5 Equations of motion: integral approach

Integral principles and, in particular, Hamilton's principle, have long occupied a prominent position in analytical mechanics. Hamilton's principle, first announced in 1834, presents a *variational principle* as the basis for the dynamical description of a holonomic system. This approach tends to view the motion as a whole and involves a search for the path in configuration space which yields a stationary value for a certain integral. As a result, one obtains the differential equations of motion.

The requirement of stationarity does not apply to nonholonomic systems. Nevertheless, one can use integral methods to obtain the equations of motion for nonholonomic systems. Here we use the integral of the variation rather than the variation of the integral. In this chapter, we shall discuss the derivation and application of these methods, particularly with respect to nonholonomic systems.

5.1 Hamilton's principle

Holonomic system

Consider a dynamical system whose motion satisfies *Lagrange's principle*, namely,

$$\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}-Q_i\right]\delta q_i=0 \tag{5.1}$$

There are n generalized coordinates and the δqs satisfy the instantaneous constraints. The kinetic energy $T(q, \dot{q}, t)$ is written for the unconstrained system, and is assumed to have at least two continuous derivatives in each of its arguments. Q_i is the generalized applied force associated with q_i.

Now integrate (5.1) with respect to time over the fixed interval t_1 to t_2. Using integration by parts, we find that

$$\int_{t_1}^{t_2}\sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i dt=-\int_{t_1}^{t_2}\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)dt+\left[\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} \tag{5.2}$$

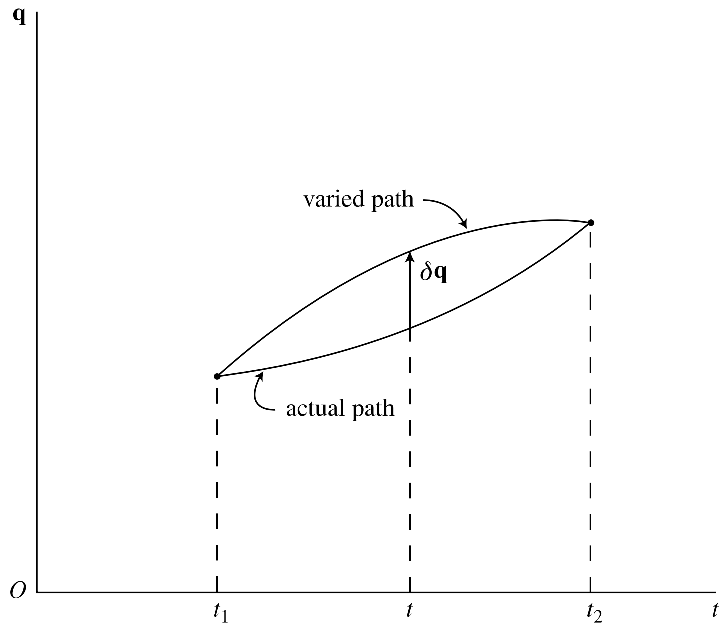

Figure 5.1.

Hence, we obtain

$$\int_{t_1}^{t_2} \left[\sum_{i=1}^{n} \frac{\partial T}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i) + \sum_{i=1}^{n} Q_i \delta q_i \right] dt = \left[\sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} \tag{5.3}$$

The δqs satisfy the m instantaneous constraint equations

$$\sum_{i=1}^{n} a_{ji}(q, t) \delta q_i = 0 \qquad (j = 1, \ldots, m) \tag{5.4}$$

and are assumed to equal zero at the fixed end points t_1 and t_2. Thus, the right-hand side of (5.3) vanishes.

The actual and varied paths in extended configuration space are shown in Fig. 5.1. The δqs are *contemporaneous* variations, that is, they take place with time held fixed. Note that, for a given actual path, the varied path is specified by the δqs which satisfy (5.4).

Let us assume that the qs and $\dot{q}$s are continuous functions of time along the actual and varied paths. Then we can write

$$\frac{d}{dt}(\delta q_i) = \delta \dot{q}_i \qquad (i = 1, \ldots, n) \tag{5.5}$$

The transposition of d and δ operators will be discussed later in the chapter.

Referring again to (5.3), note that

$$\delta T = \sum_{i=1}^{n} \frac{\partial T}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i \tag{5.6}$$

and the virtual work is

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i \tag{5.7}$$

Thus, we obtain

$$\int_{t_1}^{t_2} (\delta T + \delta W) dt = 0 \tag{5.8}$$

This important result applies to the same wide variety of dynamical systems as does Lagrange's principle as given by (5.1). We shall return to this equation when we consider nonholonomic systems.

Now let us assume that all the applied forces are associated with a potential energy function $V(q, t)$. Then $\delta W = -\delta V$ and we can write (5.8) in the form

$$\int_{t_1}^{t_2} \delta L dt = 0 \tag{5.9}$$

where the Lagrangian function $L(q, \dot{q}, t) = T - V$. Assuming a *holonomic system*, the operations of integration and variation can be interchanged. Thus, we obtain

$$\delta \int_{t_1}^{t_2} L dt = 0 \tag{5.10}$$

which is the usual form of *Hamilton's principle*.

The variation of the integral in (5.10) implies that the actual and varied paths satisfy the m constraint equations of the form

$$\sum_{i=1}^{n} q_{ji}(q, t)\dot{q}_i + a_{jt}(q, t) = 0 \qquad (j = 1, \ldots, m) \tag{5.11}$$

where these expressions are integrable in this holonomic case. On the other hand, the integral of the variation, as in (5.9), implies that the δqs in the expression for δL must satisfy the instantaneous constraints of (5.4). For holonomic systems, the varied paths satisfy both the actual and instantaneous constraint equations. The solutions of (5.10) have the property of *stationarity*; whereas the solutions of (5.9) may or may not have this property, depending on the nature of the constraints.

Now let us restate Hamilton's principle, as it applies to holonomic systems, as follows: *The actual path in configuration space followed by a holonomic dynamical system between the fixed times t_1 and t_2 is such that the integral*

$$I = \int_{t_1}^{t_2} L dt \tag{5.12}$$

is stationary with respect to path variations which vanish at the end-points.

To reiterate, the primary assumptions in the derivation of Hamilton's principle are that: (1) the variations δq_i satisfy the instantaneous constraint equations; (2) the end-points are fixed in configuration space and time; and (3) all the applied forces are derivable from a potential energy function $V(q, t)$. Note that the system need not be conservative.

An alternate approach to obtaining the equations of motion for a holonomic system is to begin with Hamilton's principle as a stationarity principle. With this as a starting point, and using the same assumptions as before, we can derive Lagrange's principle. To see how this develops, let us begin with

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt = 0 \tag{5.13}$$

Then, using (5.5) and integrating by parts, we obtain

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right] \delta q_i dt = \sum_{i=1}^{n} \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} = 0 \tag{5.14}$$

If the δqs are unconstrained, and therefore arbitrary, each coefficient must equal zero, yielding

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{5.15}$$

which is Lagrange's equation. This is also the Euler–Lagrange equation of the calculus of variations.

On the other hand, if the δqs are constrained by (5.4), then the integrand must equal zero at each instant of time since the limits t_1 and t_2 are arbitrary. Thus, we obtain Lagrange's principle, namely,

$$\sum_{i=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right] \delta q_i = 0 \tag{5.16}$$

for this case where all the generalized applied forces are obtained from the potential energy $V(q, t)$.

Nonholonomic system

Although stationarity, as expressed in Hamilton's principle, is central to the dynamical theory of holonomic systems, it does not apply to nonholonomic systems. To see how this comes about, let us consider a nonholonomic system and require that each varied path must satisfy the actual constraint equations of the general form

$$f_j(q, \dot{q}, t) = 0 \qquad (j = 1, \ldots, m) \tag{5.17}$$

These constraints are enforced by invoking the *multiplier rule*. The multiplier rule states that the constrained stationary values of the integral of (5.12) are found by considering the *free variations* of

$$I = \int_{t_1}^{t_2} \Lambda dt \tag{5.18}$$

where $\Lambda(q, \dot{q}, \mu, t)$ is the *augmented Lagrangian function* which is formed by adjoining the constraint functions to the Lagrangian function by using Lagrange multipliers. Thus,

$$\Lambda = L(q, \dot{q}, t) + \sum_{j=1}^{m} \mu_j f_j(q, \dot{q}, t) \tag{5.19}$$

where the Lagrange multipliers $\mu_j(t)$ are treated as additional variables to be determined.

The stationarity of the free variations of the integral of (5.18) results in the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}_i}\right) - \frac{\partial \Lambda}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{5.20}$$

$$\frac{\partial \Lambda}{\partial \mu_j} = f_j(q, \dot{q}, t) = 0 \qquad (j = 1, \ldots, m) \tag{5.21}$$

Note that (5.21) merely restates the constraint equations.

Now let us apply (5.20) to a nonholonomic system in which the constraint functions are linear in the $\dot{q}$s. Thus, the constraint equations have the familiar form

$$f_j(q, \dot{q}, t) = \sum_{i=1}^{n} a_{ji}(q, t)\dot{q}_i + a_{jt}(q, t) = 0 \qquad (j = 1, \ldots, m) \tag{5.22}$$

Then, using (5.19) and (5.20), we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= -\sum_{j=1}^{m} \frac{d}{dt}(\mu_j a_{ji}) + \sum_{j=1}^{m}\sum_{k=1}^{n} \mu_j \frac{\partial a_{jk}}{\partial q_i}\dot{q}_k + \sum_{j=1}^{m} \mu_j \frac{\partial a_{jt}}{\partial q_i} \\
&= -\sum_{j=1}^{m} \dot{\mu}_j a_{ji} + \sum_{j=1}^{m}\sum_{k=1}^{n} \mu_j \left(\frac{\partial a_{jk}}{\partial q_i} - \frac{\partial a_{ji}}{\partial q_k}\right)\dot{q}_k \\
&\quad + \sum_{j=1}^{m} \mu_j \left(\frac{\partial a_{jt}}{\partial q_i} - \frac{\partial a_{ji}}{\partial t}\right) \qquad (i = 1, \ldots, n)
\end{aligned} \tag{5.23}$$

These are the Euler–Lagrange equations for finding the solution path leading to a stationary value of the integral I of (5.12), where both the actual and varied paths satisfy the constraints given by (5.22). Comparing (5.23) with the known form of Lagrange's equation for this nonholonomic system, namely,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \qquad (i = 1, \ldots, n) \tag{5.24}$$

we see that, in general, the equations are different. We conclude that the requirement of stationarity leads to *incorrect dynamical equations* for the general case of *nonholonomic* constraints. Conversely, the solution path of a nonholonomic system will not, in general, result in a stationary value of the integral in (5.12).

On the other hand, if we equate $-\dot{\mu}_j$ with λ_j and set

$$\frac{\partial a_{jk}}{\partial q_i} - \frac{\partial a_{ji}}{\partial q_k} = 0 \quad \text{and} \quad \frac{\partial a_{jt}}{\partial q_i} - \frac{\partial a_{ji}}{\partial t} = 0 \quad \begin{pmatrix} i, k = 1, \ldots, n \\ j = 1, \ldots, m \end{pmatrix} \tag{5.25}$$

which are the exactness conditions, then the system is *holonomic* and (5.23) reduces to the correct equation (5.24).

The correct equations of motion of a nonholonomic system can be obtained from

$$\int_{t_1}^{t_2} \delta L dt = 0 \tag{5.26}$$

which may be considered to be the *nonholonomic form of Hamilton's principle*. It is not a stationarity principle, however, and thereby differs fundamentally from its usual holonomic form given in (5.10). Equation (5.26) assumes that: (1) the actual and varied paths are continuous functions of time and their difference $\delta \mathbf{q}$ satisfies the instantaneous constraint equations; (2) the δqs equal zero at the fixed end-points t_1 and t_2; and (3) all the applied forces arise from a potential energy function $V(q, t)$.

More generally, when the applied forces do not arise from a potential energy function, one can use

$$\int_{t_1}^{t_2} (\delta T + \delta W) dt = 0 \tag{5.27}$$

where the virtual work is

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i \tag{5.28}$$

and the δqs satisfy (5.4). As we found in the derivation of (5.8), this result is essentially an integrated form of Lagrange's principle, (5.1).

