between work and virtual work. The work done by a force $\mathbf{F}_i$ acting on the $i$th particle as it moves between points $A_i$ and $B_i$ in an inertial frame is equal to the line integral

$$W_i = \int_{A_i}^{B_i} \mathbf{F}_i \cdot d\mathbf{r}_i \tag{1.230}$$

where $\mathbf{r}_i$ is the position vector of the $i$th particle. For a system of $N$ particles, the work done in an arbitrary small displacement of the system is

$$dW = \sum_{i=1}^{N} \mathbf{F}_i \cdot d\mathbf{r}_i = \sum_{k=1}^{3N} F_k dx_k \tag{1.231}$$

where $(x_1, \ldots, x_{3N})$ are the Cartesian coordinates of the $N$ particles and the $F_k$ are the corresponding force components applied to the particles.

Now let us transform to generalized coordinates using (1.198) and (1.224). The work done on the system during a small displacement in the time interval $dt$ is

$$dW = \sum_{i=1}^{N} \sum_{j=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} dq_j + \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial t} dt \tag{1.232}$$

or, in terms of Cartesian coordinates,

$$dW = \sum_{k=1}^{3N} \sum_{j=1}^{n} F_k \frac{\partial x_k}{\partial q_j} dq_j + \sum_{k=1}^{3N} F_k \frac{\partial x_k}{\partial t} dt \tag{1.233}$$

At this point it is convenient to introduce the *velocity coefficients* $\gamma_{ij}$ and $\gamma_{it}$ defined by

$$\gamma_{ij} = \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}, \qquad \gamma_{it} = \frac{\partial \mathbf{r}_i}{\partial t} \tag{1.234}$$

Note that these coefficients are vector quantities. Now we can write (1.224) in the form

$$d\mathbf{r}_i = \sum_{j=1}^{n} \gamma_{ij}\, dq_i + \gamma_{it}\, dt \tag{1.235}$$

The corresponding velocity is

$$\mathbf{v}_i = \sum_{j=1}^{n} \gamma_{ij}\, \dot{q}_j + \gamma_{it} \tag{1.236}$$

We see that $\gamma_{ij}$ represents the sensitivity of the velocity $\mathbf{v}_i$ to changes in $\dot{q}_i$, whereas $\gamma_{it}$ is equal to the velocity $\mathbf{v}_i$ when all $q$s are held constant.

Returning now to (1.232), we can write

$$dW = \sum_{i=1}^{N} \sum_{j=1}^{n} \mathbf{F}_i \cdot \gamma_{ij}\, dq_j + \sum_{i=1}^{N} \mathbf{F}_i \cdot \gamma_{it}\, dt \tag{1.237}$$

The *virtual work* $\delta W$ due to the forces $\mathbf{F}_i$ acting on the system is obtained by setting $dt = 0$ and replacing the actual displacements $d\mathbf{r}_i$ by virtual displacements $\delta\mathbf{r}_i$. Thus we obtain the alternate forms

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_{k=1}^{3N} F_k \delta x_k \tag{1.238}$$

or

$$\delta W = \sum_{i=1}^{N} \sum_{j=1}^{n} \mathbf{F}_i \cdot \gamma_{ij}\, \delta q_j = \sum_{k=1}^{3N} \sum_{j=1}^{n} F_k \frac{\partial x_k}{\partial q_j}\, \delta q_j \tag{1.239}$$

Let us define the *generalized force* $Q_j$ associated with $q_j$ by

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \cdot \gamma_{ij} = \sum_{k=1}^{3N} F_k \frac{\partial x_k}{\partial q_j} \tag{1.240}$$

Then the virtual work can be written in the form

$$\delta W = \sum_{j=1}^{n} Q_j\, \delta q_j \tag{1.241}$$

In general, we assume that the virtual displacements are consistent with any constraints, that is, they satisfy (1.228) or (1.229). But, if the system is holonomic, it is particularly convenient to choose independent $\delta q$s.

The question arises concerning why the virtual work $\delta W$ receives so much attention in dynamical theory rather than the work $dW$ of the actual motion. The reason lies in the nature of constraint forces. An *ideal constraint* is a *workless constraint* which may be either scleronomic or rheonomic. By workless we mean that no work is done by the constraint forces in an arbitrary reversible virtual displacement that satisfies the virtual constraint equations having the form of (1.228) or (1.229). Examples of ideal constraints include frictionless constraint surfaces, or rolling contact without slipping, or a rigid massless rod connecting two particles. Another example is a knife-edge constraint that allows motion in the direction of the knife edge without friction, but does not allow motion perpendicular to the knife edge. Ideal constraint forces, such as the internal forces in a rigid body, may do work on individual particles due to a virtual displacement, but no work is done on the system as a whole because these forces occur in equal, opposite and collinear pairs.

