Virtual displacements

Suppose that the vector $\mathbf{r}_i(q,t)$ gives the location of some point in a mechanical system; for example, it might be the position vector of the ith particle written in terms of the n qs and time. Now consider an *actual* differential displacement

$$d\mathbf{r}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_j}\, dq_j + \frac{\partial \mathbf{r}_i}{\partial t} dt \tag{1.224}$$

which occurs during an infinitesimal time interval dt. If there are m holonomic constraint equations, the dqs must satisfy

$$d\phi_j = \sum_{i=1}^{n} \frac{\partial \phi_j}{\partial q_i}\, dq_i + \frac{\partial \phi_j}{\partial t}\, dt = 0 \qquad (j = 1,\ldots,m) \tag{1.225}$$

On the other hand, if there are m nonholonomic constraints, the dqs satisfy

$$\sum_{i=1}^{n} a_{ji}(q,t)\, dq_i + a_{jt}(q,t)\, dt = 0 \qquad (j = 1,\ldots,m) \tag{1.226}$$

as given previously in (1.204).

Now let us hold time fixed by setting $dt = 0$ and imagine a *virtual displacement* $\delta\mathbf{r}_i$ in ordinary three-dimensional space for each of N particles.

$$\delta\mathbf{r}_i = \sum_{j=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_i}\, \delta q_i \qquad (i = 1,\ldots,N) \tag{1.227}$$

The virtual displacement of the system can also be described by the n-dimensional vector $\delta\mathbf{q}$ in configuration space. If there are constraints acting on the system, the δqs must satisfy *instantaneous* or *virtual constraint equations* of the form

$$\sum_{i=1}^{n} \frac{\partial \phi_j}{\partial q_i}\, \delta q_i = 0 \qquad (j = 1,\ldots,m) \tag{1.228}$$

for the holonomic case, or

$$\sum_{i=1}^{n} a_{ji}\, \delta q_i = 0 \qquad (j = 1,\ldots,m) \tag{1.229}$$

for nonholonomic constraints.

A comparison of (1.228) and (1.229) with (1.225) and (1.226) shows that virtual displacements and actual displacements are different, in general, since they satisfy different constraint equations. If the constraints are *catastatic*, however; that is, if any holonomic constraints are of the form $\phi_j(q) = 0$, and if all nonholonomic constraints have $a_{jt} = 0$, then the virtual and actual small displacements satisfy the same set of constraint equations. Of course, the actual motion also satisfies the equations of motion.

The forms of (1.228) and (1.229), which are linear in the δqs, indicate that the virtual displacements lie in an $(n - m)$-dimensional hyperplane at the operating point in n-dimensional configuration space, in accordance with the constraints. For holonomic constraints, the plane is tangent to the constraint surface at the operating point.

Virtual work

The concept of virtual work is fundamental to a proper understanding of dynamical theory. First, it must be emphasized that there is a distinction between work and virtual work. The work done by a force $\mathbf{F}_i$ acting on the ith particle as it moves between points A_i and B_i in an inertial frame is equal to the line integral

$$W_i = \int_{A_i}^{B_i} \mathbf{F}_i \cdot d\mathbf{r}_i \tag{1.230}$$

where $\mathbf{r}_i$ is the position vector of the ith particle. For a system of N particles, the work done in an arbitrary small displacement of the system is

$$dW = \sum_{i=1}^{N} \mathbf{F}_i \cdot d\mathbf{r}_i = \sum_{k=1}^{3N} F_k dx_k \tag{1.231}$$

where $(x_1, \ldots, x_{3N})$ are the Cartesian coordinates of the N particles and the F_k are the corresponding force components applied to the particles.

