2 Lagrange's and Hamilton's equations

In our study of the dynamics of a system of particles, we have been concerned primarily with the Newtonian approach which is *vectorial* in nature. In general, we need to know the magnitudes and directions of the forces acting on the system, including the forces of constraint. Frequently the constraint forces are not known directly and must be included as additional unknown variables in the equations of motion. Furthermore, the calculation of particle accelerations can present kinematical difficulties.

An alternate approach is that of *analytical dynamics*, as represented by Lagrange's equations and Hamilton's equations. These methods enable one to obtain a complete set of equations of motion by differentiations of a single scalar function, namely the Lagrangian function or the Hamiltonian function. These functions include kinetic and potential energies, but ideal constraint forces are not involved. Thus, orderly procedures for obtaining the equations of motion are available and are applicable to a wide range of problems.

2.1 D'Alembert's principle and Lagrange's equations

D'Alembert's principle

Let us begin with Newton's law of motion applied to a system of N particles. For the ith particle of mass m_i and inertial position $\mathbf{r}_i$, we have

$$\mathbf{F}_i + \mathbf{R}_i - m_i\ddot{\mathbf{r}}_i = 0 \tag{2.1}$$

where $\mathbf{F}_i$ is the applied force and $\mathbf{R}_i$ is the constraint force. Now take the scalar product with a virtual displacement $\delta\mathbf{r}_i$ and sum over i. We obtain

$$\sum_{i=1}^{N}(\mathbf{F}_i + \mathbf{R}_i - m_i\ddot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i = 0 \tag{2.2}$$

This result is valid for arbitrary $\delta\mathbf{r}$s; but now assume that the $\delta\mathbf{r}$s satisfy the *instantaneous or virtual constraint equations*, namely,

$$\sum_{i=1}^{3N} a_{ji}(x,t)\delta x_i = 0 \qquad (j = 1,\ldots,m) \tag{2.3}$$

where the δxs are the Cartesian components of the $\delta \mathbf{r}$s. The virtual work of the constraint forces must vanish, that is,

$$\sum_{i=1}^{N} \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0 \tag{2.4}$$

Then (2.2) reduces to

$$\sum_{i=1}^{N} (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0 \tag{2.5}$$

This important result is the *Lagrangian form of d'Alembert's principle*. It states that the virtual work of the applied forces plus the inertia forces is zero for all virtual displacements satisfying the instantaneous constraints, that is, with time held fixed. The forces due to ideal constraints do not enter into these equations. This important characteristic will be reflected in various dynamical equations derived using d'Alembert's principle, including Lagrange's and Hamilton's equations.

In terms of Cartesian coordinates, d'Alembert's principle has the form

$$\sum_{k=1}^{3N} (F_k - m_k \ddot{x}_k) \delta x_k = 0 \tag{2.6}$$

where the δxs satisfy (2.3). Equation (2.6) is valid for any set of δxs which satisfy the instantaneous constraints. There are $(3N - m)$ independent sets of δxs, each being conveniently expressed in terms of δx_k ratios. This results in $(3N - m)$ equations of motion. An additional m equations are obtained by differentiating the constraint equations of (1.203) or, in this case,

$$\sum_{i=1}^{3N} a_{ji} \dot{x}_i + a_{jt} = 0 \qquad (j = 1, \ldots, m) \tag{2.7}$$

with respect to time. Altogether, there are now $3N$ second-order differential equations which can be solved, frequently by numerical integration, for the xs as functions of time. Note that the constraint forces have been eliminated from the equations of motion.

Example 2.1 A particle of mass m can move without friction on the inside surface of a paraboloid of revolution (Fig. 2.1)

$$\phi = x^2 + y^2 - z = 0 \tag{2.8}$$

under the action of a uniform gravitational field in the negative z direction. We wish to find the differential equations of motion using d'Alembert's principle. The applied force components are

$$F_x = 0, \qquad F_y = 0, \qquad F_z = -mg \tag{2.9}$$

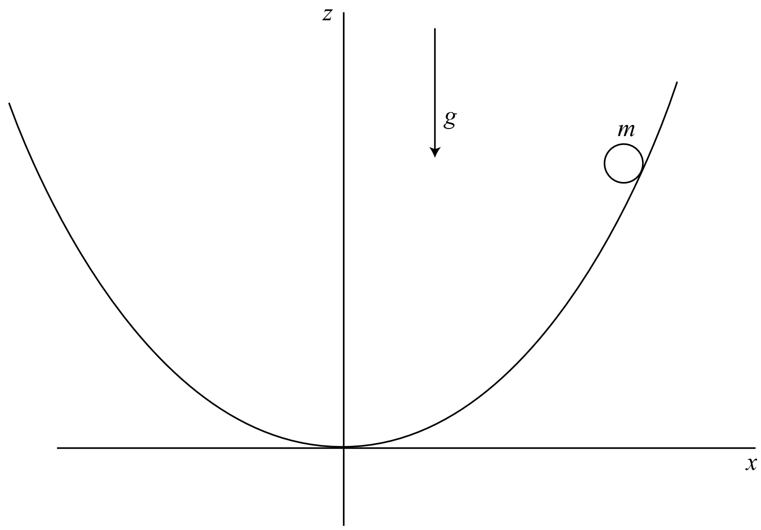

Figure 2.1.

In order to obtain the constraint equation for the virtual displacements, first write (2.8) in the differential form

$$\dot{\phi} = 2x\dot{x} + 2y\dot{y} - \dot{z} = 0 \tag{2.10}$$

The instantaneous constraint equation, obtained using (2.3), is

$$2x\delta x + 2y\delta y - \delta z = 0 \tag{2.11}$$

We need to find $3N - m = 2$ independent virtual displacements which satisfy (2.11). We can take

$$\delta y = 0, \qquad \delta z = 2x\delta x \tag{2.12}$$

and

$$\delta x = 0, \qquad \delta z = 2y\delta y \tag{2.13}$$

Then d'Alembert's principle, (2.6), results in

$$[-m\ddot{x} - 2mx(\ddot{z} + g)]\,\delta x = 0 \tag{2.14}$$

$$[-m\ddot{y} - 2my(\ddot{z} + g)]\,\delta y = 0 \tag{2.15}$$

Now δx is freely variable in (2.14), as is δy in (2.15), so their coefficients must equal zero. Thus, after dividing by $-m$, we obtain

$$\ddot{x} + 2x(\ddot{z} + g) = 0 \tag{2.16}$$

$$\ddot{y} + 2y(\ddot{z} + g) = 0 \tag{2.17}$$

These are the dynamical equations of motion. A kinematical second-order equation is obtained by differentiating (2.10) with respect to time, resulting in

$$2x\ddot{x} + 2y\ddot{y} - \ddot{z} + 2\dot{x}^2 + 2\dot{y}^2 = 0 \tag{2.18}$$

Equations (2.16)–(2.18) can be solved for $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$, which are then integrated to give the particle motion as a function of time.