It is convenient to consider the total force acting on the $i$th particle to be the sum of the *applied force* $\mathbf{F}_i$ and the *constraint force* $\mathbf{R}_i$, by which we mean an ideal constraint force. Thus, all forces that are not ideal constraint forces are classed as applied forces. Frequently the applied forces are known, but the constraint forces either are unknown or are difficult to calculate.

The advantage of using virtual displacements rather than actual displacements in dynamical analyses can be seen by considering the virtual work of all the forces acting on a system of particles. We find that

$$\delta W = \sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{R}_i) \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i \tag{1.242}$$

since

$$\sum_{i=1}^{N} \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0 \tag{1.243}$$

Thus, ideal constraint forces can be ignored in calculating the virtual work of all the forces acting on a system. On the other hand,

$$\sum_{i=1}^{N} \mathbf{R}_i \cdot d\mathbf{r}_i \neq 0 \tag{1.244}$$

in the general case, indicating that constraint forces can contribute to the work $dW$ resulting from a small actual displacement. To summarize, one can ignore the constraint forces in applying virtual work methods. This advantage will carry over to equations derived using virtual work, an example being Lagrange's equation.

**Example 1.10** In this example we will show how a generalized force can be calculated using virtual work. In general, the generalized force $Q_i$ is equal to the virtual work per unit $\delta q_i$, assuming that the other $\delta q$s are set equal to zero, that is, assuming independent $\delta q$s. This is in accordance with (1.241).

Consider a system (Fig. 1.20) consisting of two particles connected by a rigid rod of length $L$. Let $(x, y)$ be the position of particle 1, and let $\theta$ be the angle of the rod relative

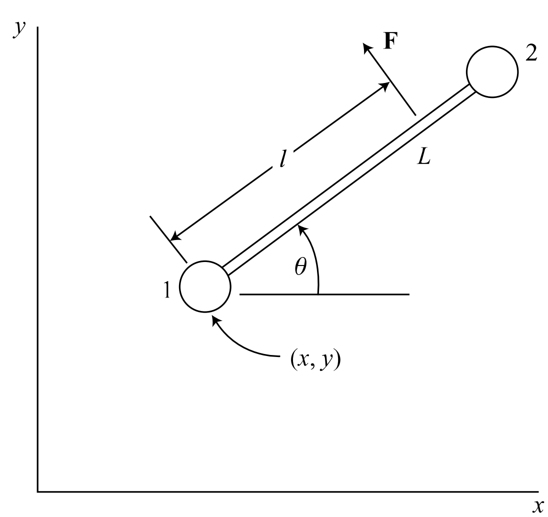


**Figure 1.20.**

to the $x$-axis. A force **F**, perpendicular to the rod, is applied at a distance $l$ from particle 1. We wish to solve for the generalized forces $Q_x$, $Q_y$, and $Q_\theta$.

First, we see that

$$\mathbf{F} = -F\sin\theta\,\mathbf{i} + F\cos\theta\,\mathbf{j} \tag{1.245}$$

where **i** and **j** are Cartesian unit vectors. The virtual displacement $\delta\mathbf{r}$ at the point of application of **F** is

$$\delta\mathbf{r} = (\delta x - l\ \sin\theta\ \delta\theta)\mathbf{i}\ + (\delta y + l\cos\theta\ \delta\theta)\mathbf{j} \tag{1.246}$$

Thus, the virtual work is

$$\delta W = \mathbf{F}\cdot\delta\mathbf{r} = -F\sin\theta\ \delta x + F\cos\theta\ \delta y + Fl\ \delta\theta \tag{1.247}$$

From (1.241) we have

$$\delta W = Q_x\ \delta x + Q_y\ \delta y + Q_\theta\ \delta\theta \tag{1.248}$$

Then, by comparing coefficients of the $\delta q$s, we find that the generalized forces are

$$Q_x = -F\sin\theta, \qquad Q_y = F\cos\theta, \qquad Q_\theta = Fl \tag{1.249}$$

Note that $Q_x$ and $Q_y$ are the $x$ and $y$ components of **F**, whereas $Q_\theta$ is the moment about particle 1.

