4 Equations of motion: differential approach

In the previous chapters, we have considered relatively familiar methods of obtaining the differential equations of motion for a mechanical system. In this chapter, we shall introduce a number of other methods, partly in order to give the student a broader view of dynamics, and partly to present some practical and efficient approaches to obtaining the differential equations of motion. The results presented here are applicable to holonomic and nonholonomic systems alike, but the emphasis will be on nonholonomic systems because greater insight is needed in finding applicable theoretical approaches for these systems. As we proceed, we will show the advantages of using quasi-velocities in the analysis of nonholonomic systems.

4.1 Quasi-coordinates and quasi-velocities

Transformation equations

It is often possible to simplify the analysis of dynamical systems by using quasi-velocities (us) rather than $\dot{q}$s as velocity variables. An example is the use of Euler's equations for the rotational motion of a rigid body. Here the velocity variables are the body-axis components of the angular velocity $\boldsymbol{\omega}$ rather than Euler angle rates. The $\boldsymbol{\omega}$ components are quasi-velocities, whereas the Euler angle rates are true $\dot{q}$s whose time integrals result in generalized coordinates. As we showed in Chapter 3, the Euler equations are simpler than the corresponding Lagrange equations written in terms of Euler angles and Euler angle rates.

The transformation equations relating the us and $\dot{q}$s can be obtained by starting with the differential form

$$d\theta_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)\,dq_i + \Psi_{jt}(q,t)\,dt \qquad (j = 1, \ldots, n) \tag{4.1}$$

where the θs are called *quasi-coordinates* and $d\theta_j = u_j dt$ or $u_j = \dot{\theta}_j$. In general, the right-hand expressions in (4.1) are *not integrable*; if they were integrable, the θs would be true generalized coordinates.

If (4.1) is divided by dt, it has the form

$$u_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)\dot{q}_i + \Psi_{jt}(q,t) \qquad (j = 1, \ldots, n) \tag{4.2}$$

as in (3.245). We assume that these n equations can be solved for the $\dot{q}$s, resulting in

$$\dot{q}_i = \sum_{j=1}^{n} \Phi_{ij}(q,t)u_j + \Phi_{it}(q,t) \qquad (i = 1, \ldots, n) \tag{4.3}$$

Notice that for this unconstrained case, there are the same number of us and $\dot{q}$s. For example, the rotational dynamics of a rigid body might have $\omega_x, \omega_y, \omega_z$ as us and the Euler angles ψ, θ, ϕ as qs. The dynamical equations would be written in terms of us and then the qs would be generated by integrating (4.3) which are kinematic equations.

Constraints

Now let us impose m nonholonomic constraints of the linear form

$$\sum_{i=1}^{n} a_{ji}(q,t)\dot{q}_i + a_{jt}(q,t) = 0 \qquad (j = 1, \ldots, m) \tag{4.4}$$

as in (1.15). This will be accomplished by changing the notation and setting the last m us in (4.2) equal to zero. Thus, we can write

$$u_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)\dot{q}_i + \Psi_{jt}(q,t) \qquad (j = 1, \ldots, n-m) \tag{4.5}$$

$$u_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)\dot{q}_i + \Psi_{jt}(q,t) = 0 \qquad (j = n-m+1, \ldots, n) \tag{4.6}$$

The first $(n - m)$ us are *independent* quasi-velocities, while the last m us are set equal to zero to enforce the m nonholonomic constraints.

As before, we assume that the $(n \times n)$ $\mathbf{\Psi}$ matrix is invertible and we can solve for the $\dot{q}$s in the form

$$\dot{q}_i = \sum_{j=1}^{n-m} \Phi_{ij}(q,t)u_j + \Phi_{it}(q,t) \qquad (i = 1, \ldots, n) \tag{4.7}$$

The upper limit on the summation is $(n - m)$ because the last m us are equal to zero. Note that the $(n \times n)$ matrix $\mathbf{\Phi}$ is

$$\mathbf{\Phi} = \mathbf{\Psi}^{-1} \tag{4.8}$$

Furthermore,

$$\Phi_{it} = -\sum_{j=1}^{n} \Phi_{ij}\Psi_{jt} \qquad (i = 1, \ldots, n) \tag{4.9}$$