Example 5.1 A flat rigid body of mass m moves in the horizontal xy-plane (Fig. 5.2). There is a knife-edge constraint at the reference point P, about which the moment of inertia

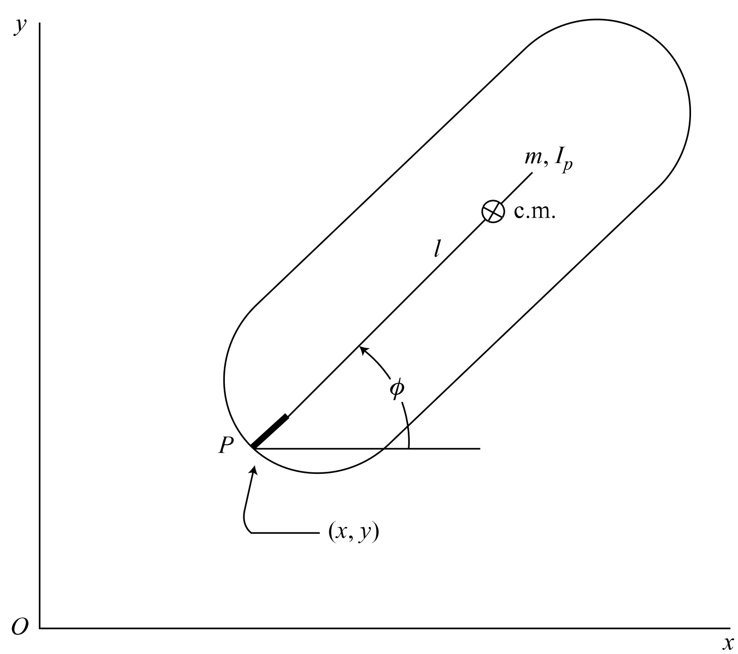

Figure 5.2.

is I_p. Assuming the center of mass is located at a distance l from P, and using (x, y, ϕ) as generalized coordinates, let us find the differential equations of motion.

We can take $V = 0$, so Hamilton's principle has the nonholonomic form

$$\int_{t_1}^{t_2} \delta T\, dt = 0 \tag{5.29}$$

The unconstrained kinetic energy is, from (3.146),

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_p\dot{\phi}^2 + ml\dot{\phi}(-\dot{x}\sin\phi + \dot{y}\cos\phi) \tag{5.30}$$

and we obtain

$$\begin{aligned}\delta T &= m\dot{x}\delta\dot{x} + m\dot{y}\delta\dot{y} + I_p\dot{\phi}\delta\dot{\phi} - ml\dot{\phi}\sin\phi\;\delta\dot{x} + ml\dot{\phi}\cos\phi\,\delta\dot{y}\\ &\quad + ml(-\dot{x}\sin\phi + \dot{y}\cos\phi)\delta\dot{\phi} + ml\dot{\phi}(-\dot{x}\cos\phi - \dot{y}\sin\phi)\delta\phi\end{aligned} \tag{5.31}$$

Noting that

$$\delta\dot{x} = \frac{d}{dt}(\delta x), \qquad \delta\dot{y} = \frac{d}{dt}(\delta y), \qquad \delta\dot{\phi} = \frac{d}{dt}(\delta\phi) \tag{5.32}$$

we find that

$$\begin{aligned}\delta T &= (m\dot{x} - ml\dot{\phi}\sin\phi)\frac{d}{dt}(\delta x) + (m\dot{y} + ml\dot{\phi}\cos\phi)\frac{d}{dt}(\delta y)\\ &\quad + (I_p\dot{\phi} - ml\dot{x}\sin\phi + ml\dot{y}\cos\phi)\frac{d}{dt}(\delta\phi) - ml\dot{\phi}(\dot{x}\cos\phi + \dot{y}\sin\phi)\delta\phi\end{aligned} \tag{5.33}$$

Now integrate by parts, noting that the δqs equal zero at the end-points. We obtain

$$\begin{aligned}\int_{t_1}^{t_2} \delta T\, dt = -\int_{t_1}^{t_2} \Bigg\{ &\frac{d}{dt}(m\dot{x} - ml\dot{\phi}\sin\phi)\delta x + \frac{d}{dt}(m\dot{y} + ml\dot{\phi}\cos\phi)\delta y\\ &+ \bigg[\frac{d}{dt}(I_p\dot{\phi} - ml\dot{x}\sin\phi + ml\dot{y}\cos\phi)\\ &+ ml\dot{\phi}(\dot{x}\cos\phi + \dot{y}\sin\phi)\bigg]\delta\phi\Bigg\}\, dt = 0\end{aligned} \tag{5.34}$$

The end-points t_1 and t_2 are arbitrary, so the integrand must be zero continuously. Thus, we obtain

$$\begin{aligned}&(m\ddot{x} - ml\ddot{\phi}\sin\phi - ml\dot{\phi}^2\cos\phi)\delta x + (m\ddot{y} + ml\ddot{\phi}\cos\phi - ml\dot{\phi}^2\sin\phi)\delta y\\ &\quad + (I_p\ddot{\phi} - ml\ddot{x}\sin\phi + ml\ddot{y}\cos\phi)\delta\phi = 0\end{aligned} \tag{5.35}$$

which is essentially Lagrange's principle of (5.1).

The nonholonomic constraint equation is

$$\dot{x}\sin\phi - \dot{y}\cos\phi = 0 \tag{5.36}$$

which states that the velocity of point P perpendicular to the knife edge is zero. The corresponding instantaneous constraint equation is

$$\sin\phi\ \delta x - \cos\phi\ \delta y = 0 \tag{5.37}$$

We can choose two independent sets of $(\delta x, \delta y, \delta\phi)$ which satisfy (5.37). Let us choose virtual displacements proportional to $(\cos\phi, \sin\phi, 0)$ and $(0, 0, 1)$. Then, from (5.35) we obtain the following two differential equations of motion:

$$m\ddot{x}\cos\phi + m\ddot{y}\sin\phi - ml\dot{\phi}^2 = 0 \tag{5.38}$$

$$I_p\ddot{\phi} - ml\ddot{x}\sin\phi + ml\ddot{y}\cos\phi = 0 \tag{5.39}$$

Alternatively, we could have noted that

$$\delta y = \tan\phi\ \delta x \tag{5.40}$$

and then considered δx and $\delta\phi$ to be independent.

We need a third differential equation which is obtained by differentiating (5.36) with respect to time.

$$\ddot{x}\sin\phi - \ddot{y}\cos\phi + \dot{x}\dot{\phi}\cos\phi + \dot{y}\dot{\phi}\sin\phi = 0 \tag{5.41}$$

Equations (5.38), (5.39), and (5.41) are linear in the $\ddot{q}$s and can be solved for $\ddot{x}$, $\ddot{y}$, and $\ddot{\phi}$, which are integrated to obtain the motion as a function of time.

The approach used in this example has yielded three second-order equations, namely, two dynamical equations and one kinematical equation. Notice that $\dot{q}$s have been used as velocity variables in the kinetic energy function; quasi-velocities should be avoided because equations similar to (5.32) will not apply.

5.2 Transpositional relations

Now let us examine the kinematical effects due to the nonintegrability of the quasi-velocity expressions and constraint equations often associated with nonholonomic systems. As before, we shall ultimately be concerned with time integrals of variational expressions, although the transpositional relations under consideration here are differential in nature. Hence, we will actually study the kinematics of differential paths in configuration space.

The d and δ operators

Let us begin with the differential form

$$d\theta_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)dq_i + \Psi_{jt}(q,t)dt \qquad (j = 1, \ldots, n) \tag{5.42}$$

In general, the differential form is not integrable, so θ_j is a quasi-coordinate. The operator d, as in dq_i, represents an infinitesimal change in the variable q_i which occurs during the infinitesimal time interval dt. If one divides (5.42) by dt, the result is

$$u_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)\dot{q}_i + \Psi_{jt}(q,t) \qquad (j = 1, \ldots, n) \tag{5.43}$$

which, again, is not integrable, in general. Here the quasi-velocity $u_j = d\theta_j/dt$.

The variational equation corresponding to (5.42) is

$$\delta\theta_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)\delta q_i \qquad (j = 1, \ldots, n) \tag{5.44}$$

The operator δ in δq_i represents an infinitesimal change in q_i which is assumed to occur with time held fixed.

Now let us consider the operations δ and d taken in sequence. Taking the total differential of (5.44), we obtain

$$d\delta\theta_j = \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial \Psi_{ji}}{\partial q_k} dq_k \delta q_i + \sum_{i=1}^{n} \frac{\partial \Psi_{ji}}{\partial t} dt \delta q_i + \sum_{i=1}^{n} \Psi_{ji} d\delta q_i \tag{5.45}$$

Similarly, taking the first variation of (5.42), and changing the summing index, we obtain

$$\delta d\theta_j = \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial \Psi_{jk}}{\partial q_i} dq_k \delta q_i + \sum_{i=1}^{n} \frac{\partial \Psi_{jt}}{\partial q_i} dt \delta q_i + \sum_{i=1}^{n} \Psi_{ji} \delta dq_i \tag{5.46}$$

where $\delta t = 0$ and $\delta dt = 0$ since the variations are *contemporaneous*.

Next, subtract (5.46) from (5.45). The result is the important transpositional relation

$$\begin{aligned} d\delta\theta_j - \delta d\theta_j = {} & \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial \Psi_{ji}}{\partial q_k} - \frac{\partial \Psi_{jk}}{\partial q_i} \right) dq_k \delta q_i + \sum_{i=1}^{n} \left(\frac{\partial \Psi_{ji}}{\partial t} - \frac{\partial \Psi_{jt}}{\partial q_i} \right) dt \delta q_i \\ & + \sum_{i=1}^{n} \Psi_{ji}(d\delta q_i - \delta dq_i) \qquad (j = 1, \ldots, n) \end{aligned} \tag{5.47}$$

The differential form of (5.42) is linear in the infinitesimal quantities dq_i and dt, but we notice that (5.47) is of second degree in these small quantities. Assuming the nonintegrability of (5.42) and (5.44), the coefficients of $dq_k \delta q_i$ and $dt \delta q_i$ in (5.47) are generally nonzero. To simplify the notation, we can let

$$F_j = \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial \Psi_{ji}}{\partial q_k} - \frac{\partial \Psi_{jk}}{\partial q_i} \right) dq_k \delta q_i + \sum_{i=1}^{n} \left(\frac{\partial \Psi_{ji}}{\partial t} - \frac{\partial \Psi_{jt}}{\partial q_i} \right) dt \delta q_i \qquad (j = 1, \ldots, n) \tag{5.48}$$

Then we obtain

$$d\delta\theta_j - \delta d\theta_j = F_j + \sum_{i=1}^{n} \Psi_{ji}(d\delta q_i - \delta dq_i) \qquad (j = 1, \ldots, n) \tag{5.49}$$

where $F_j \neq 0$, in general.

There are two principal choices which we can make concerning transpositional relations, namely, (1) $d\delta q_i - \delta dq_i = 0$ and $d\delta\theta_j - \delta d\theta_j \neq 0$, or (2) $d\delta\theta_j - \delta d\theta_j = 0$ and $d\delta q_i - \delta dq_i \neq 0$. For a system with independent qs, we usually choose $d\delta q_i - \delta dq_i = 0$ since this is consistent with a continuous varied path when integral methods are used.

Further transpositional relations can be obtained by first recalling from (4.3) that

$$dq_i = \sum_{j=1}^{n} \Phi_{ij}(q,t)d\theta_j + \Phi_{it}(q,t)dt \qquad (i = 1, \dots, n) \tag{5.50}$$

and thus

$$\delta q_i = \sum_{j=1}^{n} \Phi_{ij}(q,t)\delta\theta_j \qquad (i = 1, \dots, n) \tag{5.51}$$

Multiply (5.49) by Φ_{rj} and sum over j. Thus, we obtain

$$d\delta q_r - \delta dq_r = -\sum_{j=1}^{n} \Phi_{rj} F_j + \sum_{j=1}^{n} \Phi_{rj}(d\delta\theta_j - \delta d\theta_j) \qquad (r = 1, \dots, n) \tag{5.52}$$

where we recall that

$$\sum_{j=1}^{n} \Phi_{rj}\Psi_{ji} = \delta_{ri} \tag{5.53}$$

and δ_{ri} is the Kronecker delta. In detail, we find that

$$\begin{aligned} d\delta q_r - \delta dq_r = &-\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \Phi_{rj}\left(\frac{\partial\Psi_{ji}}{\partial q_k} - \frac{\partial\Psi_{jk}}{\partial q_i}\right) dq_k\delta q_i \\ &-\sum_{i=1}^{n}\sum_{j=1}^{n} \Phi_{rj}\left(\frac{\partial\Psi_{ji}}{\partial t} - \frac{\partial\Psi_{jt}}{\partial q_i}\right) dt\delta q_i \\ &+\sum_{j=1}^{n} \Phi_{rj}(d\delta\theta_j - \delta d\theta_j) \qquad (r = 1, \dots, n) \end{aligned} \tag{5.54}$$

Using (5.48), (5.50), and (5.51), we can write F_j in terms of the Hamel coefficients. We see that

$$\begin{aligned} F_j = &\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\left(\frac{\partial\Psi_{ji}}{\partial q_k} - \frac{\partial\Psi_{jk}}{\partial q_i}\right)\Phi_{ir}\delta\theta_r\left(\sum_{l=1}^{n}\Phi_{kl}d\theta_l + \Phi_{kt}dt\right) \\ &+\sum_{i=1}^{n}\sum_{r=1}^{n}\left(\frac{\partial\Psi_{ji}}{\partial t} - \frac{\partial\Psi_{jt}}{\partial q_i}\right)\Phi_{ir}dt\delta\theta_r \qquad (j = 1, \dots, n) \end{aligned} \tag{5.55}$$

or

$$F_j = \sum_{l=1}^{n}\sum_{r=1}^{n}\gamma_{rl}^{j} d\theta_l\delta\theta_r + \sum_{r=1}^{n}\gamma_r^{j} dt\delta\theta_r \qquad (j = 1, \dots, n) \tag{5.56}$$

where

$$\gamma_{rl}^{j} = \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial\Psi_{ji}}{\partial q_k} - \frac{\partial\Psi_{jk}}{\partial q_i}\right)\Phi_{kl}\Phi_{ir} \tag{5.57}$$

$$\gamma_{r}^{j} = \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial\Psi_{ji}}{\partial q_k} - \frac{\partial\Psi_{jk}}{\partial q_i}\right)\Phi_{kt}\Phi_{ir} + \sum_{i=1}^{n}\left(\frac{\partial\Psi_{ji}}{\partial t} - \frac{\partial\Psi_{jt}}{\partial q_i}\right)\Phi_{ir} \tag{5.58}$$

Then we can express (5.52) in the form

$$\begin{aligned} d\delta q_s - \delta d q_s = &-\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{r=1}^{n}\Phi_{sj}\gamma_{rl}^{j}d\theta_l\delta\theta_r - \sum_{j=1}^{n}\sum_{r=1}^{n}\Phi_{sj}\gamma_{r}^{j}dt\delta\theta_r \\ &+\sum_{j=1}^{n}\Phi_{sj}(d\delta\theta_j - \delta d\theta_j) \qquad (s = 1, \ldots, n) \end{aligned} \tag{5.59}$$

The basic transpositional equations such as (5.47), (5.54), and (5.59) are written for an *unconstrained system*. The application of constraints, however, will be shown to be quite simple.