Lagrange's equations

Consider a system of N particles. Let us begin the derivation with d'Alembert's principle expressed in terms of Cartesian coordinates, that is,

$$\sum_{k=1}^{3N} (F_k - m_k \ddot{x}_k)\,\delta x_k = 0 \tag{2.19}$$

The F_ks are applied force components and the δxs are consistent with the instantaneous constraints, as given in (2.3). We wish to transform to generalized coordinates. Recall that

$$\delta x_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\delta q_i \qquad (k = 1, \ldots, 3N) \tag{2.20}$$

Hence, we obtain

$$\sum_{i=1}^{n}\sum_{k=1}^{3N}\left(F_k \frac{\partial x_k}{\partial q_i} - m_k \ddot{x}_k \frac{\partial x_k}{\partial q_i}\right)\delta q_i = 0 \tag{2.21}$$

where the δqs conform to any constraints. But since $x_k = x_k(q, t)$,

$$\dot{x}_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{\partial t} \tag{2.22}$$

and we see that

$$\frac{\partial \dot{x}_k}{\partial \dot{q}_i} = \frac{\partial x_k}{\partial q_i} \tag{2.23}$$

Furthermore,

$$\frac{d}{dt}\left(\frac{\partial x_k}{\partial q_i}\right) = \sum_{j=1}^{n} \frac{\partial^2 x_k}{\partial q_j\,\partial q_i}\dot{q}_j + \frac{\partial^2 x_k}{\partial t\,\partial q_i} = \frac{\partial \dot{x}_k}{\partial q_i} \tag{2.24}$$

The kinetic energy of the system is

$$T = \frac{1}{2}\sum_{k=1}^{3N} m_k \dot{x}_k^2 \tag{2.25}$$

and the generalized momentum p_i is

$$p_i = \frac{\partial T}{\partial \dot{q}_i} = \sum_{k=1}^{3N} m_k \dot{x}_k \frac{\partial \dot{x}_k}{\partial \dot{q}_i} = \sum_{k=1}^{3N} m_k \dot{x}_k \frac{\partial x_k}{\partial q_i} \tag{2.26}$$

Therefore, we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{k=1}^{3N} m_k \ddot{x}_k \frac{\partial x_k}{\partial q_i} + \sum_{k=1}^{3N} m_k \dot{x}_k \frac{\partial \dot{x}_k}{\partial q_i} \tag{2.27}$$

However, from (2.25),

$$\frac{\partial T}{\partial q_i} = \sum_{k=1}^{3N} m_k \dot{x}_k \frac{\partial \dot{x}_k}{\partial q_i} \tag{2.28}$$

so we find that

$$\sum_{k=1}^{3N} m_k \ddot{x}_k \frac{\partial x_k}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \tag{2.29}$$

This result is the negative of the generalized inertia force Q_i^*. In other words,

$$Q_i^* = -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i} \tag{2.30}$$

From (1.240), we recall that the generalized applied force is

$$Q_i = \sum_{k=1}^{3N} F_k \frac{\partial x_k}{\partial q_i} \tag{2.31}$$

Then d'Alembert's principle, as written in (2.21) results in

$$\sum_{i=1}^{n} (Q_i + Q_i^*)\delta q_i = 0 \tag{2.32}$$

for all δqs satisfying any constraints. Thus, the virtual work of the applied plus the inertial generalized forces is equal to zero.

Using (2.30) and (2.31) with a change of sign, (2.32) takes the form

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i\right]\delta q_i = 0 \tag{2.33}$$

where, again, the δqs satisfy the instantaneous constraints. This is d'Alembert's principle expressed in terms of generalized coordinates. We will call it *Lagrange's principle*. It applies to both holonomic and nonholonomic systems, requiring only that T and Q_i be written as functions of $(q, \dot{q}, t)$. Note that constraint forces do not enter the Q_is.

Now let us assume a *holonomic system* with *independent* generalized coordinates. Then the coefficient of each δq_i must be equal to zero. We obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \qquad (i = 1, \ldots, n) \tag{2.34}$$

This is the *fundamental holonomic form of Lagrange's equation*. It results in n second-order ordinary differential equations. The Qs are generalized applied forces from any source.

Let us make the further restriction that the Qs are derivable from a potential energy function $V(q, t)$ in accordance with

$$Q_i = -\frac{\partial V}{\partial q_i} \tag{2.35}$$

Now define the *Lagrangian function*

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, t) \tag{2.36}$$

Then we can write Lagrange's equation in the *standard holonomic form*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \dots, n) \tag{2.37}$$

Equivalently, one can use

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \qquad (i = 1, \dots, n) \tag{2.38}$$

If a portion of the applied generalized force is not obtained from a potential function, we can write

$$Q_i = -\frac{\partial V}{\partial q_i} + Q_i' \tag{2.39}$$

where Q_i' is the nonpotential part of Q_i. Then we obtain Lagrange's equation in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i' \qquad (i = 1, \dots, n) \tag{2.40}$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i} + Q_i' \qquad (i = 1, \dots, n) \tag{2.41}$$

The basic forms of Lagrange's equations, given in (2.34) and (2.37) apply to holonomic systems described in terms of *independent* generalized coordinates.

When one considers nonholonomic systems, there must be more generalized coordinates than degrees of freedom. Hence, the generalized coordinates are not completely independent and, as a result, there will be generalized constraint forces C_i which are nonzero, in general. In terms of the constraint forces acting on the individual particles, we have, similar to (1.240),

$$C_i = \sum_{j=1}^{N} \mathbf{R}_j \cdot \gamma_{ji} = \sum_{k=1}^{3N} R_k \frac{\partial x_k}{\partial q_i} \tag{2.42}$$

where the R_k are Cartesian constraint force components.

Let us use the *Lagrange multiplier method* to evaluate the generalized constraint forces. First, we note that the virtual work of the constraint forces must equal zero, that is,

$$\sum_{i=1}^{n} C_i \delta q_i = 0 \tag{2.43}$$

provided that the δqs satisfy the instantaneous constraints in the form

$$\sum_{i=1}^{n} a_{ji} \delta q_i = 0 \qquad (j = 1, \dots, m) \tag{2.44}$$

Now let us multiply (2.44) by a *Lagrange multiplier* λ_j and sum over j. We obtain

$$\sum_{j=1}^{m}\sum_{i=1}^{n}\lambda_j a_{ji}\delta q_i = 0 \tag{2.45}$$

Subtract (2.45) from (2.43) with the result

$$\sum_{i=1}^{n}\left(C_i - \sum_{j=1}^{m}\lambda_j a_{ji}\right)\delta q_i = 0 \tag{2.46}$$

Up to this point, the m λs have been considered to be arbitrary, whereas the n δqs satisfy the instantaneous constraints. However, it is possible to choose the λs such that the coefficient of each δq_i vanishes. Thus, the generalized constraint force C_i is

$$C_i = \sum_{j=1}^{m}\lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.47}$$

and the δqs can be considered to be arbitrary.

To understand how m λs can be chosen to specify an arbitrary constraint force in accordance with (2.47), let us first consider a single constraint. The corresponding constraint force is perpendicular to the constraint surface at the operating point; that is, it is in the direction of the vector $\mathbf{a}_j$ whose components in n-space are the coefficients a_{ji}. This is expressed by (2.44) and we note that any virtual displacement $\delta\mathbf{q}$ must lie in the tangent plane at the operating point. The Lagrange multiplier λ_j applies equally to all components a_{ji} and so expresses the magnitude of the constraint force $\mathbf{C}_j = \lambda_j\mathbf{a}_j$.