If there are any holonomic constraints, we have

$$\phi_j(q,t) = 0 \tag{4.10}$$

and (4.6) applies, where

$$\Psi_{ji} = \frac{\partial \phi_j}{\partial q_i}, \qquad \Psi_{jt} = \frac{\partial \phi_j}{\partial t} \tag{4.11}$$

Using the notation of (4.5) and (4.6), virtual displacements are constrained in accordance with

$$\delta\theta_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)\delta q_i = 0 \qquad (j = n-m+1,\ldots,n) \tag{4.12}$$

In velocity space, we have

$$\sum_{i=1}^{n} \Psi_{ji}(q,t)\delta w_i = 0 \qquad (j = n-m+1,\ldots,n) \tag{4.13}$$

implying that any virtual velocity $\delta\mathbf{w}$ lies in the common intersection of the constraint planes in velocity space.

4.2 Maggi's equation

Derivation of Maggi's equation

Consider a mechanical system whose configuration is described by n qs and which has m nonholonomic constraints with the linear form of (4.6). One approach to the problem of obtaining differential equations of motion is to use Lagrange's equation in the multiplier form of (2.49), namely,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=n-m+1}^{n} \lambda_j \Psi_{ji} \qquad (i = 1,\ldots,n) \tag{4.14}$$

These n second-order equations plus the m first-order constraint equations comprise a total of $(n+m)$ equations to solve for the n qs and m λs as functions of time. Frequently, however, one is not interested in the λ solutions, and would prefer a method which eliminates the λs from the beginning. The use of Maggi's equation is one such method.

Let us begin with Lagrange's principle, that is,

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i\right]\delta q_i = 0 \tag{4.15}$$

where the δqs satisfy the virtual constraints of (4.12) and we note that the kinetic energy $T(q,\dot{q},t)$ is written for the *unconstrained* system. The virtual displacements of quasi-coordinates and true coordinates are related by

$$\delta\theta_j = \sum_{i=1}^{n} \Psi_{ji}(q,t)\delta q_i \qquad (j = 1,\ldots,n) \tag{4.16}$$

or, upon inversion,

$$\delta q_i = \sum_{j=1}^{n-m} \Phi_{ij}(q, t)\delta\theta_j \qquad (i = 1, \ldots, n) \tag{4.17}$$

where, from (4.12), the last m $\delta\theta$s are equal to zero.

Now substitute this expression for δq_i into (4.15). We obtain

$$\sum_{j=1}^{n-m} \sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i \right] \Phi_{ij}\delta\theta_j = 0 \tag{4.18}$$

The first $(n - m)$ $\delta\theta$s are *independent* so each coefficient must equal zero. The result is *Maggi's equation* (1896).

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \right] \Phi_{ij} = \sum_{i=1}^{n} Q_i \Phi_{ij} \qquad (j = 1, \ldots, n - m) \tag{4.19}$$

The term on the right represents the generalized applied force associated with the quasi-coordinate $\delta\theta_j$. The set of Maggi equations are $(n - m)$ second-order differential equations in $(q, \dot{q}, \ddot{q}, t)$. In addition, there are m equations of constraint, making a total of n equations to solve for the n qs as functions of time.

The Ψ_{ji} coefficients are not unique but are chosen in such a way that the first $(n - m)$ $\delta\theta$s are independent and are consistent with the constraints. In the resulting $\mathbf{\Phi}$ matrix, each column represents the amplitude ratios of an independent set of δqs which could be used in Lagrange's principle, equation (4.15). Maggi's equation uses the first $(n - m)$ columns of $\mathbf{\Phi}$ to obtain the $(n - m)$ differential equations of motion.

Example 4.1 Two particles, each of mass m, are connected by a massless rod of length l (Fig. 4.1). There is a knife-edge constraint at particle 1 which prevents a velocity component perpendicular to the rod at that point. We wish to use Maggi's equation to find the differential equations of motion.