Nonholonomic constraints

Consider a system with n generalized coordinates and m nonholonomic constraints which are linear in the $\dot{q}$s. We can represent these constraints by setting to zero the last m equations of (5.42) or (5.43). Thus, we have

$$d\theta_j = \sum_{i=1}^{n}\Psi_{ji}(q,t)dq_i + \Psi_{jt}(q,t)dt \qquad (j = 1, \ldots, n-m) \tag{5.60}$$

$$d\theta_j = \sum_{i=1}^{n}\Psi_{ji}(q,t)dq_i + \Psi_{jt}(q,t)dt = 0 \qquad (j = n-m+1, \ldots, n) \tag{5.61}$$

or, in terms of quasi-velocities,

$$u_j = \sum_{i=1}^{n}\Psi_{ji}(q,t)\dot{q}_i + \Psi_{jt}(q,t) \qquad (j = 1, \ldots, n-m) \tag{5.62}$$

$$u_j = \sum_{i=1}^{n}\Psi_{ji}(q,t)\dot{q}_i + \Psi_{jt}(q,t) = 0 \qquad (j = n-m+1, \ldots, n) \tag{5.63}$$

The m constraints are given by (5.63). Note that the first $(n-m)$ us given by (5.62) are *independent*. Thus, as velocity variables, we can use the $(n-m)$ us rather than n constrained $\dot{q}$s.

In a similar fashion, using (5.44), we can write the variational equations for the constrained system.

$$\delta\theta_j = \sum_{i=1}^{n}\Psi_{ji}(q,t)\delta q_i \qquad (j = 1, \ldots, n-m) \tag{5.64}$$

$$\delta\theta_j = \sum_{i=1}^{n}\Psi_{ji}(q,t)\delta q_i \;=\; 0 \qquad (j = n-m+1, \ldots, n) \tag{5.65}$$

The m virtual or instantaneous constraint equations are given by (5.65).

With the application of the constraints, we see that

$$d\delta\theta_j = 0, \qquad \delta d\theta_j = 0 \qquad (j = n - m + 1, \ldots, n) \tag{5.66}$$

so (5.49) becomes

$$d\delta\theta_j - \delta d\theta_j = F_j + \sum_{i=1}^{n} \Psi_{ji}(d\delta q_i - \delta d q_i) \qquad (j = 1, \ldots, n - m) \tag{5.67}$$

and (5.52) reduces to

$$d\delta q_r - \delta d q_r = -\sum_{i=1}^{n} \Phi_{ri} F_i + \sum_{j=1}^{n-m} \Phi_{rj}(d\delta\theta_j - \delta d\theta_j) \qquad (r = 1, \ldots, n) \tag{5.68}$$

The theorem of Frobenius

If the expressions for F_j given in (5.48) are all equal to zero for an unconstrained system, then the exactness conditions apply for the n differential forms of (5.42). These exactness conditions are *sufficient* for the integrability of the differential forms, implying that the θs are true generalized coordinates rather than quasi-coordinates.

A question arises concerning the necessary and sufficient conditions for the integrability of the differential forms if there are constraints. This is answered by the *Theorem of Frobenius: A system of differential equations such as (5.42) is completely integrable to yield $\theta_j = \theta_j(q, t)$ for $j = 1, \ldots, n$ if and only if $F_j = 0$ for all dq_i and δq_i satisfying the constraints*. Recall that F_j is given by (5.48).

As an example, consider a constraint which is represented by the differential form

$$d\theta = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz = 0 \tag{5.69}$$

Also,

$$\delta\theta = a(x, y, z)\delta x + b(x, y, z)\delta y + c(x, y, z)\delta z = 0 \tag{5.70}$$

This constraint is integrable, that is, *holonomic* if $F_j = 0$ or, in detail, if

$$\left(\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}\right)(dy\delta x - dx\delta y) + \left(\frac{\partial b}{\partial z} - \frac{\partial c}{\partial y}\right)(dz\delta y - dy\delta z) + \left(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z}\right)(dx\delta z - dz\delta x) = 0 \tag{5.71}$$

where, from (5.69) and (5.70),

$$dz = -\frac{1}{c}(a dx + b dy), \qquad \delta z = -\frac{1}{c}(a\delta x + b\delta y) \tag{5.72}$$

Upon making these substitutions and simplifying, we obtain

$$\left[\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x} + \frac{a}{c}\left(\frac{\partial b}{\partial z} - \frac{\partial c}{\partial y}\right) + \frac{b}{c}\left(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z}\right)\right](dy\delta x - dx\delta y) = 0 \tag{5.73}$$

where dx, δx, dy, δy are independent. Hence, we find that

$$a\left(\frac{\partial b}{\partial z}-\frac{\partial c}{\partial y}\right)+b\left(\frac{\partial c}{\partial x}-\frac{\partial a}{\partial z}\right)+c\left(\frac{\partial a}{\partial y}-\frac{\partial b}{\partial x}\right)=0 \tag{5.74}$$

This is the necessary and sufficient condition for the integrability of (5.69).

Geometrical considerations

First, let us consider an *unconstrained* system described in terms of n quasi-velocities which are given by (5.43). We wish to compare points on an actual path in n-dimensional q-space with the corresponding points on a varied path. This can be visualized by considering a small quadrilateral $ABCD$, as shown in Fig. 5.3a. The infinitesimal vector $d\mathbf{q}$ occurs on the actual path during a small time interval dt. The variation $\delta\mathbf{q}$ is an infinitesimal vector drawn from a point on the actual path to a corresponding point (at the same time) on the varied path. Thus, the operator d refers to differences that occur along a solution path with the passage of time; whereas, δ refers to differences in going from the actual path to a varied path with time held constant.

For this *unconstrained* case, the small quadrilateral $ABCD$ is closed. We see that the vector displacements ABC and ADC are equal. Thus

$$d\mathbf{q}+\delta\mathbf{q}+d\delta\mathbf{q}=\delta\mathbf{q}+d\mathbf{q}+\delta d\mathbf{q} \tag{5.75}$$

and

$$d\delta\mathbf{q}=\delta d\mathbf{q} \tag{5.76}$$

In terms of components,

$$d\delta q_i-\delta dq_i=0 \qquad (i=1,\ldots,n) \tag{5.77}$$

This result applies to a *holonomic* system for which all $F_j=0$. On the other hand, even for an unconstrained or holonomic system, if (5.77) applies, then (5.49) shows that for each $F_j\neq 0$ due to the presence of quasi-velocities, there is a corresponding transpositional term

$$d\delta\theta_j-\delta d\theta_j\neq 0 \tag{5.78}$$

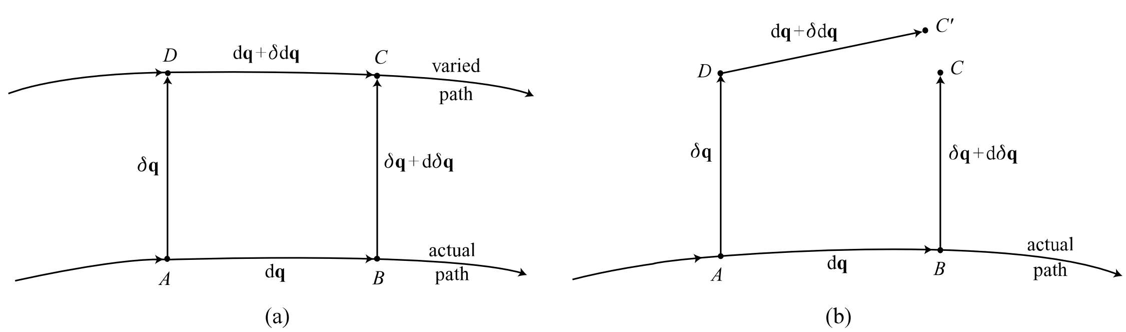

Figure 5.3.

Now let us consider the more general case in which there are *m nonholonomic constraints* that are applied to the system by setting the last m differential forms equal to zero. Thus, we assume that

$$d\theta_j = 0, \qquad \delta\theta_j = 0 \qquad (j = n - m + 1, \ldots, n) \tag{5.79}$$

as in (5.61) and (5.65). Since the virtual constraint applies at B and the actual constraint applies at D in Fig. 5.3b, we see that

$$d\delta\theta_j = 0, \qquad \delta d\theta_j = 0 \qquad (j = n - m + 1, \ldots, n) \tag{5.80}$$

and therefore

$$d\delta\theta_j - \delta d\theta_j = 0 \qquad (j = n - m + 1, \ldots, n) \tag{5.81}$$

Assuming that $F_j \neq 0$ for $j = n - m + 1, \ldots, n$, we see from (5.49) that at least m of the $d\delta q_i - \delta dq_i$ are nonzero, implying that

$$d\delta\mathbf{q} - \delta d\mathbf{q} \neq 0 \tag{5.82}$$

Thus, the quadrilateral in Fig. 5.3b does not close, and the vector in n-space directed from C' to C is equal to $d\delta\mathbf{q} - \delta d\mathbf{q}$. This lack of closure indicates that, even at the differential level, there is path dependence due to the nonintegrability of the nonholonomic constraints. For example, if there is one nonholonomic constraint, it is kinematically possible to obtain closure for all but one of the component qs. However, the remaining $d\delta q_i - \delta dq_i$ must be nonzero, resulting in nonclosure in n-space. Similarly, if there are m nonholonomic constraints, then at least m of the $d\delta q_i - \delta dq_i$ must be nonzero.