If there are m constraints, the total constraint force $\mathbf{C}$ is found by summing the individual constraint forces $\mathbf{C}_j$. We can consider the $\mathbf{a}_j$s as m independent basis vectors with the λs representing the scalar components of $\mathbf{C}$ in this m-dimensional subspace. Hence a set of m λs can always be found to represent any possible total constraint force $\mathbf{C}$.

For a system with n generalized coordinates and m nonholonomic constraints, we note first that the generalized constraint force C_i is no longer zero, in general, and must be added to Q_i to obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + C_i \qquad (i = 1, \ldots, n) \tag{2.48}$$

Then, using (2.47), the result is the *fundamental nonholonomic form of Lagrange's equation*.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=1}^{m}\lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.49}$$

If we again assume that the Qs are obtained from a potential function $V(q, t)$, we can write

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m}\lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.50}$$

This is the *standard nonholonomic form of Lagrange's equation*. Equivalently, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.51}$$

If nonpotential generalized applied forces Q_i' are present, Lagrange's equation has the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i' + \sum_{j=1}^{m} \lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.52}$$

The nonholonomic forms of Lagrange's equations presented here are also applicable to holonomic systems in which the qs are not independent. For example, if there are m holonomic constraints of the form

$$\phi_j(q, t) = 0 \qquad (j = 1, \ldots, m) \tag{2.53}$$

we take

$$a_{ji}(q, t) = \frac{\partial \phi_j}{\partial q_i} \tag{2.54}$$

and then use a nonholonomic form of Lagrange's equation such as (2.49) or (2.51).

Example 2.2 Particles A and B (Fig. 2.2), each of mass m, are connected by a rigid massless rod of length l. Particle A can move without friction on the horizontal x-axis, while particle B can move without friction on the vertical y-axis. We desire the differential equations of motion.

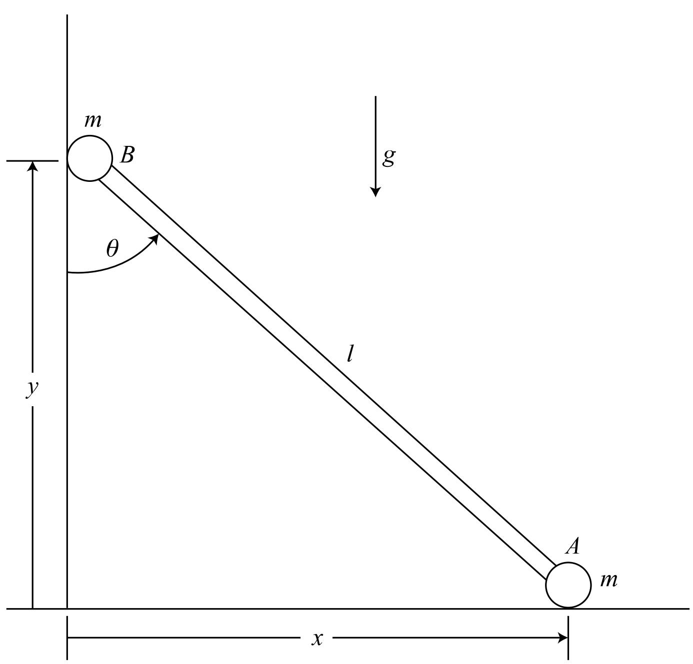

Figure 2.2.

First method Let us use the nonholonomic form of Lagrange's equation given by (2.51). The coordinates (x, y) have the holonomic constraint

$$\phi = x^2 + y^2 - l^2 = 0 \tag{2.55}$$

Differentiating with respect to time, we obtain

$$\dot{\phi} = 2x\dot{x} + 2y\dot{y} = 0 \tag{2.56}$$

which leads to

$$a_{11} = 2x, \qquad a_{12} = 2y \tag{2.57}$$

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{2.58}$$

and the potential energy is

$$V = mgy \tag{2.59}$$

The use of (2.51) results in the equations of motion

$$m\ddot{x} = 2\lambda x \tag{2.60}$$
$$m\ddot{y} = 2\lambda y - mg \tag{2.61}$$

We need a third differential equation since there are three variables (x, y, λ). Differentiating (2.56) with respect to time, and dividing by two, we obtain

$$x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = 0 \tag{2.62}$$

This equation, plus (2.60) and (2.61) are a complete set of second-order differential equations. One can solve for λ and obtain

$$\lambda = \frac{m}{2l^2}[gy - (\dot{x}^2 + \dot{y}^2)] \tag{2.63}$$

Then one can numerically integrate (2.60) and (2.61). We see that the compressive force in the rod is $2\lambda l$.

Second method A simpler approach is to choose θ as a single generalized coordinate with no constraints. We see that

$$x = l\sin\theta \tag{2.64}$$
$$y = l\cos\theta \tag{2.65}$$
$$\dot{x} = l\dot{\theta}\cos\theta \tag{2.66}$$
$$\dot{y} = -l\dot{\theta}\sin\theta \tag{2.67}$$

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\theta}^2 \tag{2.68}$$

and the potential energy is

$$V = mgy = mgl\cos\theta \tag{2.69}$$

Then we can use Lagrange's equation in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \tag{2.70}$$

to obtain the differential equation for θ which is

$$ml^2\ddot{\theta} - mgl\sin\theta = 0 \tag{2.71}$$

Third method Another possibility is to use the holonomic equation of constraint to eliminate one of the variables. For example, we might eliminate y and its derivatives and then consider x to be an independent coordinate. From (2.55), we have

$$y = \pm\sqrt{l^2 - x^2} \tag{2.72}$$

Assuming the positive sign, we obtain

$$\dot{y} = \frac{-x\dot{x}}{\sqrt{l^2 - x^2}} \tag{2.73}$$

and the kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2\left(1 + \frac{x^2}{l^2 - x^2}\right) = \frac{1}{2}m\dot{x}^2\left(\frac{l^2}{l^2 - x^2}\right) \tag{2.74}$$

The potential energy is

$$V = mgy = mg\sqrt{l^2 - x^2} \tag{2.75}$$

Then (2.70) leads to a differential equation for x, namely,

$$ml^2\left[\frac{\ddot{x}}{l^2 - x^2} + \frac{x\dot{x}^2}{(l^2 - x^2)^2}\right] - mg\frac{x}{\sqrt{l^2 - x^2}} = 0 \tag{2.76}$$

If y is actually negative, the sign of the last term is changed.

For a system with m *holonomic* constraints, one can use the constraint equations to solve for m *dependent* qs and $\dot{q}$s in terms of the corresponding $(n - m)$ independent quantities. Then one can write T and V in terms of the independent quantities only, and use simple forms of Lagrange's equation such as (2.70) to obtain $(n - m)$ equations of motion. In this example, we let x be independent and y dependent. After integrating (2.76) numerically to obtain x as a function of time, one can use (2.72) and (2.73) to obtain the motion in y.

It should be noted, however, that this procedure does not produce correct results if *nonholonomic* constraints are involved. Various approaches which do not involve Lagrange multipliers, but are applicable to nonholonomic systems, will be discussed in Chapter 4.