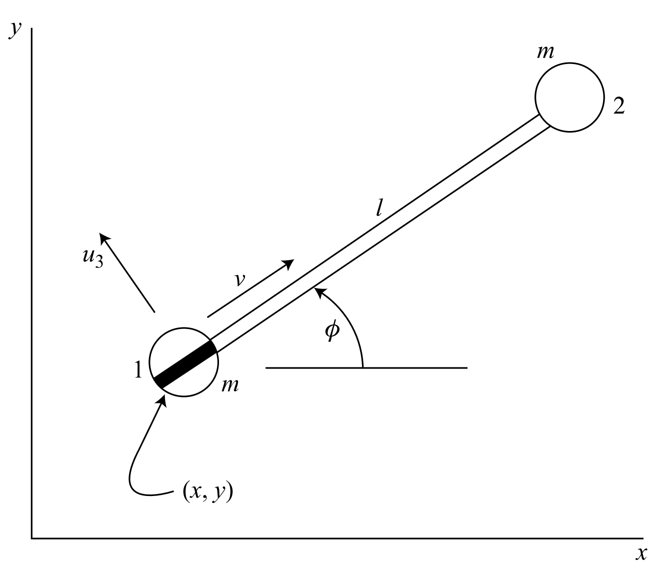

Figure 4.1.

Let (x, y, ϕ) be chosen as generalized coordinates. As quasi-velocities (us) let us take

$$u_1 = v = \dot{x}\cos\phi + \dot{y}\sin\phi \tag{4.20}$$

$$u_2 = \dot{\phi} \tag{4.21}$$

$$u_3 = -\dot{x}\sin\phi + \dot{y}\cos\phi = 0 \tag{4.22}$$

where $u_3 = 0$ is the nonholonomic constraint equation and (u_1, u_2) are independent. Thus, we obtain the coefficient matrix

$$\boldsymbol{\Psi} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \tag{4.23}$$

for the unconstrained system. The inverse matrix is

$$\boldsymbol{\Phi} = \boldsymbol{\Psi}^{-1} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ \sin\phi & 0 & \cos\phi \\ 0 & 1 & 0 \end{bmatrix} \tag{4.24}$$

We shall use Maggi's equation, (4.19), in the form

$$\sum_{i=1}^{3}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right]\Phi_{ij} = 0 \qquad (j = 1, 2) \tag{4.25}$$

because all Qs are equal to zero. The unconstrained kinetic energy is obtained by using (3.140). It is

$$T = m\left(\dot{x}^2 + \dot{y}^2 + \frac{1}{2}l^2\dot{\phi}^2 - l\dot{x}\dot{\phi}\sin\phi + l\dot{y}\dot{\phi}\cos\phi\right) \tag{4.26}$$

We find that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m(2\ddot{x} - l\ddot{\phi}\sin\phi - l\dot{\phi}^2\cos\phi), \qquad \frac{\partial T}{\partial x} = 0 \tag{4.27}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m(2\ddot{y} + l\ddot{\phi}\cos\phi - l\dot{\phi}^2\sin\phi), \qquad \frac{\partial T}{\partial y} = 0 \tag{4.28}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = m(l^2\ddot{\phi} - l\ddot{x}\sin\phi + l\ddot{y}\cos\phi - l\dot{x}\dot{\phi}\cos\phi - l\dot{y}\dot{\phi}\sin\phi) \tag{4.29}$$

$$\frac{\partial T}{\partial \phi} = m(-l\dot{x}\dot{\phi}\cos\phi - l\dot{y}\dot{\phi}\sin\phi) \tag{4.30}$$

The first equation of motion, using Maggi's equation and involving (4.27), (4.28) and the first column of the $\boldsymbol{\Phi}$ matrix, is

$$m(2\ddot{x} - l\ddot{\phi}\sin\phi - l\dot{\phi}^2\cos\phi)\cos\phi + m(2\ddot{y} + l\ddot{\phi}\cos\phi - l\dot{\phi}^2\sin\phi)\sin\phi = 0$$

or

$$m(2\ddot{x}\cos\phi + 2\ddot{y}\sin\phi - l\dot{\phi}^2) = 0 \tag{4.31}$$

The second equation of motion, involving the second column of the $\mathbf{\Phi}$ matrix, is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = m(l^2\ddot{\phi} - l\ddot{x}\sin\phi + l\ddot{y}\cos\phi) = 0 \tag{4.32}$$