For a nominal point A on the actual path, it is important to understand how the $\Psi_{ji}(q, t)$ and $\Psi_{jt}(q, t)$ coefficients are evaluated when applying the constraints at points B and D. At A we have the constraint equations

$$\sum_{i=1}^{n} \Psi_{ji} dq_i + \Psi_{jt} dt = 0 \qquad (j = n - m + 1, \ldots, n) \tag{5.83}$$

$$\sum_{i=1}^{n} \Psi_{ji} \delta q_i = 0 \qquad (j = n - m + 1, \ldots, n) \tag{5.84}$$

where the Ψ_{ji} and Ψ_{jt} coefficients are evaluated at A. At B we have

$$(\Psi_{ji})_B = \Psi_{ji} + \sum_{k=1}^{n} \frac{\partial \Psi_{ji}}{\partial q_k} dq_k + \frac{\partial \Psi_{ji}}{\partial t} dt \tag{5.85}$$

where the terms on the right are again evaluated at A. Thus, for the virtual constraint at B, we obtain

$$\sum_{i=1}^{n} \left[\Psi_{ji} + \sum_{k=1}^{n} \frac{\partial \Psi_{ji}}{\partial q_k} dq_k + \frac{\partial \Psi_{ji}}{\partial t} dt \right] (\delta q_i + d\delta q_i) = 0 \tag{5.86}$$

and, keeping terms to second order, there remains

$$\sum_{i=1}^{n} \Psi_{ji}\delta q_i + \sum_{i=1}^{n} \Psi_{ji} d\delta q_i + \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial \Psi_{ji}}{\partial q_k} dq_k \delta q_i + \sum_{i=1}^{n} \frac{\partial \Psi_{ji}}{\partial t} dt\delta q_i = 0$$
$$(j = n - m + 1, \ldots, n) \tag{5.87}$$

Similarly, at D the coefficients are

$$(\Psi_{jk})_D = \Psi_{jk} + \sum_{i=1}^{n} \frac{\partial \Psi_{jk}}{\partial q_i} \delta q_i \tag{5.88}$$

$$(\Psi_{jt})_D = \Psi_{jt} + \sum_{i=1}^{n} \frac{\partial \Psi_{jt}}{\partial q_i} \delta q_i \tag{5.89}$$

Hence, the actual constraint equation at D requires that

$$\sum_{k=1}^{n} \left[\Psi_{jk} + \sum_{i=1}^{n} \frac{\partial \Psi_{jk}}{\partial q_i} \delta q_i \right] (dq_k + \delta dq_k) + \left[\Psi_{jt} + \sum_{i=1}^{n} \frac{\partial \Psi_{jt}}{\partial q_i} \delta q_i \right] dt = 0 \tag{5.90}$$

and, omitting third-order terms, we obtain

$$\sum_{k=1}^{n} \Psi_{jk} dq_k + \sum_{k=1}^{n} \Psi_{jk} \delta dq_k + \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial \Psi_{jk}}{\partial q_i} dq_k \delta q_i + \Psi_{jt} dt + \sum_{i=1}^{n} \frac{\partial \Psi_{jt}}{\partial q_i} dt\delta q_i = 0$$
$$(j = n - m + 1, \ldots, n) \tag{5.91}$$

Now subtract (5.91) from (5.87) and recall the constraint equations at A, namely, (5.83) and (5.84). The result is

$$\sum_{i=1}^{n} \Psi_{ji}(d\delta q_i - \delta dq_i) + \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\frac{\partial \Psi_{ji}}{\partial q_k} - \frac{\partial \Psi_{jk}}{\partial q_i} \right) dq_k \delta q_i$$
$$+ \sum_{i=1}^{n} \left(\frac{\partial \Psi_{ji}}{\partial t} - \frac{\partial \Psi_{jt}}{\partial q_i} \right) dt\delta q_i = 0 \qquad (j = n - m + 1, \ldots, n) \tag{5.92}$$

This is identical to the basic equation (5.47) for the case of m nonholonomic constraints with

$$d\delta\theta_j - \delta d\theta_j = 0 \qquad (j = n - m + 1, \ldots, n) \tag{5.93}$$

We have presented a derivation which emphasizes the assumptions concerning the values of the coefficients at points B and D which are slightly displaced from the nominal reference point A. Note, however, that ultimately all calculations involve values at A; and this applies as well to calculations of F_j.

As a result of the lack of closure of the differential quadrilaterals if there are nonholonomic constraints, it is not possible to find a continuous varied path which simultaneously satisfies the actual constraints and the virtual or instantaneous constraints. One must choose one or the other. If one chooses the stationarity principle in which the varied paths are required to satisfy the actual constraints but not the virtual constraints, then incorrect equations of motion result, as we found earlier. On the other hand, if the varied paths satisfy the virtual constraints but not the actual constraints, the variational process produces correct equations of motion, as we found in the discussion of Hamilton's principle.

5.3 The Boltzmann–Hamel equation, transpositional form

Derivation

The general form of the Boltzmann–Hamel equation, as given in (4.85), is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right) - \frac{\partial T}{\partial \theta_r} + \sum_{j=1}^{n}\sum_{l=1}^{n-m} \frac{\partial T}{\partial u_j}\gamma_{rl}^{j} u_l + \sum_{j=1}^{n} \frac{\partial T}{\partial u_j}\gamma_r^j = Q_r \qquad (r = 1, \ldots, n-m) \tag{5.94}$$

where $T(q, u, t)$ is the *unconstrained* kinetic energy and Q_r is the generalized applied force associated with u_r. The $(n-m)$ us are independent and $u_r = \dot{\theta}_r$ where θ_r is a quasi-coordinate.

Multiply (5.94) by $\delta\theta_r dt$ and sum over r. We obtain

$$\sum_{r=1}^{n-m}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right) - \frac{\partial T}{\partial \theta_r} - Q_r\right]\delta\theta_r dt + \sum_{j=1}^{n}\sum_{r=1}^{n-m}\left[\sum_{l=1}^{n-m}\frac{\partial T}{\partial u_j}\gamma_{rl}^j d\theta_l \delta\theta_r + \frac{\partial T}{\partial u_j}\gamma_r^j dt\delta\theta_r\right] = 0 \tag{5.95}$$

Now recall that with the application of the m constraints,

$$d\theta_l = 0, \qquad \delta\theta_l = 0 \qquad (l = n-m+1, \ldots, n) \tag{5.96}$$

and thus, from (5.56),

$$F_j = \sum_{l=1}^{n-m}\sum_{r=1}^{n-m}\gamma_{rl}^j d\theta_l \delta\theta_r + \sum_{r=1}^{n-m}\gamma_r^j dt\delta\theta_r \qquad (j = 1, \ldots, n) \tag{5.97}$$

Hence, we find that

$$\sum_{r=1}^{n-m}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right) - \frac{\partial T}{\partial \theta_r} - Q_r\right]\delta\theta_r dt + \sum_{j=1}^{n}\frac{\partial T}{\partial u_j}F_j = 0 \tag{5.98}$$

Let us assume that

$$d\delta\theta_j - \delta d\theta_j = 0 \qquad (j = 1, \ldots, n-m) \tag{5.99}$$

so that this transpositional expression is now equal to zero for all j. Then, from (5.49), we obtain

$$F_j = -\sum_{i=1}^{n}\Psi_{ji}(d\delta q_i - \delta dq_i) \tag{5.100}$$

Finally, dividing by dt, we can write (5.98) in the form

$$\sum_{r=1}^{n-m}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right)-\frac{\partial T}{\partial \theta_r}-Q_r\right]\delta\theta_r-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial T}{\partial u_j}\Psi_{ji}\left[\frac{d}{dt}(\delta q_i)-\delta\dot{q}_i\right]=0 \tag{5.101}$$

This is the *transpositional form* of the *Boltzmann–Hamel equation*. The $(n-m)$ equations of motion are obtained by writing the transpositional expressions in terms of the $\delta\theta$s and then setting the coefficient of each $\delta\theta_r$ equal to zero. This approach results in the same equations of motion as (5.94) but is somewhat simpler to apply.

Example 5.2 A dumbbell consists of two particles, each of mass m, connected by a massless rod of length l. There is a knife-edge constraint at particle 1 (Fig. 5.4). We wish to find the equations of motion as the system moves in the horizontal xy-plane.

Let us use (5.101). The generalized coordinates are (x, y, ϕ) and the us are

$$\begin{aligned} u_1 &= v = \dot{x}\cos\phi + \dot{y}\sin\phi \\ u_2 &= \dot{\phi} \\ u_3 &= w = -\dot{x}\sin\phi + \dot{y}\cos\phi = 0 \end{aligned} \tag{5.102}$$

where the last equation is the nonholonomic constraint equation. The coefficient matrix is

$$\Psi = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \tag{5.103}$$

As quasi-coordinates, let us choose s and η, where $\dot{s} = v$ and $\dot{\eta} = w$. The angle ϕ is a true coordinate.

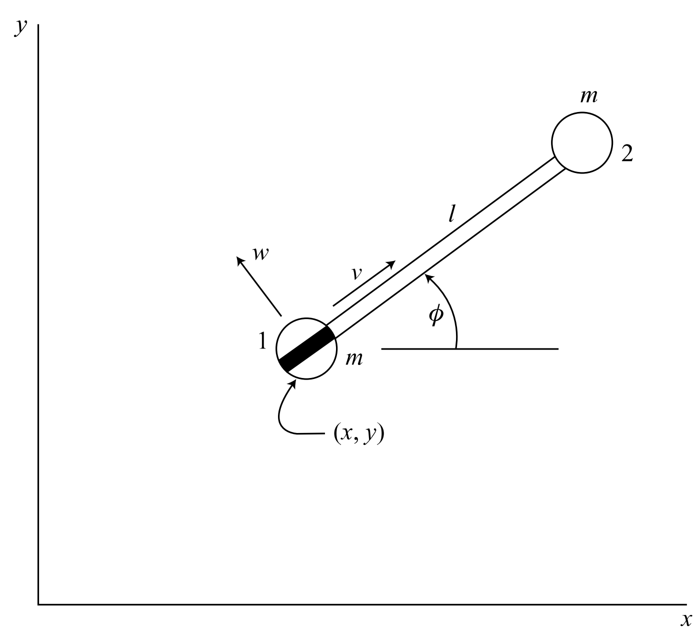

Figure 5.4.

In evaluating the transpositional terms we note that, for the unconstrained system,

$$\dot{x} = v\cos\phi - w\sin\phi \tag{5.104}$$
$$\dot{y} = v\sin\phi + w\cos\phi \tag{5.105}$$

so

$$\delta x = \cos\phi\ \delta s - \sin\phi\ \delta\eta \tag{5.106}$$
$$\delta y = \sin\phi\ \delta s + \cos\phi\ \delta\eta \tag{5.107}$$

From (5.99), we see that

$$\frac{d}{dt}(\delta s) = \delta v, \qquad \frac{d}{dt}(\delta\phi) = \delta\dot{\phi}, \qquad \frac{d}{dt}(\delta\eta) = \delta w \tag{5.108}$$

Then

$$\frac{d}{dt}(\delta x) = \cos\phi\ \delta v - \dot{\phi}\sin\phi\ \delta s - \sin\phi\ \delta w - \dot{\phi}\cos\phi\ \delta\eta \tag{5.109}$$
$$\delta\dot{x} = \cos\phi\ \delta v - v\sin\phi\ \delta\phi - \sin\phi\ \delta w - w\cos\phi\ \delta\phi \tag{5.110}$$

Now apply the actual and virtual constraint equations resulting in

$$w = 0, \qquad \delta w = 0, \qquad \delta\eta = 0 \tag{5.111}$$

and we obtain

$$\frac{d}{dt}(\delta x) - \delta\dot{x} = \sin\phi(v\delta\phi - \dot{\phi}\delta s) \tag{5.112}$$

In a similar manner, we find that

$$\frac{d}{dt}(\delta y) = \sin\phi\ \delta v + \dot{\phi}\cos\phi\ \delta s \tag{5.113}$$
$$\delta\dot{y} = \sin\phi\ \delta v + v\cos\phi\ \delta\phi \tag{5.114}$$

and thus

$$\frac{d}{dt}(\delta y) - \delta\dot{y} = \cos\phi(\dot{\phi}\delta s - v\delta\phi) \tag{5.115}$$

Notice from (5.108) that

$$\frac{d}{dt}(\delta\phi) - \delta\dot{\phi} = 0 \tag{5.116}$$

Then, with the aid of (5.103), we find that

$$\sum_{i=1}^{n} \Psi_{ji}\left[\frac{d}{dt}(\delta q_i) - \delta\dot{q}_i\right] = 0 \tag{5.117}$$

for $j = 1$ and $j = 2$. For $j = 3$, we obtain

$$\begin{aligned}\sum_{i=1}^{n} \Psi_{ji}\left[\frac{d}{dt}(\delta q_i) - \delta\dot{q}_i\right] &= (\sin^2\phi + \cos^2\phi)(\dot{\phi}\delta s - v\delta\phi)\\ &= \dot{\phi}\delta s - v\delta\phi\end{aligned} \tag{5.118}$$

The unconstrained kinetic energy is

$$T = m(v^2 + w^2) + \frac{1}{2} ml^2 \dot{\phi}^2 + ml\dot{\phi} w \tag{5.119}$$

Furthermore, all the Qs are equal to zero. Now we can substitute into the general equation (5.101), resulting in

$$2m\dot{v}\delta s + ml^2 \ddot{\phi}\delta\phi - ml\dot{\phi}(\dot{\phi}\delta s - v\delta\phi) = 0 \tag{5.120}$$

The v equation of motion, obtained by setting the coefficient of δs equal to zero, is

$$2m\dot{v} - ml\dot{\phi}^2 = 0 \tag{5.121}$$

Similarly, the coefficient of $\delta\phi$ is

$$ml^2\ddot{\phi} + mlv\dot{\phi} = 0 \tag{5.122}$$

yielding the second equation of motion. A comparison shows that this approach is somewhat less complicated than the usual Boltzmann–Hamel method.

5.4 The central equation

Derivation

Consider a system of N particles. Let us begin with d'Alembert's principle in the form

$$\sum_{i=1}^{N} (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0 \tag{5.123}$$

where $\mathbf{r}_i$ is the position vector of the ith particle and $\mathbf{F}_i$ is the applied force acting on it. The virtual displacements $\delta\mathbf{r}_i$ satisfy the instantaneous (virtual) constraints.