This example illustrates that the choice of generalized coordinates has a strong effect on the complexity of the equations of motion. In this instance, the second method is by far the simplest approach if one is interested in solving for the motion. It is not very helpful, however, in solving for various internal forces. The first method involving Lagrange multipliers is the most direct method to solve for forces.

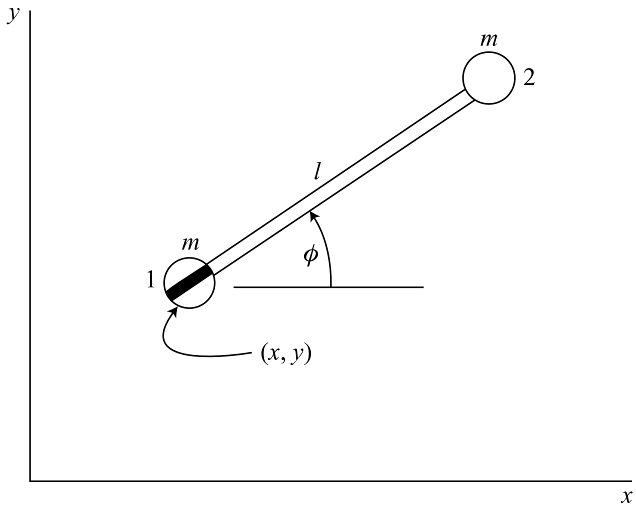

Figure 2.3.

Example 2.3 A dumbbell consists of two particles, each of mass m, connected by a rigid massless rod of length l. There is a knife-edge at particle 1, resulting in the nonholonomic constraint

$$-\dot{x}\sin\phi + \dot{y}\cos\phi = 0 \tag{2.77}$$

which states that the velocity normal to the knife-edge is zero (Fig. 2.3). Assume that the xy-plane is horizontal and therefore the potential energy V is constant. We wish to find the differential equations of motion.

There are no applied forces acting on the system, so we can use Lagrange's equation in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \tag{2.78}$$

In this case the qs are (x,y,ϕ). The kinetic energy for the unconstrained system is obtained by using (1.127), with the result

$$T = m\left(\dot{x}^2 + \dot{y}^2 + \frac{1}{2}l^2\dot{\phi}^2 - l\dot{x}\dot{\phi}\sin\phi + l\dot{y}\dot{\phi}\cos\phi\right) \tag{2.79}$$

From the constraint equation we note that

$$a_{11} = -\sin\phi, \qquad a_{12} = \cos\phi, \qquad a_{13} = 0 \tag{2.80}$$

Now

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m(2\ddot{x} - l\ddot{\phi}\sin\phi - l\dot{\phi}^2\cos\phi) \tag{2.81}$$

and $\partial T/\partial x = 0$. Using (2.78), the x equation is

$$m(2\ddot{x} - l\ddot{\phi}\sin\phi - l\dot{\phi}^2\cos\phi) = -\lambda\sin\phi \tag{2.82}$$

Similarly,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m(2\ddot{y} + l\ddot{\phi}\cos\phi - l\dot{\phi}^2\sin\phi) \tag{2.83}$$

and $\partial T/\partial y = 0$. Hence, the y equation is

$$m\,(2\ddot{y} + l\ddot{\phi}\cos\phi - l\dot{\phi}^2\sin\phi) = \lambda\cos\phi \tag{2.84}$$

Finally,

$$\frac{\partial T}{\partial \dot{\phi}} = m\,(l^2\dot{\phi} - l\dot{x}\sin\phi + l\dot{y}\cos\phi) \tag{2.85}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = m\,(l^2\ddot{\phi} - l\ddot{x}\sin\phi + l\ddot{y}\cos\phi - l\dot{x}\dot{\phi}\cos\phi - l\dot{y}\dot{\phi}\sin\phi) \tag{2.86}$$

$$\frac{\partial T}{\partial \phi} = m\,(-l\dot{x}\dot{\phi}\cos\phi - l\dot{y}\dot{\phi}\sin\phi) \tag{2.87}$$

Thus, the ϕ equation is

$$m\,(l^2\ddot{\phi} - l\ddot{x}\sin\phi + l\ddot{y}\cos\phi) = 0 \tag{2.88}$$

Equations (2.82), (2.84), and (2.88) are the dynamical equations of motion. In addition, we can differentiate the constraint equation with respect to time and obtain

$$-\ddot{x}\sin\phi + \ddot{y}\cos\phi - \dot{x}\dot{\phi}\cos\phi - \dot{y}\dot{\phi}\sin\phi = 0 \tag{2.89}$$

These four equations are linear in $(\ddot{x},\ddot{y},\ddot{\phi},\lambda)$ and can be solved for these variables, which are then integrated to yield $x(t)$, $y(t)$, $\phi(t)$, and $\lambda(t)$.

In general, for a system with n generalized coordinates and m nonholonomic constraint equations, the Lagrangian method results in n second-order differential equations of motion plus m equations of constraint. These $(n + m)$ equations are solved for the n qs and m λs as functions of time. In later chapters, we will discuss methods which are more efficient in the sense of requiring fewer equations to describe a nonholonomic system.

2.2 Hamilton's equations

Canonical equations

The generalized momentum p_i, as given by (1.291), can be made more general by defining

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{2.90}$$

where $L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, \dot{q}, t)$, and $V(q, \dot{q}, t)$ is a *velocity-dependent* potential energy function. This might be used, for example, to account for electromagnetic forces acting on moving charged particles. In our development, however, we will continue to assume that the potential energy has the form $V(q, t)$, and therefore that

$$p_i = \sum_{j=1}^{n} m_{ij}(q, t)\dot{q}_j + a_i(q, t) \qquad (i = 1, \ldots, n) \tag{2.91}$$

in agreement with (1.292). We assume that the inertia matrix $[m_{ij}]$ is positive definite and hence has an inverse. Thus, (2.91) can be solved for the $\dot{q}$s with the result

$$\dot{q}_j = \sum_{i=1}^{n} b_{ji}(q, t)(p_i - a_i) \qquad (j = 1, \ldots, n) \tag{2.92}$$

where the matrix $[b_{ji}] = [m_{ij}]^{-1}$.

The *Hamiltonian function* is defined by

$$H(q, p, t) = \sum_{i=1}^{n} p_i\dot{q}_i - L(q, \dot{q}, t) \tag{2.93}$$

The $\dot{q}$s on the right-hand side of (2.93) are expressed in terms of ps by using (2.92). Consider an arbitrary variation of the Hamiltonian function $H(q, p, t)$.

$$\delta H = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial t}\delta t \tag{2.94}$$

From (2.93), we have

$$\begin{aligned} \delta H &= \sum_{i=1}^{n} p_i\delta\dot{q}_i + \sum_{i=1}^{n} \dot{q}_i\delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i - \frac{\partial L}{\partial t}\delta t \\ &= \sum_{i=1}^{n} \dot{q}_i\delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i - \frac{\partial L}{\partial t}\delta t \end{aligned} \tag{2.95}$$

where (2.90) has been used.