Now let us transform to generalized coordinates using (1.198) and (1.224). The work done on the system during a small displacement in the time interval dt is

$$dW = \sum_{i=1}^{N} \sum_{j=1}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} dq_j + \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial t} dt \tag{1.232}$$

or, in terms of Cartesian coordinates,

$$dW = \sum_{k=1}^{3N} \sum_{j=1}^{n} F_k \frac{\partial x_k}{\partial q_j} dq_j + \sum_{k=1}^{3N} F_k \frac{\partial x_k}{\partial t} dt \tag{1.233}$$

At this point it is convenient to introduce the *velocity coefficients* γ_{ij} and γ_{it} defined by

$$\gamma_{ij} = \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}, \qquad \gamma_{it} = \frac{\partial \mathbf{r}_i}{\partial t} \tag{1.234}$$

Note that these coefficients are vector quantities. Now we can write (1.224) in the form

$$d\mathbf{r}_i = \sum_{j=1}^{n} \gamma_{ij}\, dq_i + \gamma_{it}\, dt \tag{1.235}$$

The corresponding velocity is

$$\mathbf{v}_i = \sum_{j=1}^{n} \gamma_{ij}\, \dot{q}_j + \gamma_{it} \tag{1.236}$$

We see that γ_{ij} represents the sensitivity of the velocity $\mathbf{v}_i$ to changes in $\dot{q}_i$, whereas γ_{it} is equal to the velocity $\mathbf{v}_i$ when all qs are held constant.

Returning now to (1.232), we can write

$$dW = \sum_{i=1}^{N}\sum_{j=1}^{n} \mathbf{F}_i \cdot \boldsymbol{\gamma}_{ij}\, dq_j + \sum_{i=1}^{N} \mathbf{F}_i \cdot \boldsymbol{\gamma}_{it}\, dt \tag{1.237}$$

The *virtual work* δW due to the forces $\mathbf{F}_i$ acting on the system is obtained by setting $dt = 0$ and replacing the actual displacements $d\mathbf{r}_i$ by virtual displacements $\delta\mathbf{r}_i$. Thus we obtain the alternate forms

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_{k=1}^{3N} F_k \delta x_k \tag{1.238}$$

or

$$\delta W = \sum_{i=1}^{N}\sum_{j=1}^{n} \mathbf{F}_i \cdot \boldsymbol{\gamma}_{ij}\, \delta q_j = \sum_{k=1}^{3N}\sum_{j=1}^{n} F_k \frac{\partial x_k}{\partial q_j}\, \delta q_j \tag{1.239}$$

Let us define the *generalized force* Q_j associated with q_j by

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \cdot \boldsymbol{\gamma}_{ij} = \sum_{k=1}^{3N} F_k \frac{\partial x_k}{\partial q_j} \tag{1.240}$$

Then the virtual work can be written in the form

$$\delta W = \sum_{j=1}^{n} Q_j\, \delta q_j \tag{1.241}$$

In general, we assume that the virtual displacements are consistent with any constraints, that is, they satisfy (1.228) or (1.229). But, if the system is holonomic, it is particularly convenient to choose independent δqs.

The question arises concerning why the virtual work δW receives so much attention in dynamical theory rather than the work dW of the actual motion. The reason lies in the nature of constraint forces. An *ideal constraint* is a *workless constraint* which may be either scleronomic or rheonomic. By workless we mean that no work is done by the constraint forces in an arbitrary reversible virtual displacement that satisfies the virtual constraint equations having the form of (1.228) or (1.229). Examples of ideal constraints include frictionless constraint surfaces, or rolling contact without slipping, or a rigid massless rod connecting two particles. Another example is a knife-edge constraint that allows motion in the direction of the knife edge without friction, but does not allow motion perpendicular to the knife edge. Ideal constraint forces, such as the internal forces in a rigid body, may do work on individual particles due to a virtual displacement, but no work is done on the system as a whole because these forces occur in equal, opposite and collinear pairs.

It is convenient to consider the total force acting on the ith particle to be the sum of the *applied force* $\mathbf{F}_i$ and the *constraint force* $\mathbf{R}_i$, by which we mean an ideal constraint force. Thus, all forces that are not ideal constraint forces are classed as applied forces. Frequently the applied forces are known, but the constraint forces either are unknown or are difficult to calculate.

The advantage of using virtual displacements rather than actual displacements in dynamical analyses can be seen by considering the virtual work of all the forces acting on a system of particles. We find that

$$\delta W = \sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{R}_i) \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i \tag{1.242}$$

since

$$\sum_{i=1}^{N} \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0 \tag{1.243}$$

Thus, ideal constraint forces can be ignored in calculating the virtual work of all the forces acting on a system. On the other hand,

$$\sum_{i=1}^{N} \mathbf{R}_i \cdot d\mathbf{r}_i \neq 0 \tag{1.244}$$

in the general case, indicating that constraint forces can contribute to the work dW resulting from a small actual displacement. To summarize, one can ignore the constraint forces in applying virtual work methods. This advantage will carry over to equations derived using virtual work, an example being Lagrange's equation.