We can write the kinematic identity

$$\begin{aligned} \frac{d}{dt}(\mathbf{v}_i \cdot \delta\mathbf{r}_i) &= \ddot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i + \mathbf{v}_i \cdot \frac{d}{dt}(\delta\mathbf{r}_i) \\ &= \ddot{\mathbf{r}}_i \cdot \delta\mathbf{r}_i + \delta\left(\frac{\mathbf{v}_i \cdot \mathbf{v}_i}{2}\right) + \mathbf{v}_i \cdot \left[\frac{d}{dt}(\delta\mathbf{r}_i) - \delta\mathbf{v}_i\right] \end{aligned} \tag{5.124}$$

where the particle velocity $\mathbf{v}_i = \dot{\mathbf{r}}_i$. Now let

$$\delta P = \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \delta\mathbf{r}_i \tag{5.125}$$

$$\delta T = \sum_{i=1}^{N} m_i \delta\left(\frac{\mathbf{v}_i \cdot \mathbf{v}_i}{2}\right) \tag{5.126}$$

$$\delta D = \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \left[\frac{d}{dt}(\delta\mathbf{r}_i) - \delta\mathbf{v}_i\right] \tag{5.127}$$

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i \tag{5.128}$$

From (5.123)–(5.128), we obtain the *central equation* of Heun and Hamel:

$$\frac{d}{dt}(\delta P) = \delta T + \delta W + \delta D \tag{5.129}$$

Explicit form

The central equation has the same general validity as d'Alembert's principle from which it was derived. The principal assumption is that the $\delta\mathbf{r}$s satisfy the virtual constraints. There is a need, however, to put the central equation in a more convenient, usable form.

First, let us transform to generalized coordinates. In terms of velocity coefficients, we have

$$\mathbf{v}_i = \sum_{k=1}^{n} \gamma_{ik}(q,t)\,\dot{q}_k + \gamma_{it}(q,t) \qquad (i = 1,\ldots,N) \tag{5.130}$$

$$\delta\mathbf{r}_i = \sum_{k=1}^{n} \gamma_{ik}(q,t)\,\delta q_k \qquad (i = 1,\ldots,N) \tag{5.131}$$

Thus, we find that

$$\frac{d}{dt}(\delta\mathbf{r}_i) = \sum_{k=1}^{n} \gamma_{ik}\frac{d}{dt}(\delta q_k) + \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial\gamma_{il}}{\partial q_k}\dot{q}_k\delta q_l + \sum_{l=1}^{n}\frac{\partial\gamma_{il}}{\partial t}\delta q_l \tag{5.132}$$

and

$$\delta\mathbf{v}_i = \sum_{k=1}^{n} \gamma_{ik}\delta\dot{q}_k + \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial\gamma_{ik}}{\partial q_l}\dot{q}_k\delta q_l + \sum_{l=1}^{n}\frac{\partial\gamma_{it}}{\partial q_l}\delta q_l \tag{5.133}$$

But

$$\gamma_{ik} = \frac{\partial\mathbf{v}_i}{\partial\dot{q}_k} = \frac{\partial\mathbf{r}_i}{\partial q_k}, \qquad \gamma_{it} = \frac{\partial\mathbf{r}_i}{\partial t} \tag{5.134}$$

so we obtain

$$\frac{\partial\gamma_{il}}{\partial q_k} = \frac{\partial^2\mathbf{r}_i}{\partial q_k\partial q_l} = \frac{\partial\gamma_{ik}}{\partial q_l}, \qquad \frac{\partial\gamma_{il}}{\partial t} = \frac{\partial^2\mathbf{r}_i}{\partial t\partial q_l} = \frac{\partial\gamma_{it}}{\partial q_l} \tag{5.135}$$

for $i = 1,\ldots,N$ and $k, l = 1,\ldots,n$. Thus, from (5.132)–(5.135), we find that

$$\frac{d}{dt}(\delta\mathbf{r}_i) - \delta\mathbf{v}_i = \sum_{k=1}^{n}\gamma_{ik}\left[\frac{d}{dt}(\delta q_k) - \delta\dot{q}_k\right] \tag{5.136}$$

The kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i\mathbf{v}_i\cdot\mathbf{v}_i \tag{5.137}$$

where the ith particle velocity $\mathbf{v}_i$ is given by (5.130). Thus, for the *unconstrained* kinetic energy $T(q, \dot{q}, t)$, we obtain

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{n} m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_k} = \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \gamma_{ik} \tag{5.138}$$

Then, from (5.127), (5.136), and (5.138), we find that

$$\delta D = \sum_{k=1}^{n} \frac{\partial T}{\partial \dot{q}_k} \left[\frac{d}{dt}(\delta q_k) - \delta \dot{q}_k \right] \tag{5.139}$$

Let us consider the general case where the *unconstrained* kinetic energy is written in terms of quasi-velocities, that is, let $T = T^*(q, u, t)$. Then

$$\delta T^* = \sum_{i=1}^{n} \frac{\partial T^*}{\partial q_i} \delta q_i + \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \delta u_j \tag{5.140}$$

where

$$\delta q_i = \sum_{j=1}^{n} \Phi_{ij} \delta \theta_j \tag{5.141}$$

$$\delta u_j = \frac{d}{dt}(\delta \theta_j) - \left[\frac{d}{dt}(\delta \theta_j) - \delta u_j \right] \tag{5.142}$$

Thus,

$$\delta T^* = \sum_{j=1}^{n} \left[\frac{\partial T^*}{\partial \theta_j} \delta \theta_j + \frac{\partial T^*}{\partial u_j} \frac{d}{dt}(\delta \theta_j) \right] - \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \left[\frac{d}{dt}(\delta \theta_j) - \delta u_j \right] \tag{5.143}$$

where we use the notation

$$\frac{\partial T^*}{\partial \theta_j} = \sum_{i=1}^{n} \frac{\partial T^*}{\partial q_i} \Phi_{ij} \qquad (j = 1, \ldots, n) \tag{5.144}$$

The virtual work of the applied forces is

$$\delta W = \sum_{j=1}^{n} Q_r \delta \theta_r \tag{5.145}$$

where Q_r is the applied generalized force associated with u_r or $\delta\theta_r$.

Now let us integrate the central equation with respect to time over the fixed interval t_1 to t_2. The system follows the actual path in q-space; and the δqs, which equal zero at the end-points, satisfy the virtual constraints.

$$\int_{t_1}^{t_2} \left[\frac{d}{dt}(\delta P) - \delta T^* - \delta W - \delta D \right] dt = 0 \tag{5.146}$$

First consider

$$\int_{t_1}^{t_2} \frac{d}{dt}(\delta P) dt = [\delta P]_{t_1}^{t_2} = \left[\sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \delta \mathbf{r}_i \right]_{t_1}^{t_2} = 0 \tag{5.147}$$

where we note that $\delta \mathbf{r}_i = 0$ at the end-points. Next, consider the integral of δT^*. We note that, upon integrating the second term of (5.143) by parts, we obtain

$$\int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \frac{d}{dt}(\delta\theta_j)\, dt = \left[\sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \delta\theta_j \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{r=1}^{n} \frac{d}{dt}\left(\frac{\partial T^*}{\partial u_r}\right) \delta\theta_r dt \tag{5.148}$$

where the $\delta\theta$s equal zero at the end-points. Thus, we find that

$$\int_{t_1}^{t_2} \delta T^* dt = -\int_{t_1}^{t_2} \sum_{r=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T^*}{\partial u_r}\right) - \frac{\partial T^*}{\partial \theta_r} \right] \delta\theta_r dt - \int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \left[\frac{d}{dt}(\delta\theta_j) - \delta u_j \right] dt \tag{5.149}$$

Now we can write (5.146) in the form

$$\int_{t_1}^{t_2} \left\{ \sum_{r=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T^*}{\partial u_r}\right) - \frac{\partial T^*}{\partial \theta_r} - Q_r \right] \delta\theta_r + \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \left[\frac{d}{dt}(\delta\theta_j) - \delta u_j \right] \right.$$
$$\left. - \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \left[\frac{d}{dt}(\delta q_i) - \delta \dot{q}_i \right] \right\} dt = 0 \tag{5.150}$$

The end times t_1 and t_2 are arbitrary, so the integrand must equal zero continuously, resulting in the general equation

$$\sum_{r=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T^*}{\partial u_r}\right) - \frac{\partial T^*}{\partial \theta_r} - Q_r \right] \delta\theta_r + \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \left[\frac{d}{dt}(\delta\theta_j) - \delta u_j \right]$$
$$- \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \left[\frac{d}{dt}(\delta q_i) - \delta \dot{q}_i \right] = 0 \tag{5.151}$$

Up to this point, no assumptions have been made concerning the transpositional terms in (5.151). But now let us assume that

$$\frac{d}{dt}(\delta\theta_j) - \delta u_j = 0 \qquad (j = 1, \ldots, n) \tag{5.152}$$

Furthermore, let us apply the m virtual constraints by letting $\delta\theta_r = 0$ for $r = n - m + 1, \ldots, n$. The remaining $(n - m)$ $\delta\theta$s are *independent*. Then we obtain the *explicit form of the central equation*:

$$\sum_{r=1}^{n-m} \left[\frac{d}{dt}\left(\frac{\partial T^*}{\partial u_r}\right) - \frac{\partial T^*}{\partial \theta_r} - Q_r \right] \delta\theta_r - \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \left[\frac{d}{dt}(\delta q_i) - \delta \dot{q}_i \right] = 0 \tag{5.153}$$

Notice that this result is similar to (5.101), the transpositional form of the Boltzmann–Hamel equation.

The $(n - m)$ differential equations of motion are obtained by writing the last term of (5.153) as a function of the $\delta\theta$s, and then setting the coefficient of each $\delta\theta_r$ in (5.153) equal to zero. As mentioned earlier, the kinetic energy $T^*(q, u, t)$ may be written for the *constrained* system, that is, the order of differentiation and the application of constraints makes no difference in this case. But the kinetic energy $T(q, \dot{q}, t)$ must be written for the *unconstrained* system.

The Boltzmann–Hamel equation can be derived from the general result given in (5.151). Let us begin by first noting that

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \Psi_{ji} \qquad (i = 1, \dots, n) \tag{5.154}$$

and then recalling from (5.47) that

$$\frac{d}{dt}(\delta\theta_j) - \delta u_j - \sum_{i=1}^{n} \Psi_{ji} \left[\frac{d}{dt}(\delta q_i) - \delta \dot{q}_i \right] = \sum_{i=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial \Psi_{ji}}{\partial q_k} - \frac{\partial \Psi_{jk}}{\partial q_i} \right) \dot{q}_k \delta q_i$$
$$+ \sum_{i=1}^{n} \left(\frac{\partial \Psi_{ji}}{\partial t} - \frac{\partial \Psi_{jt}}{\partial q_i} \right) \delta q_i \qquad (j = 1, \dots, n) \tag{5.155}$$

Moreover,

$$\dot{q}_k = \sum_{l=1}^{n} \Phi_{kl} u_l + \Phi_{kt} \qquad (k = 1, \dots, n) \tag{5.156}$$

$$\delta q_i = \sum_{r=1}^{n} \Phi_{ir} \delta\theta_r \qquad (i = 1, \dots, n) \tag{5.157}$$

Then (5.151) takes the form

$$\sum_{r=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial T^*}{\partial u_r} \right) - \frac{\partial T^*}{\partial \theta_r} - Q_r + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial T^*}{\partial u_j} \left(\frac{\partial \Psi_{ji}}{\partial q_k} - \frac{\partial \Psi_{jk}}{\partial q_i} \right) \right.$$
$$\left. \times \left(\sum_{l=1}^{n} \Phi_{kl} u_l + \Phi_{kt} \right) \Phi_{ir} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \left(\frac{\partial \Psi_{ji}}{\partial t} - \frac{\partial \Psi_{jt}}{\partial q_i} \right) \Phi_{ir} \right] \delta\theta_r = 0 \tag{5.158}$$

Now apply the instantaneous constraints by setting $\delta\theta_r = 0$ for $r = n - m + 1, \dots, n$. The remaining $(n - m)$ $\delta\theta$s are *independent* so each coefficient must equal zero. Thus we obtain

$$\frac{d}{dt} \left(\frac{\partial T^*}{\partial u_r} \right) - \frac{\partial T^*}{\partial \theta_r} + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial T^*}{\partial u_j} \left(\frac{\partial \Psi_{ji}}{\partial q_k} - \frac{\partial \Psi_{jk}}{\partial q_i} \right) \Phi_{kl} \Phi_{ir} u_l$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial T^*}{\partial u_j} \left(\frac{\partial \Psi_{ji}}{\partial q_k} - \frac{\partial \Psi_{jk}}{\partial q_i} \right) \Phi_{kt} \Phi_{ir}$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \left(\frac{\partial \Psi_{ji}}{\partial t} - \frac{\partial \Psi_{jt}}{\partial q_i} \right) \Phi_{ir} = Q_r \qquad (r = 1, \dots, n - m) \tag{5.159}$$

This is a detailed form of the Boltzmann–Hamel equation, as in (4.82). Here the derivation involves an integral method using the central equation. We note from (5.155) that, in spite of transpositional terms in the integral expressed in (5.150), the varied path is actually determined by δqs or $\delta\theta$s which satisfy the virtual constraints.