Now assume that δqs, δps, and δt are independently variable; that is, there are no kinematic constraints. Then, equating corresponding coefficients in (2.94) and (2.95), we obtain

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad (i = 1, \ldots, n) \tag{2.96}$$

$$\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, \ldots, n) \tag{2.97}$$

and

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \tag{2.98}$$

Next, introduce the standard holonomic form of Lagrange's equation, (2.37), which can be written as

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \qquad (i = 1, \ldots, n) \tag{2.99}$$

Then we can write

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, \ldots, n) \tag{2.100}$$

These $2n$ first-order equations are known as *Hamilton's canonical equations*. The first n equations express the $\dot{q}$s as linear functions of the ps, as in (2.92). The final n equations contain the laws of motion for the system. Because of the symmetry in form of Hamilton's canonical equations, there is a tendency to accord the qs and ps equal status and think of the qs and ps together as a $2n$-dimensional *phase vector*. Thus, the motion of a system can be represented by a path or trajectory in $2n$-dimensional *phase space*.

Comparing the standard holonomic form of Lagrange's equations with Hamilton's canonical equations, we find that they are equivalent in that both require independent qs and apply to the same mechanical systems. Hamilton's equations are $2n$ first-order equations rather than the n second-order equations of Lagrange. However, it should be pointed out that most computer representations require the conversion of higher-order equations to a larger number of first-order equations.

Form of the Hamiltonian function

Let us return to a further consideration of the Hamiltonian function $H(q, p, t)$, as given by (2.93). Using (2.91), we see that

$$\begin{aligned} \sum_{i=1}^{n} p_i \dot{q}_i &= \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j + \sum_{i=1}^{n} a_i \dot{q}_i \\ &= 2T_2 + T_1 \end{aligned} \tag{2.101}$$

where we recall the expressions for T_2 and T_1 given in (1.259) and (1.262). Now

$$L = T - V = T_2 + T_1 + T_0 - V \tag{2.102}$$

Hence, using (2.93), we find that

$$H = T_2 - T_0 + V \tag{2.103}$$

For the particular case of a *scleronomic system*, $T_1 = T_0 = 0$ and $T = T_2$, so

$$H = T + V \tag{2.104}$$

that is, the Hamiltonian function is equal to the total energy.

Now let us write T_2 in the form

$$T_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(p_i - a_i)(p_j - a_j)$$
$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}p_i p_j - \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_i p_j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_i a_j \tag{2.105}$$

where we note that the matrix $[b_{ij}]$ is symmetric and the inverse of $[m_{ij}]$. Then, using (2.103), we can group the terms in the Hamiltonian function according to their degree in p. We can write

$$H = H_2 + H_1 + H_0 \tag{2.106}$$

where

$$H_2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}p_i p_j \tag{2.107}$$

$$H_1 = -\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_i p_j \tag{2.108}$$

$$H_0 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_i a_j - T_0 + V \tag{2.109}$$

and we note that b_{ij}, a_i, T_0 and V are all functions of (q, t).

Other Hamiltonian equations

We have seen that Hamilton's canonical equations apply to the same systems as the standard holonomic form of Lagrange's equation, (2.37). Other forms of Lagrange's equations have their Hamiltonian counterparts. For example, if nonpotential generalized forces Q'_i are present in a holonomic system, we have the Hamiltonian equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q'_i \qquad (i = 1, \ldots, n) \tag{2.110}$$

For a *nonholonomic system*, we have, corresponding to (2.50),

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{m} \lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.111}$$

This assumes that all the applied forces arise from a potential energy $V(q, t)$. On the other hand, if there are nonpotential forces in a nonholonomic system, Hamilton's equations have the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q'_i + \sum_{j=1}^{n} \lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.112}$$

The nonholonomic forms of Hamilton's equations are solved in conjunction with constraint equations of the form

$$\sum_{i=1}^{n} a_{ji}\dot{q}_i + a_{jt} = 0 \qquad (j = 1, \ldots, m) \tag{2.113}$$

Thus, we have a total of $(2n + m)$ first-order differential equations to solve for n qs, n ps, and m λs as functions of time.

It is well to notice that, as in the Lagrangian case, the nonholonomic forms of Hamilton's equations also apply to holonomic systems in which the motions of the qs are restricted by holonomic constraints.

Example 2.4 A massless disk of radius r has a particle of mass m embedded at a distance $\frac{1}{2}r$ from the center O (Fig. 2.4). The disk rolls without slipping down a plane inclined at an angle α from the horizontal. We wish to obtain the differential equations of motion.

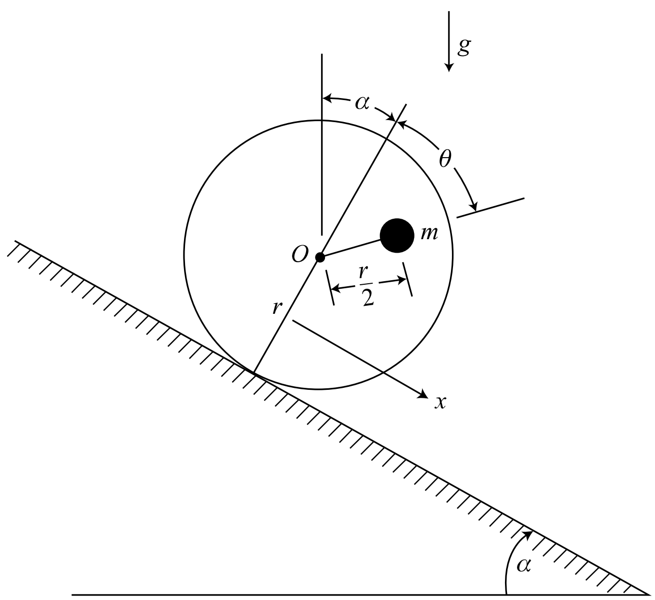

Figure 2.4.

First method Let us employ Hamilton's canonical equations, as given by (2.100), and choose θ as the single generalized coordinate. The particle velocity $\boldsymbol{v}$ is the vector sum of the velocity of the center O and the velocity of the particle relative to O. Thus, we obtain

$$v^2 = \left(r\dot{\theta}\right)^2 + \left(\frac{1}{2}r\dot{\theta}\right)^2 + r^2\dot{\theta}^2\cos\theta \tag{2.114}$$

and the kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\left(\frac{5}{4} + \cos\theta\right)\dot{\theta}^2 \tag{2.115}$$

Now the generalized momentum p_θ is

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = mr^2\left(\frac{5}{4} + \cos\theta\right)\dot{\theta} \tag{2.116}$$

and we obtain

$$\dot{\theta} = \frac{p_\theta}{mr^2\left(\frac{5}{4} + \cos\theta\right)} \tag{2.117}$$

The kinetic energy as a function of (q, p) is

$$T = \frac{p_\theta^2}{2mr^2\left(\frac{5}{4} + \cos\theta\right)} \tag{2.118}$$

The potential energy is

$$V = mg\left[-r\theta\sin\alpha + \frac{1}{2}r\cos(\theta + \alpha)\right] \tag{2.119}$$

The Hamiltonian function is, in general,

$$H = T_2 - T_0 + V \tag{2.120}$$

but, in this case, it is equal to the total energy.

$$H = \frac{p_\theta^2}{2mr^2\left(\frac{5}{4} + \cos\theta\right)} + mg\left[-r\theta\sin\alpha + \frac{1}{2}r\cos(\theta + \alpha)\right] \tag{2.121}$$

The first canonical equation results in

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2\left(\frac{5}{4} + \cos\theta\right)} \tag{2.122}$$

which is a restatement of (2.117). The second canonical equation is

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{-p_\theta^2\sin\theta}{2mr^2\left(\frac{5}{4} + \cos\theta\right)^2} + mgr\left[\sin\alpha + \frac{1}{2}\sin(\theta + \alpha)\right] \tag{2.123}$$

These two canonical equations are together equivalent to the single second-order equation in θ which would be obtained by using Lagrange's equation.