## Principle of virtual work

A system of particles is in static equilibrium if each particle of the system is in static equilibrium. A particle is in static equilibrium if it is motionless at the initial time $t = 0$, and if its acceleration remains zero for all $t \geq 0$.

Now consider a *catastatic system* of particles; that is, all transformation equations from inertial $x$s to $q$s do not contain time explicitly. This implies that all particles are at rest if all $\dot{q}$s equal zero. For such a system we can state the *principle of virtual work: The necessary and sufficient condition for the static equilibrium of an initially motionless catastatic system which is subject to ideal bilateral constraints is that zero virtual work is done by the applied forces in moving through an arbitrary virtual displacement satisfying the constraints.*

To explain this principle, first assume that the catastatic system is in static equilibrium. Then

$$\mathbf{F}_i + \mathbf{R}_i = 0 \qquad (i = 1, \dots, N) \tag{1.250}$$

implying that each particle has zero acceleration. Now take the dot product with $\delta\mathbf{r}_i$ and sum over $i$. We obtain

$$\sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{R}_i)\cdot\delta\mathbf{r}_i = 0 \tag{1.251}$$

and we assume that the $\delta\mathbf{r}$s satisfy the constraints. The virtual work of the constraint forces equals zero, that is,

$$\sum_{i=1}^{N} \mathbf{R}_i \cdot \delta\mathbf{r}_i = 0 \tag{1.252}$$

Hence we find that

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0 \tag{1.253}$$

if the system is in static equilibrium. This is the necessary condition.

Now suppose that the system is not in static equilibrium, implying that $\mathbf{F}_i + \mathbf{R}_i \neq 0$ for at least one particle. Then it is always possible to find a virtual displacement such that

$$\sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{R}_i) \cdot \delta\mathbf{r}_i \neq 0 \tag{1.254}$$

since sufficient degrees of freedom remain. Using (1.252) we conclude that a virtual displacement can always be found that results in $\delta W \neq 0$ if the system is not in static equilibrium. Thus, if $\delta W = 0$ for all possible $\delta\mathbf{r}$s, the system must be in static equilibrium; this is the sufficient condition.

We have assumed a catastatic system. It is possible that a particular system that is not catastatic could, nevertheless, have a position of static equilibrium if $a_{jt}$ and $\partial x_k/\partial t$ are not both identically zero, but are equal to zero at the position of static equilibrium. This is a rare situation, however.

## Kinetic energy

Earlier we found that the kinetic energy of a system of $N$ particles can be expressed in the form

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2}\sum_{k=1}^{3N} m_k \dot{x}_k^2 \tag{1.255}$$

where $(x_1, x_2, x_3)$ is the inertial Cartesian position of the first particle whose mass is $m_1 = m_2 = m_3$, and similar notation is used to indicate the position and mass of each of the other particles. We wish to express the same kinetic energy in terms of $q$s, $\dot{q}$s, and possibly time. To accomplish this, we use the transformation equation (1.198) to obtain

$$\dot{x}_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{\partial t} \qquad (k = 1, \ldots, 3N) \tag{1.256}$$

Then we find that the kinetic energy is

$$T(q, \dot{q}, t) = \frac{1}{2}\sum_{k=1}^{3N} m_k \left(\sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{\partial t}\right)^2 \tag{1.257}$$

Let us express this kinetic energy in terms of homogeneous functions of the $\dot{q}$s, that is, separating the various powers of $\dot{q}_i$. We can write

$$T = T_2 + T_1 + T_0 \tag{1.258}$$

where the quadratic portion is

$$T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\,\dot{q}_i\dot{q}_j \tag{1.259}$$

The mass coefficients $m_{ij}$ are

$$m_{ij}(q,t) = m_{ji} = \sum_{k=1}^{3N} m_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j} \qquad (i, j = 1, \ldots, n) \tag{1.260}$$

Note that

$$m_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i\,\partial \dot{q}_j} \tag{1.261}$$

Similarly, the portion that is linear in the $\dot{q}$s is

$$T_1 = \sum_{i=1}^{n} a_i\dot{q}_i \tag{1.262}$$

where

$$a_i(q,t) = \sum_{i=1}^{3N} m_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial t} \qquad (i = 1, \ldots, n) \tag{1.263}$$

Finally, that portion of the kinetic energy which is not a function of the $\dot{q}$s is

$$T_0(q,t) = \frac{1}{2}\sum_{k=1}^{3N} m_k \left(\frac{\partial x_k}{\partial t}\right)^2 \tag{1.264}$$

For a scleronomic or catastatic system, $\partial x_k/\partial t = 0$, so both $T_1$ and $T_0$ vanish and $T = T_2$.