Example 5.3 A thin uniform disk of mass m and radius r rolls without slipping on the horizontal xy-plane (Fig. 5.5). Let us find the differential equations of motion, using the explicit central equation, namely,

$$\sum_{r=1}^{n-m}\left[\frac{d}{dt}\left(\frac{\partial T^*}{\partial u_r}\right)-\frac{\partial T^*}{\partial \theta_r}-Q_r\right]\delta\theta_r-\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\left[\frac{d}{dt}(\delta q_i)-\delta\dot{q}_i\right]=0 \qquad (5.160)$$

As generalized coordinates, let us choose $(\phi, \theta, \psi, x, y)$ where (ϕ, θ, ψ) are Euler angles and where (x, y) is the location of the contact point C. The independent us are $(\omega_d, \dot{\theta}, \Omega)$. There are two additional us that are set equal to zero to represent the nonholonomic constraints. Thus, we have

$$u_1 = \omega_d = \dot{\phi}\sin\theta \qquad (5.161)$$

$$u_2 = \dot{\theta} \qquad (5.162)$$

$$u_3 = \Omega = \dot{\phi}\cos\theta + \dot{\psi} \qquad (5.163)$$

$$u_4 = r\dot{\psi}\cos\phi + \dot{x} = 0 \qquad (5.164)$$

$$u_5 = r\dot{\psi}\sin\phi + \dot{y} = 0 \qquad (5.165)$$

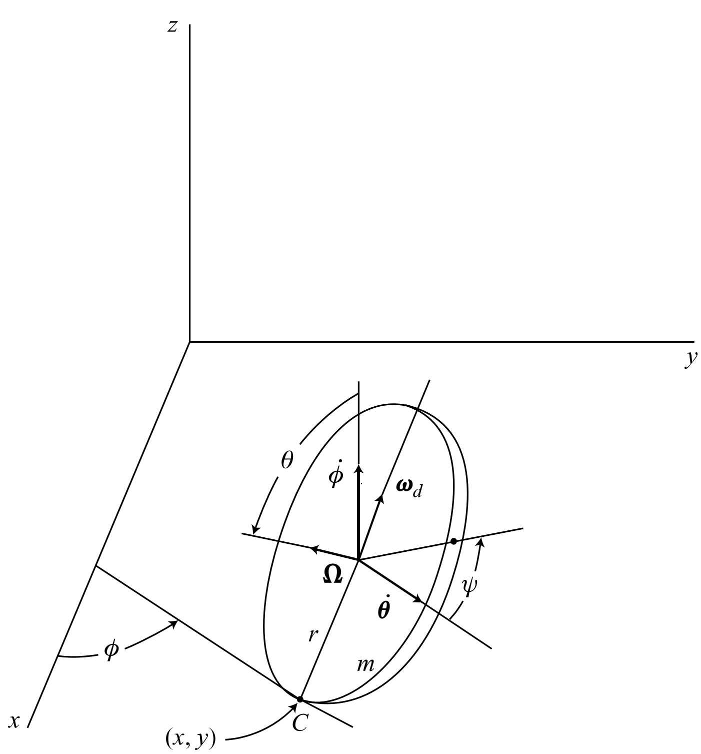

Figure 5.5.

The two constraint equations state that there is no slipping at the contact point C. With this assumption, the contact point moves with a speed $r\dot{\psi}$ on the xy-plane. The rotations $\dot{\phi}$ and $\dot{\theta}$ occur about C and do not cause it to move.

The *constrained* kinetic energy $T^*(q, u)$ is equal to the sum of the translational and rotational portions. We obtain

$$\begin{aligned} T^* &= \frac{1}{2}mr^2(\dot{\theta}^2 + \Omega^2) + \frac{1}{2}\left(\frac{mr^2}{2}\right)\Omega^2 + \frac{1}{2}\left(\frac{mr^2}{4}\right)\left(\omega_d^2 + \dot{\theta}^2\right) \\ &= \frac{1}{8}mr^2u_1^2 + \frac{5}{8}mr^2u_2^2 + \frac{3}{4}mr^2u_3^2 \end{aligned} \tag{5.166}$$

where we note that the moments of inertia about the center are

$$I_a = \frac{1}{2}mr^2, \qquad I_t = \frac{1}{4}mr^2 \tag{5.167}$$

The *unconstrained* kinetic energy $T(q, \dot{q})$ is more complicated. Again we sum the translational and rotational parts, obtaining

$$\begin{aligned} T &= \frac{1}{2}m[(\dot{x} - r\dot{\phi}\cos\phi\cos\theta + r\dot{\theta}\sin\phi\sin\theta)^2 + (\dot{y} - r\dot{\phi}\sin\phi\cos\theta - r\dot{\theta}\cos\phi\sin\theta)^2 \\ &\quad + r^2\dot{\theta}^2\cos^2\theta] + \frac{1}{8}mr^2(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2) + \frac{1}{4}mr^2(\dot{\phi}\cos\theta + \dot{\psi})^2 \\ &= \frac{1}{8}mr^2\dot{\phi}^2(1 + 5\cos^2\theta) + \frac{5}{8}mr^2\dot{\theta}^2 + \frac{1}{4}mr^2\dot{\psi}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &\qquad + \frac{1}{2}mr^2\dot{\phi}\dot{\psi}\cos\theta - mr\dot{\phi}\dot{x}\cos\phi\cos\theta + mr\dot{\theta}\dot{x}\sin\phi\sin\theta \\ &\qquad - mr\dot{\phi}\dot{y}\sin\phi\cos\theta - mr\dot{\theta}\dot{y}\cos\phi\sin\theta \end{aligned} \tag{5.168}$$

We wish to express the last term of (5.160) in terms of us and $\delta\theta$s. If (5.161)–(5.165) are solved for the $\dot{q}$s, the result is

$$\dot{\phi} = u_1 \csc\theta \tag{5.169}$$

$$\dot{\theta} = u_2 \tag{5.170}$$

$$\dot{\psi} = -u_1 \cot\theta + u_3 \tag{5.171}$$

$$\dot{x} = ru_1 \cos\phi \cot\theta - ru_3 \cos\phi + u_4 \tag{5.172}$$

$$\dot{y} = ru_1 \sin\phi \cot\theta - ru_3 \sin\phi + u_5 \tag{5.173}$$

where $u_4 = u_5 = 0$. Similarly,

$$\delta\phi = \csc\theta\ \delta\theta_1 \tag{5.174}$$

$$\delta\theta = \delta\theta_2 \tag{5.175}$$

$$\delta\psi = -\cot\theta\ \delta\theta_1 + \delta\theta_3 \tag{5.176}$$

$$\delta x = r\cos\phi\cot\theta\ \delta\theta_1 - r\cos\phi\ \delta\theta_3 \tag{5.177}$$

$$\delta y = r\sin\phi\cot\theta\ \delta\theta_1 - r\sin\phi\ \delta\theta_3 \tag{5.178}$$

In addition we assume that, for all j,

$$\frac{d}{dt}(\delta\theta_j) - \delta u_j = 0 \tag{5.179}$$

Then we find that

$$\frac{d}{dt}(\delta\phi) - \delta\dot{\phi} = \csc\theta\cot\theta(u_1\delta\theta_2 - u_2\delta\theta_1) \tag{5.180}$$

$$\frac{\partial T}{\partial\dot{\phi}} = \frac{1}{4}mr^2u_1\sin\theta + \frac{3}{2}mr^2u_3\cos\theta \tag{5.181}$$

and therefore

$$\frac{\partial T}{\partial\dot{\phi}}\left[\frac{d}{dt}(\delta\phi) - \delta\dot{\phi}\right] = \frac{1}{4}mr^2\cot\theta(u_1 + 6u_3\cot\theta)(u_1\delta\theta_2 - u_2\delta\theta_1) \tag{5.182}$$

Since (5.162) is integrable and (5.179) applies, we obtain

$$\frac{d}{dt}(\delta\theta) - \delta\dot{\theta} = 0 \tag{5.183}$$

Similarly, we find that

$$\frac{\partial T}{\partial\dot{\psi}}\left[\frac{d}{dt}(\delta\psi) - \delta\dot{\psi}\right] = -\frac{1}{2}mr^2u_3\csc^2\theta(u_1\delta\theta_2 - u_2\delta\theta_1) \tag{5.184}$$

and

$$\begin{aligned}\frac{\partial T}{\partial\dot{x}}\left[\frac{d}{dt}(\delta x) - \delta\dot{x}\right] &= mr^2(u_2\sin\phi\cos\phi\csc\theta - u_3\cos^2\phi\csc^2\theta)(u_1\delta\theta_2 - u_2\delta\theta_1)\\ &\quad + mr^2(u_2\sin^2\phi - u_3\sin\phi\cos\phi\csc\theta)(u_1\delta\theta_3 - u_3\delta\theta_1)\end{aligned} \tag{5.185}$$

Furthermore,

$$\begin{aligned}\frac{\partial T}{\partial\dot{y}}\left[\frac{d}{dt}(\delta y) - \delta\dot{y}\right] &= -mr^2(u_2\sin\phi\cos\phi\csc\theta + u_3\sin^2\phi\csc^2\theta)(u_1\delta\theta_2 - u_2\delta\theta_1)\\ &\quad + mr^2(u_2\cos^2\phi + u_3\sin\phi\cos\phi\csc\theta)(u_1\delta\theta_3 - u_3\delta\theta_1)\end{aligned} \tag{5.186}$$

Adding (5.182)–(5.186), we obtain

$$\begin{aligned}\sum_{i=1}^{5}\frac{\partial T}{\partial\dot{q}_i}\left[\frac{d}{dt}(\delta q_i) - \delta\dot{q}_i\right] &= mr^2\left(\frac{1}{4}u_1\cot\theta - \frac{3}{2}u_3\right)(u_1\delta\theta_2 - u_2\delta\theta_1)\\ &\quad + mr^2u_2(u_1\delta\theta_3 - u_3\delta\theta_1)\\ &= mr^2\left(-\frac{1}{4}u_1u_2\cot\theta + \frac{1}{2}u_2u_3\right)\delta\theta_1\\ &\quad + mr^2\left(\frac{1}{4}u_1^2\cot\theta - \frac{3}{2}u_1u_3\right)\delta\theta_2\\ &\quad + mr^2u_1u_2\delta\theta_3\end{aligned} \tag{5.187}$$

Referring to (5.166), we have

$$\sum_{r=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T^*}{\partial u_r}\right)-\frac{\partial T^*}{\partial \theta_r}-Q_r\right]\delta\theta_r$$
$$=\frac{1}{4}mr^2\dot{u}_1\delta\theta_1+\left(\frac{5}{4}mr^2\dot{u}_2+mgr\cos\theta\right)\delta\theta_2+\frac{3}{2}mr^2\dot{u}_3\delta\theta_3 \tag{5.188}$$

where

$$Q_1=0,\qquad Q_2=-mgr\cos\theta,\qquad Q_3=0 \tag{5.189}$$

Finally, using the explicit form of the central equation, we obtain the equations of motion. The u_1 equation, found by setting the coefficient of $\delta\theta_1$ equal to zero, is

$$\frac{1}{4}mr^2\dot{u}_1+\frac{1}{4}mr^2u_1u_2\cot\theta-\frac{1}{2}mr^2u_2u_3=0 \tag{5.190}$$

Similarly, the u_2 equation is

$$\frac{5}{4}mr^2\dot{u}_2-\frac{1}{4}mr^2u_1^2\cot\theta+\frac{3}{2}mr^2u_1u_3=-mgr\cos\theta \tag{5.191}$$

The u_3 equation is

$$\frac{3}{2}mr^2\dot{u}_3-mr^2u_1u_2=0 \tag{5.192}$$

Using the original notation for angular velocity components, we can write the equations of motion as follows:

$$\frac{1}{4}mr^2\dot{\omega}_d+\frac{1}{4}mr^2\omega_d\dot{\theta}\cot\theta-\frac{1}{2}mr^2\dot{\theta}\Omega=0 \tag{5.193}$$

$$\frac{5}{4}mr^2\ddot{\theta}-\frac{1}{4}mr^2\omega_d^2\cot\theta+\frac{3}{2}mr^2\omega_d\Omega=-mgr\cos\theta \tag{5.194}$$

$$\frac{3}{2}mr^2\dot{\Omega}-mr^2\omega_d\dot{\theta}=0 \tag{5.195}$$

These equations are identical with (4.259)–(4.261) obtained earlier.