In addition, (4.22), the constraint equation, can be differentiated with respect to time to yield,

$$\ddot{x}\sin\phi - \ddot{y}\cos\phi + \dot{x}\dot{\phi}\cos\phi + \dot{y}\dot{\phi}\sin\phi = 0 \tag{4.33}$$

In equations (4.31)–(4.33) we have three equations linear in the $\ddot{q}$s. Thus, we can solve for $\ddot{x}$, $\ddot{y}$, $\ddot{\phi}$ which are then integrated numerically to obtain the response as a function of time.

Example 4.2 Now consider a more complex problem, the rolling motion of a disk on a horizontal plane. Suppose that the classical Euler angles (ϕ, θ, ψ) are used to specify the orientation of a disk of mass m and radius r, whose contact point C has Cartesian coordinates (x, y), (Fig. 4.2).

Let us choose the independent us to be the Euler angle rates. Thus,

$$u_1 = \dot{\phi}, \qquad u_2 = \dot{\theta}, \qquad u_3 = \dot{\psi} \tag{4.34}$$

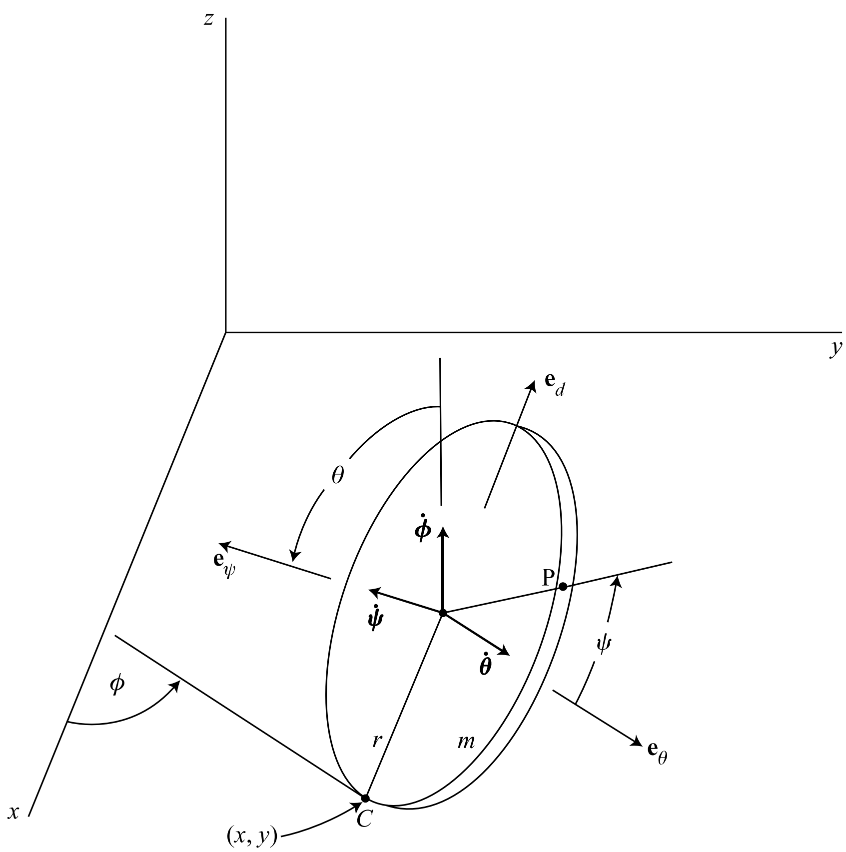

Figure 4.2.