Now recall from (1.236) that the velocity of the $k$th particle, expressd in terms of velocity coefficients is

$$\mathbf{v}_k = \sum_{i=1}^{n} \gamma_{ki}(q,t)\,\dot{q}_i + \gamma_{kt}(q,t) \tag{1.265}$$

where

$$\gamma_{ki} = \frac{\partial \mathbf{v}_k}{\partial \dot{q}_i}, \qquad \gamma_{kt} = \frac{\partial \mathbf{r}_k}{\partial t} \tag{1.266}$$

Thus, the kinetic energy is

$$T = \frac{1}{2}\sum_{k=1}^{N} m_k \left(\sum_{i=1}^{n} \gamma_{ki}\,\dot{q}_i + \gamma_{kt}\right)^2 \tag{1.267}$$

and we find that

$$m_{ij} = m_{ji} = \sum_{k=1}^{N} m_k \gamma_{ki} \cdot \gamma_{kj} \tag{1.268}$$

$$a_i = \sum_{k=1}^{N} m_k \gamma_{ki} \cdot \gamma_{kt} \tag{1.269}$$

$$T_0 = \frac{1}{2} \sum_{k=1}^{N} m_k \gamma_{kt}^2 \tag{1.270}$$

Note that $\gamma_{kt} \equiv 0$ for a scleronomic or catastatic system.

## Potential energy

For a system of $N$ particles whose configuration is given in terms of $3N$ Cartesian coordinates, the force $F_k$ obtained from a potential energy function $V(x, t)$ is

$$F_k = -\frac{\partial V}{\partial x_k} \tag{1.271}$$

in accordance with (1.144). The virtual work due to these applied forces is

$$\delta W = \sum_{k=1}^{3N} F_k \, \delta x_k = -\sum_{k=1}^{3N} \frac{\partial V}{\partial x_k} \delta x_k = -\delta V \tag{1.272}$$

Now transform from $x$s to $q$s using (1.198). The potential energy has the form $V(q, t)$ and we obtain

$$\delta W = -\delta V = -\sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j \tag{1.273}$$

In general, we know from (1.241) that

$$\delta W = \sum_{j=1}^{n} Q_j \, \delta q_j \tag{1.274}$$

Upon comparing coefficients of the $\delta q$s, we find that the generalized force $Q_j$ due to $V(q, t)$ is

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{1.275}$$

For the particular case in which $V = V(q)$ and all the applied forces are obtained using (1.275), the total energy $T + V$ is conserved.

Finally, from (1.273) and the principle of virtual work we see that an initially motionless conservative system with bilateral constraints is in static equilibrium if and only if

$$\delta V = 0 \tag{1.276}$$

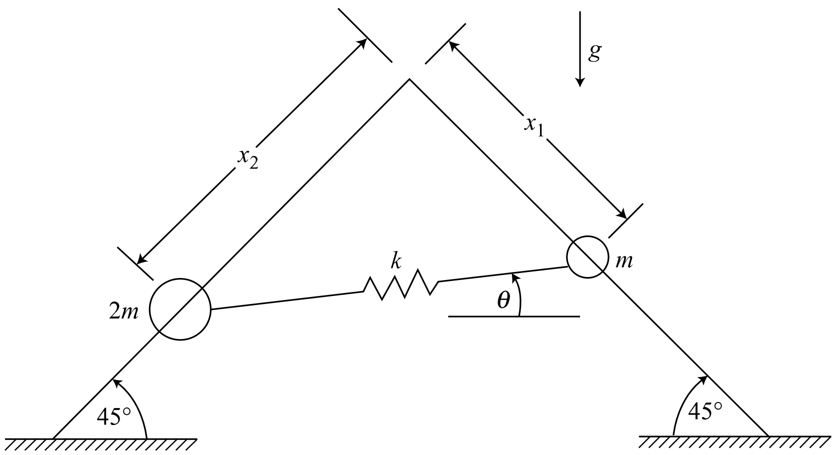


**Figure 1.21.**

for all virtual displacements consistent with the constraints. Any such static equilibrium configuration is *stable* if it occurs at a local minimum of the potential energy, considered as a function of the $q$s.