5.5 Suslov's principle

Dependent and independent coordinates

Thus far, we have assumed that the linear nonholonomic constraint equations have the form

$$\sum_{i=1}^{n}a_{ji}(q,t)\dot{q}_i+a_{jt}(q,t)=0\qquad (j=1,\ldots,m) \tag{5.196}$$

This form has the characteristic that all the qs are treated equally. Now let us change the viewpoint and arbitrarily designate m of the qs as *dependent*; whereas, the remaining

$(n-m)$ qs are termed *independent*. Equation (5.196) can be solved for the dependent $\dot{q}_D$s in terms of the independent $\dot{q}_I$s. We obtain m nonholonomic constraint equations written in the form

$$\dot{q}_D = \sum_{I=1}^{n-m} B_{DI}(q,t)\dot{q}_I + B_{Dt}(q,t) \qquad (D = n-m+1,\ldots,n) \tag{5.197}$$

or, more briefly,

$$\dot{q}_D = \phi_D(q, \dot{q}_I, t) \qquad (D = n-m+1,\ldots,n) \tag{5.198}$$

Suslov's principle

Let us begin with the general nonholonomic form of Hamilton's principle, namely,

$$\int_{t_1}^{t_2} (\delta T + \delta W)dt = 0 \tag{5.199}$$

where each varied path satisfies the virtual constraints, and the δqs are equal to zero at the end-points in extended q-space (Fig. 5.1).

Now introduce the *constrained* kinetic energy $T^0(q, \dot{q}_I, t)$ which is obtained from the unconstrained kinetic energy $T(q, \dot{q}, t)$ by substituting for the $\dot{q}_D$s using the constraint equations (5.197). For any realizable motion, the energies T^0 and T are equal. Their variations δT^0 and δT are not equal, however. To see how this comes about, consider

$$\delta T = \sum_{i=1}^{n} \frac{\partial T}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i}\delta \dot{q}_i \tag{5.200}$$

For a given varied path, we can take

$$\frac{d}{dt}(\delta q_i) - \delta \dot{q}_i = 0 \qquad (i = 1,\ldots,n) \tag{5.201}$$

and therefore we obtain that, in (5.199),

$$\delta T = \sum_{i=1}^{n} \frac{\partial T}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) \tag{5.202}$$

For the constrained system, however, we know that at least m of the transpositional expressions of (5.201) must be nonzero. Let us assume that

$$\frac{d}{dt}(\delta q_I) - \delta \dot{q}_I = 0 \qquad (I = 1,\ldots,n-m) \tag{5.203}$$

$$\frac{d}{dt}(\delta q_D) - \delta \dot{q}_D \neq 0 \qquad (D = n-m+1,\ldots,n) \tag{5.204}$$

where

$$\delta \dot{q}_D = \delta \phi_D = \sum_{i=1}^{n} \frac{\partial \phi_D}{\partial q_i}\delta q_i + \sum_{I=1}^{n-m} \frac{\partial \phi_D}{\partial \dot{q}_I}\delta \dot{q}_I \tag{5.205}$$

Then we see that

$$\delta T^0 = \sum_{i=1}^{n} \frac{\partial T}{\partial q_i}\delta q_i + \sum_{I=1}^{n-m} \frac{\partial T}{\partial \dot{q}_I}\frac{d}{dt}(\delta q_I) + \sum_{D=n-m+1}^{n} \frac{\partial T}{\partial \dot{q}_D}\delta \dot{q}_D \tag{5.206}$$

A comparison of (5.202) and (5.206) results in

$$\delta T = \delta T^0 + \sum_{D=n-m+1}^{n} \frac{\partial T}{\partial \dot{q}_D}\left[\frac{d}{dt}(\delta q_D) - \delta \dot{q}_D\right] \tag{5.207}$$

Now substitute the expression for δT from (5.207) into (5.199) which is Hamilton's principle. The result is

$$\int_{t_1}^{t_2} \left\{\delta T^0 + \delta W + \sum_{D=n-m+1}^{n} \frac{\partial T}{\partial \dot{q}_D}\left[\frac{d}{dt}(\delta q_D) - \delta \dot{q}_D\right]\right\} dt = 0 \tag{5.208}$$

This is *Suslov's principle*.

The virtual work is

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i = \sum_{I=1}^{n-m} Q_I \delta q_I + \sum_{D=n-m+1}^{n} Q_D \delta q_D \tag{5.209}$$

where the Qs are generalized applied forces for the case of unconstrained δqs. We actually have constrained δqs, however, with

$$\delta q_D = \sum_{I=1}^{n-m} B_{DI} \delta q_I \tag{5.210}$$

Hence, the virtual work becomes

$$\delta W = \sum_{I=1}^{n-m} Q_I \delta q_I + \sum_{I=1}^{n-m} \sum_{D=n-m+1}^{n} Q_D B_{DI} \delta q_I \tag{5.211}$$

or

$$\delta W = \sum_{I=1}^{n-m} Q_I^0 \delta q_I \tag{5.212}$$

where the generalized applied force associated with q_I in the *constrained* system is

$$Q_I^0 = Q_I + \sum_{D=n-m+1}^{n} Q_D B_{DI} \qquad (I = 1, \ldots, n-m) \tag{5.213}$$

Equations of motion

To obtain the differential equations of motion from Suslov's principle, first let us write

$$\delta T^0 = \sum_{I=1}^{n-m} \frac{\partial T^0}{\partial q_I}\delta q_I + \sum_{D=n-m+1}^{n} \frac{\partial T^0}{\partial q_D}\delta q_D + \sum_{I=1}^{n-m} \frac{\partial T^0}{\partial \dot{q}_I}\frac{d}{dt}(\delta q_I) \tag{5.214}$$

where we have used (5.203). Now integrate δT^0 with respect to time, using integration by parts on the last term and noting that δq_I vanishes at t_1 and t_2. Upon multiplying by -1 and recalling (5.210), the integral of (5.208) can be written in the form

$$\int_{t_1}^{t_2} \left\{ \sum_{I=1}^{n-m} \left[\frac{d}{dt}\left(\frac{\partial T^0}{\partial \dot{q}_I}\right) - \left(\frac{\partial T^0}{\partial q_I} + \sum_{D=n-m+1}^{n} \frac{\partial T^0}{\partial q_D} B_{DI}\right) - Q_I^0 \right] \delta q_I \right.$$
$$\left. - \sum_{D=n-m+1}^{n} \frac{\partial T}{\partial \dot{q}_D}\left[\frac{d}{dt}(\delta q_D) - \delta \dot{q}_D\right] \right\} dt = 0 \tag{5.215}$$

The transpositional term can be written as a homogeneous linear expression in the δq_Is which are independent. Hence, the integrand must vanish for all t, resulting in the *explicit form of Suslov's principle*:

$$\sum_{I=1}^{n-m} \left[\frac{d}{dt}\left(\frac{\partial T^0}{\partial \dot{q}_I}\right) - \left(\frac{\partial T^0}{\partial q_I} + \sum_{D=n-m+1}^{n} \frac{\partial T^0}{\partial q_D} B_{DI}\right) - Q_I^0 \right] \delta q_I$$
$$- \sum_{D=n-m+1}^{n} \frac{\partial T}{\partial \dot{q}_D}\left[\frac{d}{dt}(\delta q_D) - \delta \dot{q}_D\right] = 0 \tag{5.216}$$

The equations of motion are obtained by expressing the last term in terms of δq_Is and then equating to zero the coefficient of each δq_I. This results in $(n - m)$ second-order differential equations. There are also m kinematical constraint equations, giving a total of n equations to solve for the n qs.

Comparing this result with the explicit central equation, (5.153), we see that the Suslov equation has the advantage that the last summation is over the m dependent $\dot{q}$s only rather than over the complete set of n. On the other hand, Suslov's equation has the possible disadvantage that it does not allow quasi-velocities to be used.

Example 5.4 As an example of the application of Suslov's equation (5.216), consider the motion of a dumbbell with a knife-edge constraint sliding on the horizontal xy-plane (Fig. 5.6). As generalized coordinates, let us choose (ϕ, x, y), where (ϕ, x) are considered independent and y is dependent.

The nonholonomic constraint equation is

$$\dot{q}_D = \dot{y} = \dot{x} \tan \phi \tag{5.217}$$

and the corresponding virtual constraint is

$$\delta q_D = \delta y = \delta x \tan \phi \tag{5.218}$$

The unconstrained kinetic energy, from (1.127), is

$$T = m(\dot{x} + \dot{y}^2) + \frac{1}{2} m l^2 \dot{\phi}^2 + m l \dot{\phi}(-\dot{x} \sin \phi + \dot{y} \cos \phi) \tag{5.219}$$

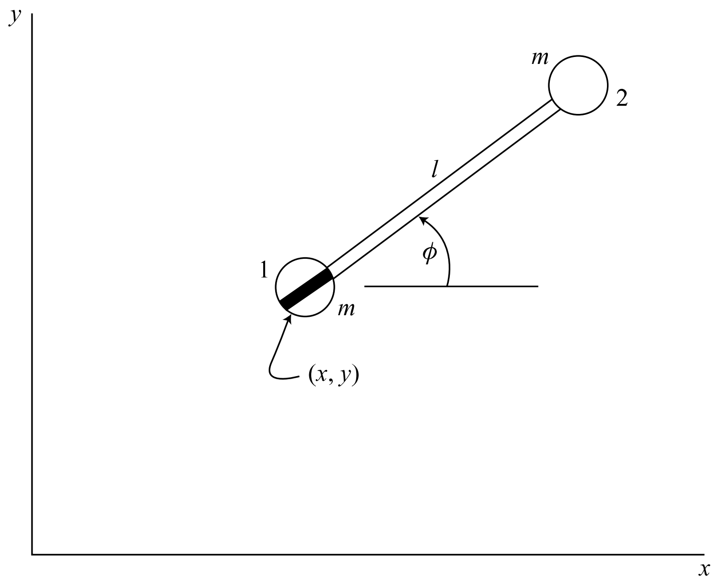

Figure 5.6.

The constrained kinetic energy, upon substitution from (5.217), is

$$T^0 = m(1 + \tan^2\phi)\dot{x}^2 + \frac{1}{2}ml^2\dot{\phi}^2 = m\dot{x}^2\sec^2\phi + \frac{1}{2}ml^2\dot{\phi}^2 \tag{5.220}$$

We see that

$$\frac{d}{dt}(\delta y) - \delta\dot{y} = \sec^2\phi\,(\dot{\phi}\delta x - \dot{x}\delta\phi) \tag{5.221}$$

where we note from (5.203) that

$$\frac{d}{dt}(\delta x) - \delta\dot{x} = 0 \tag{5.222}$$

Then, using (5.217), we obtain

$$\frac{\partial T}{\partial\dot{y}}\left[\frac{d}{dt}(\delta y) - \delta\dot{y}\right] = (2m\dot{x}\tan\phi + ml\dot{\phi}\cos\phi)\sec^2\phi\,(\dot{\phi}\delta x - \dot{x}\delta\phi) \tag{5.223}$$

Differentiation of the constrained kinetic energy yields

$$\frac{\partial T^0}{\partial\phi} = 2m\dot{x}^2\sec^2\phi\tan\phi, \qquad \frac{\partial T^0}{\partial x} = \frac{\partial T^0}{\partial y} = 0 \tag{5.224}$$

$$\frac{\partial T^0}{\partial\dot{\phi}} = ml^2\dot{\phi}, \qquad \frac{\partial T^0}{\partial\dot{x}} = 2m\dot{x}\sec^2\phi \tag{5.225}$$

Also,

$$Q_\phi^0 = 0, \qquad Q_x^0 = 0 \tag{5.226}$$

Finally, a substitution in Suslov's equation, (5.216), results in

$$(ml^2\ddot{\phi} + ml\dot{\phi}\dot{x}\sec\phi)\delta\phi + (2m\ddot{x} + 2m\dot{\phi}\dot{x}\tan\phi - ml\dot{\phi}^2\cos\phi)\sec^2\phi\;\delta x = 0 \tag{5.227}$$

The virtual displacements $\delta\phi$ and δx are arbitrary and $\sec^2\phi$ is not equal to zero, so each coefficient must equal zero. Thus, we obtain the equations of motion

$$ml^2\ddot{\phi} + ml\dot{\phi}\dot{x}\sec\phi = 0 \tag{5.228}$$

$$2m\ddot{x} + 2m\dot{\phi}\dot{x}\tan\phi - ml\dot{\phi}^2\cos\phi = 0 \tag{5.229}$$

Note that $\dot{x}\sec\phi$ is equal to the longitudinal velocity component of the particles, that is, the velocity along the rod.

These equations of motion agree with (4.31) and (4.32) of Example 4.1 on page 220, obtained using Maggi's equation, if one substitutes the differentiated constraint equation, namely

$$\ddot{y} = \ddot{x}\tan\phi + \dot{\phi}\dot{x}\sec^2\phi \tag{5.230}$$

Example 5.5 A uniform disk of mass m and radius r rolls without slipping on the horizontal xy-plane (Fig. 5.7). We wish to obtain the equations of motion using the Suslov equation, (5.216).