The constraint equations, which enforce no slipping at the contact point, are

$$u_4 = \dot{x} + r\dot{\psi}\cos\phi = 0 \tag{4.35}$$

$$u_5 = \dot{y} + r\dot{\psi}\sin\phi = 0 \tag{4.36}$$

Maggi's equation is

$$\sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right]\Phi_{ij} = \sum_{i=1}^{n} Q_i\Phi_{ij} \qquad (j = 1, \ldots, n-m) \tag{4.37}$$

where, in this case, the only nonzero Q is

$$Q_2 = -mgr\cos\theta \tag{4.38}$$

The coefficients in (4.34)–(4.36) result in

$$\boldsymbol{\Psi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & r\cos\phi & 1 & 0 \\ 0 & 0 & r\sin\phi & 0 & 1 \end{bmatrix} \tag{4.39}$$

for the order $(\dot{\phi}, \dot{\theta}, \dot{\psi}, \dot{x}, \dot{y})$. The inverse matrix is

$$\boldsymbol{\Phi} = \boldsymbol{\Psi}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -r\cos\phi & 1 & 0 \\ 0 & 0 & -r\sin\phi & 0 & 1 \end{bmatrix} \tag{4.40}$$

In terms of the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the velocity of the center of the unconstrained disk is

$$\begin{aligned} \mathbf{v} &= (\dot{x} - r\dot{\phi}\cos\phi\cos\theta + r\dot{\theta}\sin\phi\sin\theta)\mathbf{i} \\ &\quad + (\dot{y} - r\dot{\phi}\sin\phi\cos\theta - r\dot{\theta}\cos\phi\sin\theta)\mathbf{j} + r\dot{\theta}\cos\theta\ \mathbf{k} \end{aligned} \tag{4.41}$$

The angular velocity is

$$\boldsymbol{\omega} = (\dot{\phi}\cos\theta + \dot{\psi})\mathbf{e}_\psi + \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\ \mathbf{e}_d \tag{4.42}$$

In accordance with Koenig's theorem, the kinetic energy is

$$T = \frac{1}{2}mv^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\cdot\boldsymbol{\omega} \tag{4.43}$$

where the axial and transverse moments of inertia are

$$I_a = \frac{1}{2}mr^2, \qquad I_t = \frac{1}{4}mr^2 \tag{4.44}$$

about the center of the disk. Thus, the total unconstrained kinetic energy is

$$\begin{aligned} T = &\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{8}mr^2\dot{\phi}^2(1 + 5\cos^2\theta) + \frac{5}{8}mr^2\dot{\theta}^2 + \frac{1}{4}mr^2\dot{\psi}^2 \\ &+ \frac{1}{2}mr^2\dot{\phi}\dot{\psi}\cos\theta - mr\dot{x}\dot{\phi}\cos\phi\cos\theta + mr\dot{x}\dot{\theta}\sin\phi\sin\theta \\ &- mr\dot{y}\dot{\phi}\sin\phi\cos\theta - mr\dot{y}\dot{\theta}\cos\phi\sin\theta \end{aligned} \tag{4.45}$$

Next, let us apply the Lagrangian operator with respect to each of the generalized coordinates. We obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = &\frac{1}{4}mr^2\ddot{\phi}(1 + 5\cos^2\theta) + \frac{1}{2}mr^2\ddot{\psi}\cos\theta \\ &- mr\ddot{x}\cos\phi\cos\theta - mr\ddot{y}\sin\phi\cos\theta \\ &- \frac{5}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta - \frac{1}{2}mr^2\dot{\theta}\dot{\psi}\sin\theta \end{aligned} \tag{4.46}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = &\frac{5}{4}mr^2\ddot{\theta} + mr\ddot{x}\sin\phi\sin\theta - mr\ddot{y}\cos\phi\sin\theta \\ &+ \frac{5}{4}mr^2\dot{\phi}^2\sin\theta\cos\theta + \frac{1}{2}mr^2\dot{\phi}\dot{\psi}\sin\theta \end{aligned} \tag{4.47}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = \frac{1}{2}mr^2\ddot{\psi} + \frac{1}{2}mr^2\ddot{\phi}\cos\theta - \frac{1}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta \tag{4.48}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = &m\ddot{x} - mr\ddot{\phi}\cos\phi\cos\theta \\ &+ mr\ddot{\theta}\sin\phi\sin\theta + mr\dot{\phi}^2\sin\phi\cos\theta \\ &+ mr\dot{\theta}^2\sin\phi\cos\theta + 2mr\dot{\phi}\dot{\theta}\cos\phi\sin\theta \end{aligned} \tag{4.49}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} = &m\ddot{y} - mr\ddot{\phi}\sin\phi\cos\theta \\ &- mr\ddot{\theta}\cos\phi\sin\theta - mr\dot{\phi}^2\cos\phi\cos\theta \\ &- mr\dot{\theta}^2\cos\phi\cos\theta + 2mr\dot{\phi}\dot{\theta}\sin\phi\sin\theta \end{aligned} \tag{4.50}$$