**Example 1.11** Particles of mass $m$ and $2m$ can slide freely on two rigid rods inclined at 45° with the horizontal (Fig. 1.21). They are connected by a linear spring of stiffness $k$. We wish to solve for the inclination $\theta$ of the spring and its tensile force $F$ at the position of static equilibrium.

Let us use the principle of virtual work. The applied forces acting on the system are the spring force $F$ and those due to gravity. The virtual work is

$$\delta W = \left[\frac{mg}{\sqrt{2}} - F\cos\left(\frac{\pi}{4}+\theta\right)\right]\delta x_1 + \left[\sqrt{2}\,mg - F\cos\left(\frac{\pi}{4}-\theta\right)\right]\delta x_2 = 0 \tag{1.277}$$

Since $\delta x_1$ and $\delta x_2$ are independent virtual displacements, their coefficients must each be zero. Thus we obtain

$$F = \frac{mg}{\sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right)} = \frac{\sqrt{2}\,mg}{\cos\left(\frac{\pi}{4}-\theta\right)} \tag{1.278}$$

Hence, at equilibrium,

$$\cos\left(\frac{\pi}{4}-\theta\right) = 2\cos\left(\frac{\pi}{4}+\theta\right) \tag{1.279}$$

or

$$\cos\theta + \sin\theta = 2(\cos\theta - \sin\theta) \tag{1.280}$$

Thus,

$$\tan\theta = \frac{1}{3} \quad \text{or} \quad \theta = 18.43^\circ \tag{1.281}$$

Then, from (1.278), the tensile force $F$ is

$$F = \frac{2mg}{\cos\theta + \sin\theta} = \frac{\sqrt{10}mg}{2} = 1.5811mg \tag{1.282}$$

Notice that, at the equilibrium configuration, the angle $\theta$ and force $F$ do not depend on the spring stiffness $k$. So let us consider the case in which $k$ is infinite; that is, replace the spring by a massless rigid rod of length $L$. Then $\delta x_1$ and $\delta x_2$ are not longer independent. The force $F$ becomes a constraint force which does not enter into the calculation of $\theta$.

The principle of virtual work gives the equilibrium condition

$$\delta V = -\frac{mg}{\sqrt{2}}(\delta x_1 + 2\delta x_2) = 0 \tag{1.283}$$

The constraint relation between $\delta x_1$ and $\delta x_2$ can be found by noting that the length of the rod is unchanged during a virtual displacement.

$$\delta x_1 \cos\left(\frac{\pi}{4} + \theta\right) + \delta x_2 \cos\left(\frac{\pi}{4} - \theta\right) = 0$$

or

$$(\cos\theta - \sin\theta)\,\delta x_1 + (\cos\theta + \sin\theta)\,\delta x_2 = 0 \tag{1.284}$$

We need to express an arbitrary virtual displacement in terms of $\delta\theta$ which is unconstrained. To accomplish this, consider a small rotation of the system about its instantaneous center. This results in

$$\delta\theta = \frac{\sqrt{2}\,\delta x_2}{L(\cos\theta - \sin\theta)} = \frac{-\sqrt{2}\,\delta x_1}{L(\cos\theta + \sin\theta)} \tag{1.285}$$

Then the equilibrium condition of (1.283) can be written in the form

$$\delta V = \frac{mgL}{2}(3\sin\theta - \cos\theta)\,\delta\theta = 0 \tag{1.286}$$

since $\delta\theta \neq 0$, in general, we conclude that

$$\tan\theta = \frac{1}{3} \tag{1.287}$$

as we obtained previously. The expression for $\delta V$ can be integrated to yield the potential energy.

$$V = -\frac{mgL}{2}(\sin\theta + 3\cos\theta) \tag{1.288}$$

Then we find that

$$\frac{\partial^2 V}{\partial\theta^2} = \frac{mgL}{2}(\sin\theta + 3\cos\theta) = \sqrt{2.5}mgL \tag{1.289}$$

Since this result is positive, the equilibrium is stable for the system consisting of two particles connected by a rigid rod.