Example 1.10 In this example we will show how a generalized force can be calculated using virtual work. In general, the generalized force Q_i is equal to the virtual work per unit δq_i, assuming that the other δqs are set equal to zero, that is, assuming independent δqs. This is in accordance with (1.241).

Consider a system (Fig. 1.20) consisting of two particles connected by a rigid rod of length L. Let (x, y) be the position of particle 1, and let θ be the angle of the rod relative

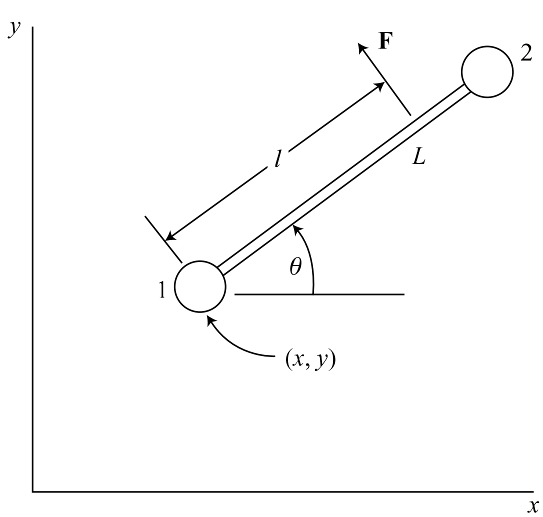

Figure 1.20.

to the x-axis. A force **F**, perpendicular to the rod, is applied at a distance l from particle 1. We wish to solve for the generalized forces Q_x, Q_y, and Q_θ.

First, we see that

$$\mathbf{F} = -F\sin\theta\,\mathbf{i} + F\cos\theta\,\mathbf{j} \tag{1.245}$$

where **i** and **j** are Cartesian unit vectors. The virtual displacement $\delta\mathbf{r}$ at the point of application of **F** is

$$\delta\mathbf{r} = (\delta x - l\ \sin\theta\ \delta\theta)\mathbf{i}\ + (\delta y + l\cos\theta\ \delta\theta)\mathbf{j} \tag{1.246}$$

Thus, the virtual work is

$$\delta W = \mathbf{F}\cdot\delta\mathbf{r} = -F\sin\theta\ \delta x + F\cos\theta\ \delta y + Fl\ \delta\theta \tag{1.247}$$

From (1.241) we have

$$\delta W = Q_x\ \delta x + Q_y\ \delta y + Q_\theta\ \delta\theta \tag{1.248}$$

Then, by comparing coefficients of the δqs, we find that the generalized forces are

$$Q_x = -F\sin\theta, \qquad Q_y = F\cos\theta, \qquad Q_\theta = Fl \tag{1.249}$$

Note that Q_x and Q_y are the x and y components of **F**, whereas Q_θ is the moment about particle 1.

Principle of virtual work

A system of particles is in static equilibrium if each particle of the system is in static equilibrium. A particle is in static equilibrium if it is motionless at the initial time $t = 0$, and if its acceleration remains zero for all $t \geq 0$.

Now consider a *catastatic system* of particles; that is, all transformation equations from inertial xs to qs do not contain time explicitly. This implies that all particles are at rest if all $\dot{q}$s equal zero. For such a system we can state the *principle of virtual work: The necessary and sufficient condition for the static equilibrium of an initially motionless catastatic system which is subject to ideal bilateral constraints is that zero virtual work is done by the applied forces in moving through an arbitrary virtual displacement satisfying the constraints.*

To explain this principle, first assume that the catastatic system is in static equilibrium. Then

$$\mathbf{F}_i + \mathbf{R}_i = 0 \qquad (i = 1, \ldots, N) \tag{1.250}$$

implying that each particle has zero acceleration. Now take the dot product with $\delta\mathbf{r}_i$ and sum over i. We obtain

$$\sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{R}_i)\cdot\delta\mathbf{r}_i = 0 \tag{1.251}$$