A substitution into (4.37), using the first three columns of the $\mathbf{\Phi}$ matrix, yields the Maggi equations. The ϕ equation is

$$\begin{aligned} &\frac{1}{4}mr^2\ddot{\phi}(1 + 5\cos^2\theta) + \frac{1}{2}mr^2\ddot{\psi}\cos\theta - mr\ddot{x}\cos\phi\cos\theta \\ &- mr\ddot{y}\sin\phi\cos\theta - \frac{5}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta - \frac{1}{2}mr^2\dot{\theta}\dot{\psi}\sin\theta = 0 \end{aligned} \tag{4.51}$$

The θ equation is

$$\frac{5}{4}mr^2\ddot{\theta} + mr\ddot{x}\sin\phi\sin\theta - mr\ddot{y}\cos\phi\sin\theta$$
$$+\frac{5}{4}mr^2\dot{\phi}^2\sin\theta\cos\theta + \frac{1}{2}mr^2\dot{\phi}\dot{\psi}\sin\theta = -mgr\cos\theta \tag{4.52}$$

The ψ equation is

$$\frac{1}{2}mr^2\ddot{\psi} + \frac{3}{2}mr^2\ddot{\phi}\cos\theta - \frac{5}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta - mr(\ddot{x}\cos\phi + \ddot{y}\sin\phi) = 0 \tag{4.53}$$

There are two additional equations obtained by differentiating the constraint equations (4.35) and (4.36) with respect to time.

$$\ddot{x} + r\ddot{\psi}\cos\phi - r\dot{\phi}\dot{\psi}\sin\phi = 0 \tag{4.54}$$
$$\ddot{y} + r\ddot{\psi}\sin\phi + r\dot{\phi}\dot{\psi}\cos\phi = 0 \tag{4.55}$$

In (4.51)–(4.55) we have five differential equations which are linear in the $\ddot{q}$s. They can be solved for the $\ddot{q}$s and then integrated to yield the motion as given by ϕ, θ, ψ, x and y as functions of time.

A simplification can be made if we solve (4.54) for $\ddot{x}$ and (4.55) for $\ddot{y}$, and then substitute into the Maggi equations. In (4.51) we find that

$$-mr\cos\theta(\ddot{x}\cos\phi + \ddot{y}\sin\phi) = mr^2\ddot{\psi}\cos\theta \tag{4.56}$$

and the ϕ equation reduces to

$$\frac{1}{4}mr^2\ddot{\phi}(1 + 5\cos^2\theta) + \frac{3}{2}mr^2\ddot{\psi}\cos\theta$$
$$-\frac{5}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta - \frac{1}{2}mr^2\dot{\theta}\dot{\psi}\sin\theta = 0 \tag{4.57}$$

Similarly,

$$mr\sin\theta(\ddot{x}\sin\phi - \ddot{y}\cos\phi) = mr^2\dot{\phi}\dot{\psi}\sin\theta \tag{4.58}$$

and the θ equation becomes

$$\frac{5}{4}mr^2\ddot{\theta} + \frac{5}{4}mr^2\dot{\phi}^2\sin\theta\cos\theta + \frac{3}{2}mr^2\dot{\phi}\dot{\psi}\sin\theta = -mgr\cos\theta \tag{4.59}$$

Finally, from (4.53) and (4.56), we find that the ψ equation is

$$\frac{3}{2}mr^2\ddot{\psi} + \frac{3}{2}mr^2\ddot{\phi}\cos\theta - \frac{5}{2}mr^2\dot{\phi}\dot{\theta}\sin\theta = 0 \tag{4.60}$$

In (4.57), (4.59), and (4.60) we have reduced the original set of five differential equations to three dynamical equations in the Euler angles. This is a minimum set for this system.