The Maggi equations lead to a reduced set of $(n-m)$ second-order dynamical equations compared to the Lagrangian approach, but quasi-velocities do not appear in these equations. In addition, $m$ differentiated constraint equations are needed, making a total of $n$ second-order differential equations to be solved for the $n$ $q$s as functions of time.

On the other hand, the Boltzmann–Hamel equation produces a minimum set of $(n-m)$ first-order dynamical equations which are written in terms of quasi-velocities. Thus, the final equations have the ideal form. The procedure suffers, however, because the kinetic energy must be written for the unconstrained system having $n$ degrees of freedom rather than the constrained system with $(n-m)$ degrees of freedom, and in addition, the basic equation with its multiple summations is complicated.

In the remaining portion of this chapter, we shall introduce several additional methods of obtaining the dynamical equations of motion, and will look into their computational efficiency, as illustrated by example problems.

## D'Alembert's principle

Let us begin with the Lagrangian form of d'Alembert's principle for a system of $N$ particles, as given by (2.5).

$$\sum_{i=1}^{N}(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i = 0 \tag{4.124}$$

Here $\mathbf{r}_i$ is the position vector of the $i$th particle and $\mathbf{F}_i$ is the applied force acting on that particle. The virtual displacement $\delta\mathbf{r}_i$ is consistent with the $m$ instantaneous constraints.

Let us assume that the particle velocities are given in terms of $(n-m)$ independent quasi-velocities in accordance with

$$\mathbf{v}_i = \sum_{j=1}^{n-m}\gamma_{ij}(q,t)u_j + \gamma_{it}(q,t) \qquad (i = 1,\ldots,N) \tag{4.125}$$

where the $\gamma$s are *velocity coefficients*. The virtual displacement $\delta\mathbf{r}_i$ is

$$\delta\mathbf{r}_i = \sum_{j=1}^{n-m}\gamma_{ij}(q,t)\delta\theta_j \tag{4.126}$$

where $\theta_j$ is a quasi-coordinate and $u_j = \dot{\theta}_j$. Then (4.124) becomes

$$\sum_{j=1}^{n-m}\sum_{i=1}^{N}(\mathbf{F}_i - m_i\dot{\mathbf{v}}_i)\cdot\gamma_{ij}\delta\theta_j = 0 \tag{4.127}$$

The virtual work of the applied forces is

$$\delta W = \sum_{i=1}^{N}\mathbf{F}_i\cdot\delta\mathbf{r}_i = \sum_{j=1}^{n-m}Q_j\delta\theta_j \tag{4.128}$$

so, using (4.126), we find that the generalized applied force corresponding to $u_j$ or $\delta\theta_j$ is

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \cdot \boldsymbol{\gamma}_{ij} \qquad (j = 1, \ldots, n - m) \tag{4.129}$$

The corresponding generalized inertia force is

$$Q_j^* = -\sum_{i=1}^{N} m_i \dot{\mathbf{v}}_i \cdot \boldsymbol{\gamma}_{ij} \qquad (j = 1, \ldots, n - m) \tag{4.130}$$

Then (4.127) can be written in the form

$$\sum_{j=1}^{n-m} (Q_j + Q_j^*)\delta\theta_j = 0 \tag{4.131}$$

Since the $\delta\theta$s are independent for $j = 1, \ldots, n - m$, we obtain

$$Q_j + Q_j^* = 0 \qquad (j = 1, \ldots, n - m) \tag{4.132}$$

These $(n - m)$ equations, written in terms of $u$s and $\dot{u}$s, are sometimes known as *Kane's equations*.

For our purposes, we can write

$$\sum_{i=1}^{N} m_i \dot{\mathbf{v}}_i \cdot \boldsymbol{\gamma}_{ij} = Q_j \qquad (j = 1, \ldots, n - m) \tag{4.133}$$

We shall call this the *general dynamical equation for a system of particles*. As we have seen, it derives directly from d'Alembert's principle. It consists of a minimum set of $(n - m)$ first-order differential equations in the $u$s, since $\dot{\mathbf{v}}_i$, in general, will be a function of $(q, u, \dot{u}, t)$ and is linear in the $\dot{u}$s. In addition, there are $n$ first-order kinematical equations of the form

$$\dot{q}_i = \sum_{j=1}^{n-m} \Phi_{ij}(q, t)u_j + \Phi_{it}(q, t) \qquad (i = 1, \ldots, n) \tag{4.134}$$

Thus, there are a total of $(2n - m)$ first-order equations to solve for the $n$ $q$s and $(n - m)$ $u$s as functions of time. Notice that the constraint equations do not enter explicitly, but rather implicitly through the choice of independent $u$s. Furthermore, one does not need to solve for the constraint forces.