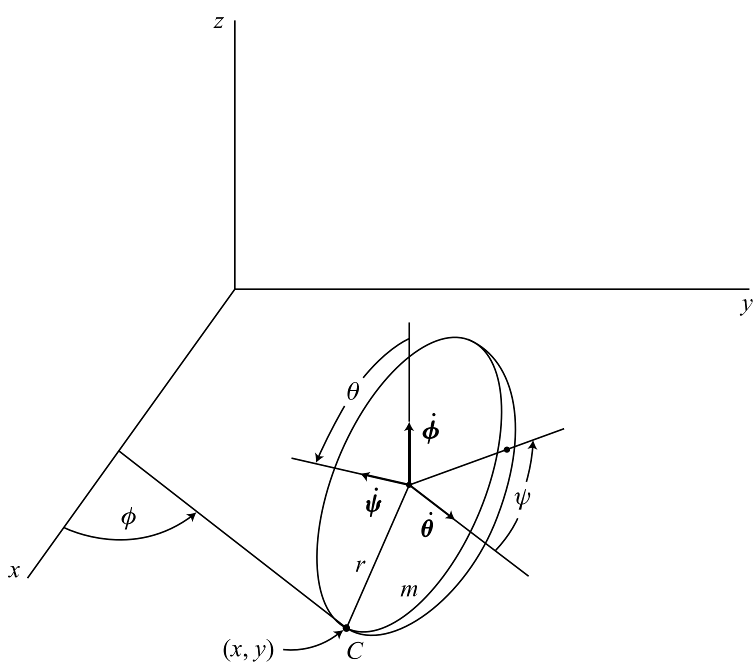

Figure 5.7.

The configuration of the system is given by the location of the contact point C and the Euler angles. Thus, we can choose $(\phi, \theta, \psi, x, y)$ as generalized coordinates, with the first three as independent q_Is and the last two as dependent q_Ds. The constraint equations of the form $\dot{q}_D = \phi_D(q, \dot{q}_I)$ are

$$\dot{x} = -r\dot{\psi}\cos\phi \tag{5.231}$$

$$\dot{y} = -r\dot{\psi}\sin\phi \tag{5.232}$$

and the corresponding virtual displacements are

$$\delta x = -r\cos\phi\,\delta\psi \tag{5.233}$$

$$\delta y = -r\sin\phi\,\delta\psi \tag{5.234}$$

The unconstrained kinetic energy is equal to that due to the translation of the center of mass, plus that due to rotation about the center of mass. Thus, we obtain

$$\begin{aligned} T &= \frac{1}{2}m[(\dot x\cos\phi + \dot y\sin\phi - r\dot\phi\cos\theta)^2 \\ &\quad + (-\dot x\sin\phi + \dot y\cos\phi - r\dot\theta\sin\theta)^2 + r^2\dot\theta^2\cos^2\theta] \\ &\quad + \frac{1}{8}mr^2(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{4}mr^2(\dot\phi\cos\theta + \dot\psi)^2 \\ &= \frac{1}{8}mr^2\dot\phi^2(1+5\cos^2\theta) + \frac{5}{8}mr^2\dot\theta^2 + \frac{1}{4}mr^2\dot\psi^2 + \frac{1}{2}m(\dot x^2 + \dot y^2) \\ &\quad + \frac{1}{2}mr^2\dot\phi\dot\psi\cos\theta - mr\dot\phi\cos\theta(\dot x\cos\phi + \dot y\sin\phi) \\ &\quad + mr\dot\theta\sin\theta(\dot x\sin\phi - \dot y\cos\phi) \end{aligned} \tag{5.235}$$

The constrained kinetic energy, obtained by substituting the constraint equations for $\dot x$ and $\dot y$ into (5.235), is

$$T^0 = \frac{1}{8}mr^2\dot\phi^2(1+5\cos^2\theta) + \frac{5}{8}mr^2\dot\theta^2 + \frac{3}{4}mr^2\dot\psi^2 + \frac{3}{2}mr^2\dot\phi\dot\psi\cos\theta \tag{5.236}$$

This assumes that there is no slipping at the contact point C.

The generalized applied forces for the constrained system are

$$Q_1^0 = 0, \qquad Q_2^0 = -mgr\cos\theta, \qquad Q_3^0 = 0 \tag{5.237}$$

In accordance with (5.203), we have

$$\frac{d}{dt}(\delta\phi) - \delta\dot\phi = 0, \qquad \frac{d}{dt}(\delta\theta) - \delta\dot\theta = 0, \qquad \frac{d}{dt}(\delta\psi) - \delta\dot\psi = 0 \tag{5.238}$$

Then, differentiation of (5.231) and (5.233) results in

$$\frac{d}{dt}(\delta x) - \delta\dot x = r\sin\phi(\dot\phi\delta\psi - \dot\psi\delta\phi) \tag{5.239}$$

Similarly,

$$\frac{d}{dt}(\delta y) - \delta\dot y = -r\cos\phi(\dot\phi\delta\psi - \dot\psi\delta\phi) \tag{5.240}$$

Furthermore,

$$\begin{aligned} \frac{\partial T}{\partial \dot x} &= m\dot x - mr\dot\phi\cos\phi\cos\theta + mr\dot\theta\sin\phi\sin\theta \\ &= -mr\dot\phi\cos\phi\cos\theta + mr\dot\theta\sin\phi\sin\theta - mr\dot\psi\cos\phi \end{aligned} \tag{5.241}$$

and

$$\frac{\partial T}{\partial \dot{y}} = -mr\dot{\phi}\sin\phi\cos\theta - mr\dot{\theta}\cos\phi\sin\theta - mr\dot{\psi}\sin\phi \tag{5.242}$$

Hence, we find that

$$\frac{\partial T}{\partial \dot{x}}\left[\frac{d}{dt}(\delta x) - \delta\dot{x}\right] + \frac{\partial T}{\partial \dot{y}}\left[\frac{d}{dt}(\delta y) - \delta\dot{y}\right] = mr^2\dot{\theta}\sin\theta(\dot{\phi}\delta\psi - \dot{\psi}\delta\phi) \tag{5.243}$$

To obtain the ϕ equation of motion, we first evaluate

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T^0}{\partial \dot{\phi}}\right) - \frac{\partial T^0}{\partial \phi} &= \frac{1}{4}mr^2\ddot{\phi}(1 + 5\cos^2\theta) + \frac{3}{2}mr^2\ddot{\psi}\cos\theta \\ &\quad - \frac{5}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta - \frac{3}{2}mr^2\dot{\theta}\dot{\psi}\sin\theta\end{aligned} \tag{5.244}$$

Then, using Suslov's equation, the coefficient of $\delta\phi$ is the ϕ equation:

$$\begin{aligned}&\frac{1}{4}mr^2\ddot{\phi}(1 + 5\cos^2\theta) + \frac{3}{2}mr^2\ddot{\psi}\cos\theta - \frac{5}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta \\ &\quad - \frac{1}{2}mr^2\dot{\theta}\dot{\psi}\sin\theta = 0\end{aligned} \tag{5.245}$$

Next, we obtain

$$\frac{d}{dt}\left(\frac{\partial T^0}{\partial \dot{\theta}}\right) - \frac{\partial T^0}{\partial \theta} = \frac{5}{4}mr^2\ddot{\theta} + \frac{5}{4}mr^2\dot{\phi}^2\sin\theta\cos\theta + \frac{3}{2}mr^2\dot{\phi}\dot{\psi}\sin\theta \tag{5.246}$$

and the θ equation is

$$\frac{5}{4}mr^2\ddot{\theta} + \frac{5}{4}mr^2\dot{\phi}^2\sin\theta\cos\theta + \frac{3}{2}mr^2\dot{\phi}\dot{\psi}\sin\theta = -mgr\cos\theta \tag{5.247}$$

Finally,

$$\frac{d}{dt}\left(\frac{\partial T^0}{\partial \dot{\psi}}\right) - \frac{\partial T^0}{\partial \psi} = \frac{3}{2}mr^2\ddot{\psi} + \frac{3}{2}mr^2\ddot{\phi}\cos\theta - \frac{3}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta \tag{5.248}$$

and the ψ equation is

$$\frac{3}{2}mr^2\ddot{\psi} + \frac{3}{2}mr^2\ddot{\phi}\cos\theta - \frac{5}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta = 0 \tag{5.249}$$

These equations of motion are equivalent to those obtained in Example 4.2 on page 222 using Maggi's equation, but the Suslov approach is simpler and more direct.

5.6 Summary of integral methods

A review of integral methods in the analysis of nonholonomic systems must begin with Hamilton's principle, namely,

$$\int_{t_1}^{t_2} (\delta T + \delta W)dt = 0 \tag{5.250}$$

Here the kinetic energy $T(q, \dot{q}, t)$ is unconstrained and

$$\delta T = \sum_{i=1}^{n} \frac{\partial T}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i) \tag{5.251}$$

since we assume that

$$\frac{d}{dt}(\delta q_i) - \delta \dot{q}_i = 0 \tag{5.252}$$

Furthermore, the varied path determined by the δqs satisfies the virtual constraints of (5.4) but does not satisfy, in general, the actual constraints of (5.11).

Hamilton's principle leads to equations of motion of the form given by Lagrange's principle or Maggi's equation.

The other integral methods can be derived directly from Hamilton's principle. For example, if the kinetic energy in terms of quasi-velocities is $T^*(q, u, t)$ and we use the substitution

$$\delta T = \delta T^* + \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \left[\frac{d}{dt}(\delta q_i) - \delta \dot{q}_i \right] \tag{5.253}$$

in (5.250), the result after integration by parts reduces to the explicit form of the central equation, (5.153).

On the other hand, if we designate the qs as either independent or dependent, and write the *constrained* kinetic energy $T^0(q, \dot{q}_I, t)$, then the substitution

$$\delta T = \delta T^0 + \sum_{D=n-m+1}^{n} \frac{\partial T}{\partial \dot{q}_D} \left[\frac{d}{dt}(\delta q_D) - \delta \dot{q}_D \right] \tag{5.254}$$

in (5.250) results in the explicit form of Suslov's principle, (5.216). Here we assume that

$$\frac{d}{dt}(\delta q_I) - \delta \dot{q}_I = 0 \qquad (I = 1, \ldots, n - m) \tag{5.255}$$

All these methods utilize varied paths, determined by the δqs, which satisfy the virtual constraints but not the actual constraints. Furthermore, for a given system and varied path, the integrals used in the various methods will all be equal to δP at any given time t, where $t_1 < t < t_2$. Thus, differing notation and variables are being used to express the same basic integral of Hamilton's principle. The differing approaches, however, can result in one method being easier than another, depending upon the particular system being analyzed.

5.7 Bibliography

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5.8 Problems

5.1. Consider the nonholonomic form of Hamilton's principle, as given by (5.27). Show that the integral

$$\int_{t_1}^{t} (\delta T + \delta W) dt$$

where $t_1 < t < t_2$, is equal to δP of the central equation (5.129).

5.2. (a) Show that, for a system of N particles,

$$\delta P = \sum_{j=1}^{n} \frac{\partial T^*}{\partial u_j} \delta\theta_j$$

where $T^*(q, u, t)$ is the unconstrained kinetic energy written in terms of quasi-velocities. (b) Starting with the central equation in the form of (5.129), and without using the fixed end-points and integration by parts of the integral method, derive (5.151) and the explicit form of the central equation.

5.3. A nonholonomic system has m constraints that are expressed in the dependent form

$$\dot{q}_k = \sum_{i=1}^{n-m} B_{ki}(q)\dot{q}_i \qquad (k = n - m + 1, \ldots, n)$$

Assuming that the independent qs satisfy

$$\frac{d}{dt}(\delta q_i) - \delta\dot{q}_i = 0 \qquad (i = 1, \ldots, n - m)$$

show that, for the dependent qs,

$$\frac{d}{dt}(\delta q_k) - \delta\dot{q}_k = \sum_{i=1}^{n-m}\sum_{j=1}^{n-m} \beta_{ij}^k \dot{q}_j \delta q_i \qquad (k = n - m + 1, \ldots, n)$$

where

$$\beta_{ij}^k = \frac{\partial B_{ki}}{\partial q_j} - \frac{\partial B_{kj}}{\partial q_i} + \sum_{l=n-m+1}^{n} \left(\frac{\partial B_{ki}}{\partial q_l} B_{lj} - \frac{\partial B_{kj}}{\partial q_l} B_{li} \right)$$

A substitution of this result into the Suslov equation (5.216) results in *Voronets' equation*:

$$\frac{d}{dt}\left(\frac{\partial T^0}{\partial \dot{q}_i}\right) - \left(\frac{\partial T^0}{\partial q_i} + \sum_{k=n-m+1}^{n} \frac{\partial T^0}{\partial q_k} B_{ki}\right) - \sum_{j=1}^{n-m}\sum_{k=n-m+1}^{n} \frac{\partial T}{\partial \dot{q}_k} \beta_{ij}^k \dot{q}_j = Q_i^0$$

where

$$Q_i^0 = Q_i + \sum_{k=n-m+1}^{n} Q_k B_{ki} \qquad (i = 1, \ldots, n - m)$$