Second method Let us use the same system to illustrate the use of Lagrange multipliers with Hamilton's equations, as in (2.111). We will use (x, θ) as generalized coordinates, where x is the displacement of the center of the disk. There is a holonomic constraint

$$\dot{x} - r\dot{\theta} = 0 \tag{2.124}$$

which expresses the nonslipping condition. The kinetic and potential energies, however, must be written for the *unconstrained* system, that is, assuming the possibility of slipping.

The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \frac{1}{4}r^2\dot{\theta}^2 + r\dot{x}\dot{\theta}\cos\theta\right) \tag{2.125}$$

and the potential energy is

$$V = mg\left[-x\sin\alpha + \frac{1}{2}r\cos(\theta+\alpha)\right] \tag{2.126}$$

The generalized momenta are

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x} + \frac{1}{2}mr\dot{\theta}\cos\theta \tag{2.127}$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{2}mr\dot{x}\cos\theta + \frac{1}{4}mr^2\dot{\theta} \tag{2.128}$$

Equations (2.127) and (2.128) can be solved for $\dot{x}$ and $\dot{\theta}$. We obtain

$$\dot{x} = \frac{rp_x - 2p_\theta\cos\theta}{mr\sin^2\theta} \tag{2.129}$$

$$\dot{\theta} = \frac{4p_\theta - 2rp_x\cos\theta}{mr^2\sin^2\theta} \tag{2.130}$$

Substituting these expressions for $\dot{x}$ and $\dot{\theta}$ into the kinetic energy equation, we obtain, after some algebraic simplification, the Hamiltonian function

$$H = T + V = \frac{1}{2mr^2\sin^2\theta}\left(r^2p_x^2 + 4p_\theta^2 - 4rp_xp_\theta\cos\theta\right) + V \tag{2.131}$$

where V is given by (2.126).

We shall use Hamilton's equations in the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{m}\lambda_j a_{ji} \qquad (i = 1, \dots, n) \tag{2.132}$$

where, from (2.124), the constraint coefficients are

$$a_{11} = 1, \qquad a_{12} = -r \tag{2.133}$$

The $\dot{x}$ equation is

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{rp_x - 2p_\theta\cos\theta}{mr\sin^2\theta} \tag{2.134}$$

The $\dot{p}_x$ equation is

$$\dot{p}_x = -\frac{\partial H}{\partial x} + \lambda = -\frac{\partial V}{\partial x} + \lambda = mg\sin\alpha + \lambda \tag{2.135}$$

The $\dot{\theta}$ equation is

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{4p_\theta - 2rp_x\cos\theta}{mr^2\sin^2\theta} \tag{2.136}$$

Finally, the $\dot{p}_\theta$ equation is

$$\begin{aligned}\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} - r\lambda \\ &= \frac{1}{mr^2 \sin^3\theta}\left[\left(r^2 p_x^2 + 4p_\theta^2\right)\cos\theta - 2rp_x p_\theta(1+\cos^2\theta)\right] \\ &\quad + \frac{1}{2}mgr\sin(\theta+\alpha) - r\lambda \end{aligned} \tag{2.137}$$

These four first-order Hamiltonian equations plus the constraint equation (2.124) can be integrated numerically to solve for the 2 qs, $2p$s, and λ as functions of time.

For a *scleronomic system* such as the one we have been considering, relatively simple matrix equations can be used to obtain the Hamiltonian function. The kinetic energy is quadratic in the $\dot{q}$s and of the form

$$T = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{m}\dot{\mathbf{q}} \tag{2.138}$$

where $\mathbf{m}$ is the $n \times n$ matrix of generalized mass coefficients. The generalized momenta are given by the matrix equation

$$\mathbf{p} = \mathbf{m}\dot{\mathbf{q}} \tag{2.139}$$

and, conversely,

$$\dot{\mathbf{q}} = \mathbf{b}\mathbf{p} \tag{2.140}$$

where $\mathbf{b} = \mathbf{m}^{-1}$. For scleronomic systems, the Hamiltonian function is equal to the total energy, or

$$H = T + V = \frac{1}{2}\mathbf{p}^{\mathrm{T}}\mathbf{b}\mathbf{p} + V \tag{2.141}$$

For the system of this example, we have

$$\mathbf{m} = \begin{bmatrix} m & \frac{1}{2}mr\cos\theta \\ \frac{1}{2}mr\cos\theta & \frac{1}{4}mr^2 \end{bmatrix} \tag{2.142}$$

and

$$\mathbf{b} = \frac{4}{mr^2\sin^2\theta}\begin{bmatrix} \frac{1}{4}r^2 & -\frac{1}{2}r\cos\theta \\ -\frac{1}{2}r\cos\theta & 1 \end{bmatrix} \tag{2.143}$$

in agreement with (2.131) and (2.141).

2.3 Integrals of the motion

We have found that for a dynamical system whose configuration is given by n independent generalized coordinates, the Lagrangian method results in n second-order differential equations of motion with time as the independent variable. Any general analytical solution of these equations of motion contains $2n$ constants of integration which are usually evaluated from the $2n$ initial conditions. One method of expressing the general solution is to find $2n$ independent functions of the form

$$g_j(q, \dot{q}, t) = \alpha_j \qquad (j = 1, \ldots, 2n) \tag{2.144}$$

where the αs are constants. The $2n$ functions are called *integrals or constants of the motion*. Each function g_j maintains a constant value α_j during the actual motion of the system. In principle, these $2n$ equations can be solved for the qs and $\dot{q}$s as functions of the αs and t, that is,

$$q_i = q_i(\alpha, t) \qquad (i = 1, \ldots, n) \tag{2.145}$$

$$\dot{q}_i = \dot{q}_i(\alpha, t) \qquad (i = 1, \ldots, n) \tag{2.146}$$

where these solutions satisfy (2.144).

Usually it is not possible to obtain a full set of $2n$ integrals of the motion by any direct process. Nevertheless, the presence of a few integrals such as those representing conservation of energy or momentum are very useful in characterizing the motion of a system.

If one uses the Hamiltonian approach to the equations of motion, integrals of the motion have the form

$$f_j(q, p, t) = \alpha_j \qquad (j = 1, \ldots, 2n) \tag{2.147}$$

Under the proper conditions, the Hamiltonian function itself can be an integral of the motion.

Conservative system

A common example of an integral of the motion is the *energy integral* $E(q, \dot{q}, t)$ which is quadratic in the $\dot{q}$s and is expressed in units of energy. It satisfies the equation

$$E(q, \dot{q}, t) = h \tag{2.148}$$

where h is a constant that is normally evaluated from initial conditions. For scleronomic systems with $T = T_2$, the energy integral, if it exists, is equal to the sum of the kinetic and potential energies. More generally, however, the energy integral is not equal to the total energy. Usually it is not an explicit function of time.