## Rigid body equations

Equation (4.133) can be generalized for the case of a system of $N$ rigid bodies (Fig. 4.3). Suppose that the $i$th rigid body has a reference point $P_i$, fixed in the body, a mass $m_i$, and an inertia dyadic $\mathbf{I}_i$ about $P_i$. The applied forces acting on the $i$th body are equivalent to a force $\mathbf{F}_i$ acting at $P_i$, plus a couple of moment $\mathbf{M}_i$. In terms of quasi-velocities, we can write the velocity of the reference point $P_i$ as

$$\mathbf{v}_i = \sum_{j=1}^{n-m} \boldsymbol{\gamma}_{ij}(q, t)u_j + \boldsymbol{\gamma}_{it}(q, t) \tag{4.135}$$

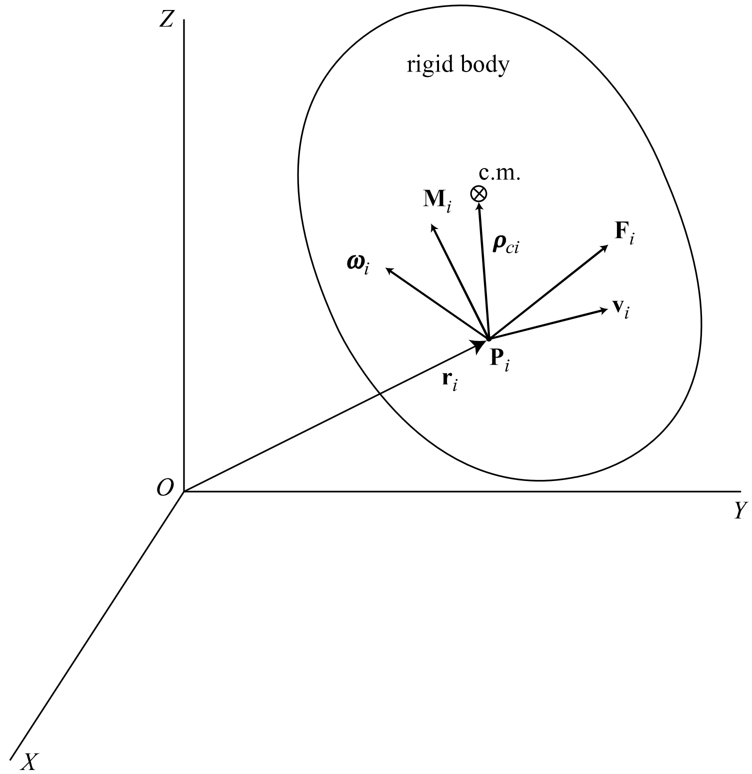


**Figure 4.3.**

The angular velocity of the $i$th body is

$$\boldsymbol{\omega}_i = \sum_{j=1}^{n-m} \boldsymbol{\beta}_{ij}(q,t)u_j + \boldsymbol{\beta}_{it}(q,t) \tag{4.136}$$

where the $\boldsymbol{\beta}$s are *angular velocity coefficients*. The generalized force associated with $u_j$ is

$$Q_j = \sum_{i=1}^{N} (\mathbf{F}_i \cdot \boldsymbol{\gamma}_{ij} + \mathbf{M}_i \cdot \boldsymbol{\beta}_{ij}) \qquad (j = 1, \dots, n-m) \tag{4.137}$$

Note that constraint forces do not enter into $Q_j$ if the $(n-m)$ $u$s are independent.

The differential equations of motion are obtained from (4.131) where, for this system of rigid bodies, the generalized inertia force is

$$Q_j^* = \sum_{i=1}^{N} (\mathbf{F}_i^* \cdot \boldsymbol{\gamma}_{ij} + \mathbf{M}_i^* \cdot \boldsymbol{\beta}_{ij}) \qquad (j = 1, \dots, n-m) \tag{4.138}$$

The inertia force for the $i$th body is equal to the negative of the mass times the acceleration of the center of mass, that is,

$$\mathbf{F}_i^* = -m_i(\dot{\mathbf{v}}_i + \ddot{\boldsymbol{\rho}}_{ci}) \qquad (i = 1, \dots, N) \tag{4.139}$$