Let us define a *conservative system* as a dynamical system for which an energy integral can be found. To obtain sufficient conditions for the existence of an energy integral, let us consider a system which is described by the standard nonholonomic form of Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.149}$$

This equation is valid for holonomic or nonholonomic systems whose applied forces are derivable from a potential energy function $V(q, t)$.

Multiply (2.149) by $\dot{q}_i$ and sum over i. The result is

$$\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}\right]\dot{q}_i=\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_j a_{ji}\dot{q}_i \tag{2.150}$$

We note that

$$\sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i=\frac{d}{dt}\left[\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right]-\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \tag{2.151}$$

where

$$\frac{d}{dt}\left[\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right]=2\dot{T}_2+\dot{T}_1 \tag{2.152}$$

Furthermore,

$$\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i=\dot{L}-\sum_{i=1}^{n}\frac{\partial L}{\partial q_i}\dot{q}_i-\frac{\partial L}{\partial t} \tag{2.153}$$

and

$$\dot{L}=\dot{T}_2+\dot{T}_1+\dot{T}_0-\dot{V} \tag{2.154}$$

Hence, from (2.150)–(2.154), we obtain

$$\dot{T}_2-\dot{T}_0+\dot{V}=\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_j a_{ji}\dot{q}_i-\frac{\partial L}{\partial t} \tag{2.155}$$

If the right-hand side of (2.155) remains equal to zero as the motion proceeds, then $E=T_2-T_0+V$ is an energy integral and the system is conservative. The first term on the right-hand side will be zero if the nonholonomic constraint equations of the form of (2.113) have all $a_{jt}=0$; that is; if they are all *catastatic*. The second term on the right-hand side will be zero if neither T nor V is an explicit function of time.

To summarize, a system having holonomic or nonholonomic constraints will be *conservative* if it meets the following conditions:

1. The standard form of Lagrange's equation, as given by (2.149) applies.
2. All constraints can be written in the form

$$\sum_{i=1}^{n}a_{ji}\dot{q}_i=0 \qquad (j=1,\ldots,m) \tag{2.156}$$

   that is, all $a_{jt}=0$.
3. The Lagrangian function $L=T-V$ is not an explicit function of time.

These are *sufficient conditions* for a conservative system. For systems with *independent* qs, Lagrange's equation has the simpler form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{2.157}$$

If this equation applies, and if $\partial L/\partial t = 0$, then the system is *conservative*.

Now let us consider the Hamiltonian approach to conservative systems. Suppose that a system can be described by Hamilton's equations of the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{n} \lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{2.158}$$

The Hamiltonian function will be a constant of the motion if

$$\dot{H} = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\dot{p}_i + \frac{\partial H}{\partial t} = 0 \tag{2.159}$$

Substituting from (2.158) into (2.159), we find that

$$\dot{H} = \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_j a_{ji}\dot{q}_i + \frac{\partial H}{\partial t} \tag{2.160}$$

We see that the system will be conservative and $H(q, p)$ will be a constant of the motion if

$$\sum_{i=1}^{n} a_{ji}\dot{q}_i = 0 \qquad (j = 1, \ldots, m) \tag{2.161}$$

that is, if $a_{jt} = 0$ for all j, and if the Hamiltonian function is not an explicit function of time, implying that

$$\frac{\partial H}{\partial t} = 0 \tag{2.162}$$

These conditions are equivalent to those found earlier with the Lagrangian approach. Thus, the energy integral is

$$H = T_2 - T_0 + V = E \tag{2.163}$$

Finally, it should be noted from the first equality of (2.159) that if the canonical equations (2.100) apply, then

$$\dot{H} = \frac{\partial H}{\partial t} \tag{2.164}$$

whether the system is conservative or not.

Example 2.5 A particle of mass m can slide without friction on a rigid wire in the form of a circle of radius r, as shown in Fig. 2.5. The circular wire rotates about a vertical axis

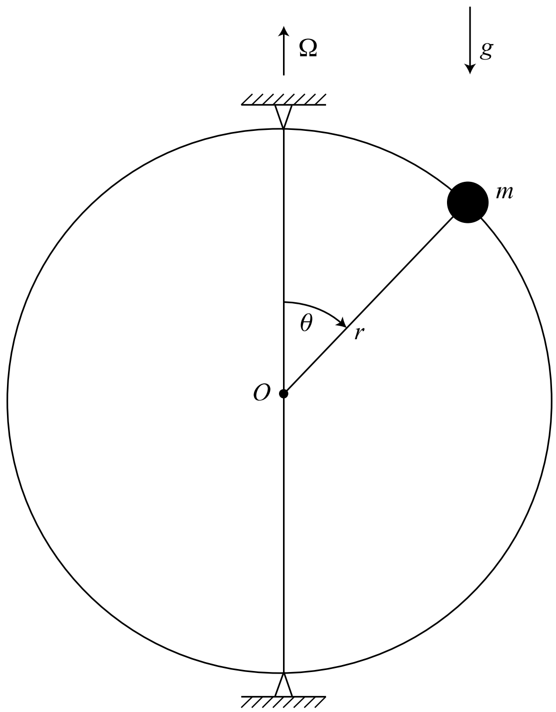

Figure 2.5.

through the center O with a constant angular velocity Ω. We wish to determine if this system is conservative.

First, notice that this is a rheonomic holonomic system. It is possible, however, to choose a single independent generalized coordinate θ. The system is described by Lagrange's equation of the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{2.165}$$

as in (2.157). The Lagrangian function is

$$L = T - V = \frac{1}{2}m\,(r^2\dot{\theta}^2 + r^2\Omega^2 \sin^2\theta) - mgr\cos\theta \tag{2.166}$$

which is not an explicit function of time. Hence, the sufficient conditions for a conservative system are met. The energy integral, which is constant during the motion is

$$E = T_2 - T_0 + V = \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}mr^2\Omega^2\sin^2\theta + mgr\cos\theta \tag{2.167}$$

This, of course, is different from the total energy $T + V$.

It is interesting to study the same system using the Cartesian coordinates (x, y, z) to specify the position of the particle. Let the Cartesian frame be fixed in space with its origin at the center O and with the positive z-axis pointing upward and lying on the axis of rotation. Choose the time reference such that, at $t = 0$, the circular wire lies in the xz-plane. The transformation equations are

$$x = r \sin\theta \cos \Omega t \tag{2.168}$$

$$y = r \sin\theta \sin \Omega t \tag{2.169}$$

$$z = r \cos\theta \tag{2.170}$$

Notice the explicit functions of time, confirming that the system is rheonomic.

The Lagrangian function has the simple form

$$L = T - V = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \tag{2.171}$$

which is not an explicit function of time. There are two holonomic constraints, namely,

$$\phi_1 = x^2 + y^2 + z^2 - r^2 = 0 \tag{2.172}$$

$$\phi_2 = x \tan \Omega t - y = 0 \tag{2.173}$$

Upon differentiation with respect to time, we obtain forms that are linear in the $\dot{q}$s, that is,

$$\dot{\phi}_1 = 2\,(x\dot{x} + y\dot{y} + z\dot{z}) = 0 \tag{2.174}$$

$$\dot{\phi}_2 = \dot{x} \tan \Omega t - \dot{y} + \Omega x \sec^2 \Omega t = 0 \tag{2.175}$$

The second constraint equation has $a_{jt} \neq 0$, so the sufficient conditions for a conservative system are not met with this Cartesian formulation. Here the energy function would equal the total energy $T + V$, which is not constant because there is a nonzero driving moment about the vertical axis.

Nevertheless, the energy function found earlier in (2.167) remains constant and is a valid energy integral. When expressed in terms of Cartesian coordinates, it is

$$E = \frac{1}{2}m[(\dot{x} \cos \Omega t + \dot{y} \sin \Omega t)^2 + \dot{z}^2] - \frac{1}{2}m\Omega^2(x^2 + y^2) + mgz \tag{2.176}$$

We conclude that the system should be classed as *conservative* even though it does not meet the sufficient conditions in the Cartesian formulation.

Ignorable coordinates

Consider a *holonomic* system which is described by Hamilton's canonical equations.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, \ldots, n) \tag{2.177}$$

Now suppose that the Hamiltonian function has the form
$H(q_{k+1}, \ldots, q_n, p_1, \ldots, p_n, t)$, that is, the first k qs do not appear. Then we find that

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = 0 \qquad (i = 1, \ldots, k) \tag{2.178}$$

and therefore the first k generalized momenta are

$$p_i = \beta_i \qquad (i = 1, \ldots, k) \tag{2.179}$$

where the βs are constants. The k qs which do not appear in the Hamiltonian function are called *ignorable coordinates*. The constant ps are *integrals of the motion*, in accordance with (2.147).

The Hamiltonian function can now be written in the form $H(q_{k+1}, \ldots, q_n, p_{k+1}, \ldots, p_n, \beta_1, \ldots, \beta_k, t)$. The canonical equations, assuming k ignorable coordinates, are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = k+1, \ldots, n) \tag{2.180}$$

Thus, k degrees of freedom corresponding to the k ignorable coordinates have been removed from the equations of motion. The motion of the ignorable coordinates can be recovered by integrating

$$\dot{q}_i = \frac{\partial H}{\partial \beta_i} \qquad (i = 1, \ldots, k) \tag{2.181}$$

Now let us consider ignorable coordinates from the Lagrangian viewpoint. We assume a holonomic system whose equations of motion have the standard Lagrangian form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{2.182}$$

First, recall from (2.97) that

$$\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, \ldots, n) \tag{2.183}$$

Hence, if a certain ignorable coordinate q_i is missing from the Hamiltonian function, it will also be missing from the Lagrangian function. If the first k qs are ignorable, the Lagrangian L will be a function of $(q_{k+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$. We would like to eliminate the ignorable $\dot{q}$s from the Lagrangian formulation in order to reduce the number of degrees of freedom in the equations of motion.

This goal may be accomplished by first defining a *Routhian function* $R(q_{k+1}, \ldots, q_n, \dot{q}_{k+1}, \ldots, \dot{q}_n, \beta_1, \ldots, \beta_k, t)$ as follows:

$$R = L - \sum_{i=1}^{k} \beta_i \dot{q}_i \tag{2.184}$$

where the ignorable $\dot{q}$s have been eliminated by solving the k equations

$$\frac{\partial L}{\partial \dot{q}_i} = \beta_i \qquad (i = 1, \ldots, k) \tag{2.185}$$

for $(\dot{q}_1, \ldots, \dot{q}_k)$ in terms of $(q_{k+1}, \ldots q_n, \dot{q}_{k+1}, \ldots, \dot{q}_n, \beta_1, \ldots, \beta_k, t)$.

Now let us make an arbitrary variation in the Routhian function, including variations in the βs and time. We obtain

$$\delta R = \sum_{i=k+1}^{n} \frac{\partial R}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n} \frac{\partial R}{\partial \dot{q}_i}\delta \dot{q}_i + \sum_{i=1}^{k} \frac{\partial R}{\partial \beta_i}\delta \beta_i + \frac{\partial R}{\partial t}\delta t \tag{2.186}$$

Next, take the variation of the right-hand side of (2.184). We have

$$\delta\left(L-\sum_{i=1}^{k}\beta_i\dot q_i\right)=\sum_{i=k+1}^{n}\frac{\partial L}{\partial q_i}\delta q_i+\sum_{i=1}^{k}\frac{\partial L}{\partial \dot q_i}\delta\dot q_i+\sum_{i=k+1}^{n}\frac{\partial L}{\partial \dot q_i}\delta\dot q_i+\frac{\partial L}{\partial t}\delta t$$
$$-\sum_{i=1}^{k}\beta_i\delta\dot q_i-\sum_{i=1}^{k}\dot q_i\delta\beta_i \qquad (2.187)$$

Using (2.185), this simplifies to

$$\delta\left(L-\sum_{i=1}^{k}\beta_i\dot q_i\right)=\sum_{i=k+1}^{n}\frac{\partial L}{\partial q_i}\delta q_i+\sum_{i=k+1}^{n}\frac{\partial L}{\partial \dot q_i}\delta\dot q_i-\sum_{i=1}^{k}\dot q_i\delta\beta_i+\frac{\partial L}{\partial t}\delta t \qquad (2.188)$$

We assume that the varied quantities in (2.186) and (2.188) are independent, and therefore the corresponding coefficients must be equal. Thus,

$$\frac{\partial L}{\partial q_i}=\frac{\partial R}{\partial q_i}\qquad (i=k+1,\ldots,n) \qquad (2.189)$$

$$\frac{\partial L}{\partial \dot q_i}=\frac{\partial R}{\partial \dot q_i}\qquad (i=k+1,\ldots,n) \qquad (2.190)$$

$$\dot q_i=-\frac{\partial R}{\partial \beta_i}\qquad (i=1,\ldots,k) \qquad (2.191)$$

$$\frac{\partial L}{\partial t}=\frac{\partial R}{\partial t} \qquad (2.192)$$

Now let us substitute from (2.189) and (2.190) into Lagrange's equation, (2.182). We obtain

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot q_i}\right)-\frac{\partial R}{\partial q_i}=0\qquad (i=k+1,\ldots,n) \qquad (2.193)$$

These equations are of the form of Lagrange's equation with the Routhian function replacing the Lagrangian function. There are $(n-k)$ second-order equations in the nonignorable variables. Thus, the Routhian procedure has succeeded in eliminating the ignorable coordinates from the equations of motion and, in effect, has reduced the number of degrees of freedom to $(n-k)$. Usually there is no need to solve for the ignorable coordinates, but, if necessary, they can be recovered by integrating (2.191). The k integrals of the motion associated with the ignorable coordinates are given by (2.185) and are equal to the corresponding generalized momenta.

Example 2.6 Consider the same system as in Example 2.5 on page 94 (Fig. 2.5) except that it can rotate *freely* about the fixed vertical axis through the center, the angle of rotation being ϕ. We find that (ϕ,θ) are the generalized coordinates and the Lagrangian function is

$$L=T-V=\frac{1}{2}mr^2\dot\phi^2\sin^2\theta+\frac{1}{2}mr^2\dot\theta^2-mgr\cos\theta \qquad (2.194)$$