4.6 The Gibbs–Appell equation

System of particles

Let us consider a system of N particles whose configuration is specified by n qs. The particle velocities for a system with m nonholonomic constraints are given by $(n - m)$ independent us. Thus, the absolute velocity of the ith particle is

$$\mathbf{v}_i = \sum_{j=1}^{n-m} \gamma_{ij}(q, t) u_j + \gamma_{it}(q, t) \qquad (i = 1, \dots, N) \tag{4.275}$$

and the *constrained* kinetic energy is

$$T(q, u, t) = \frac{1}{2} \sum_{i=1}^{N} m_i \mathbf{v}_i^2 \tag{4.276}$$

The absolute particle accelerations, obtained by differentiating (4.275), are

$$\dot{\mathbf{v}}_i = \sum_{j=1}^{n-m} (\gamma_{ij} \dot{u}_j + \dot{\gamma}_{ij} u_j) + \dot{\gamma}_{it} \qquad (i = 1, \dots, N) \tag{4.277}$$

where the $\dot{\gamma}$s are linear functions of the us due to the equations

$$\dot{q}_k = \sum_{l=1}^{n-m} \Phi_{kl} u_l + \Phi_{kt} \qquad (k = 1, \dots, n) \tag{4.278}$$

Thus, we see that the acceleration $\dot{\mathbf{v}}_i$ is linear in the $\dot{u}$s and quadratic in the us.

Now let us introduce the Gibbs–Appell function

$$S(q, u, \dot{u}, t) = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\mathbf{v}}_i^2 \tag{4.279}$$

which is obtained by substituting $\dot{\mathbf{v}}_i$ from (4.277) for $\mathbf{v}_i$ in the kinetic energy expression of (4.276). We see that

$$\frac{\partial S}{\partial \dot{u}_j} = \sum_{i=1}^{N} m_i \dot{\mathbf{v}}_i \cdot \frac{\partial \dot{\mathbf{v}}_i}{\partial \dot{u}_j} = \sum_{i=1}^{N} m_i \dot{\mathbf{v}}_i \cdot \gamma_{ij} \qquad (j = 1, \dots, n - m) \tag{4.280}$$

But, from the general dynamical equation (4.133), we have

$$\sum_{i=1}^{N} m_i \dot{\mathbf{v}}_i \cdot \gamma_{ij} = Q_j \qquad (j = 1, \dots, n - m) \tag{4.281}$$

Finally, from (4.280) and (4.281) we obtain

$$\frac{\partial S}{\partial \dot{u}_j} = Q_j \qquad (j = 1, \dots, n - m) \tag{4.282}$$

which is the *Gibbs–Appell equation* for a system of particles. This equation was discovered by Gibbs in 1879 and was studied in detail by Appell in an 1899 publication. It provides a minimal set of dynamical equations which are applicable to systems with quasi-velocities and nonholonomic constraints.

To emphasize an important point, recall that the Gibbs–Appell function is obtained by substituting $\dot{\mathbf{v}}_i$ for $\mathbf{v}_i$ in the kinetic energy expression, where $\dot{\mathbf{v}}_i$ is the absolute acceleration of the ith particle. One cannot in general, obtain S by writing $T(q, \dot{q}, t)$ and then substituting $\ddot{q}$s for $\dot{q}$s. Furthermore, since (4.282) involves differentiations with respect to the $\dot{u}$s, any terms in $S(q, u, \dot{u}, t)$ which do not contain $\dot{u}$s can be omitted.

Example 4.10 Let us return to the dumbbell problem of Fig. 4.6. As independent quasi-velocities consistent with the knife-edge constraint we choose

$$u_1 = v, \qquad u_2 = \dot{\phi} \tag{4.283}$$

The particle velocities are

$$\mathbf{v}_1 = v\mathbf{e}_t, \qquad \mathbf{v}_2 = v\mathbf{e}_t + l\dot{\phi}\mathbf{e}_n \tag{4.284}$$

and the corresponding accelerations are

$$\dot{\mathbf{v}}_1 = \dot{v}\mathbf{e}_t + v\dot{\phi}\mathbf{e}_n \tag{4.285}$$

$$\dot{\mathbf{v}}_2 = (\dot{v} - l\dot{\phi}^2)\mathbf{e}_t + (l\ddot{\phi} + v\dot{\phi})\mathbf{e}_n \tag{4.286}$$

The resulting Gibbs–Appell function is

$$S = \frac{1}{2}m\left(\dot{\mathbf{v}}_1^2 + \dot{\mathbf{v}}_2^2\right) = \frac{1}{2}m[\dot{v}^2 + v^2\dot{\phi}^2 + (\dot{v} - l\dot{\phi}^2)^2 + (l\ddot{\phi} + v\dot{\phi})^2] \tag{4.287}$$

The generalized applied forces are

$$Q_1 = 0, \qquad Q_2 = 0 \tag{4.288}$$

Now we can apply (4.282) and obtain the following equations of motion:

$$\frac{\partial S}{\partial \dot{u}_1} = \frac{\partial S}{\partial \dot{v}} = m(2\dot{v} - l\dot{\phi}^2) = 2m\dot{v} - ml\dot{\phi}^2 = 0 \tag{4.289}$$

$$\frac{\partial S}{\partial \dot{u}_2} = \frac{\partial S}{\partial \ddot{\phi}} = ml(l\ddot{\phi} + v\dot{\phi}) = ml^2\ddot{\phi} + mlv\dot{\phi} = 0 \tag{4.290}$$

It is apparent that, for this problem, the Gibbs–Appell method is quite efficient in producing the differential equations of motion.

System of rigid bodies

Now let us generalize the Gibbs–Appell function to give correct equations of motion for a system of N rigid bodies when (4.282) is used. Let $\mathbf{v}_i$ be the velocity of the reference point of the ith body, and let $\mathbf{I}_i$ be the inertia dyadic about this reference point. The total kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}_i^2 + \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{\omega}_i \cdot \mathbf{I}_i \cdot \boldsymbol{\omega}_i + \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \dot{\boldsymbol{\rho}}_{ci} \tag{4.291}$$

A Gibbs–Appell function which yields correct equations of motion for this system of rigid bodies is

$$S(q, u, \dot{u}, t) = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\mathbf{v}}_i^2 + \sum_{i=1}^{N}\left[\frac{1}{2}\dot{\boldsymbol{\omega}}_i \cdot \mathbf{I}_i \cdot \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\mathbf{I}_i \cdot \boldsymbol{\omega}_i) \cdot \dot{\boldsymbol{\omega}}_i\right] + \sum_{i=1}^{N} m_i \dot{\mathbf{v}}_i \cdot \ddot{\boldsymbol{\rho}}_{ci} \tag{4.292}$$

where

$$\ddot{\boldsymbol{\rho}}_{ci} = \dot{\boldsymbol{\omega}}_i \times \boldsymbol{\rho}_{ci} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{\rho}_{ci}) \tag{4.293}$$

and we note that $\mathbf{I}_i$ is symmetric. In the evaluation of $\partial S/\partial \dot{u}_j$, only terms involving $\dot{\mathbf{v}}_i$, $\dot{\boldsymbol{\omega}}_i$, or $\ddot{\boldsymbol{\rho}}_{ci}$ need be considered. Recall that

$$\frac{\partial \dot{\mathbf{v}}_i}{\partial \dot{u}_j} = \frac{\partial \mathbf{v}_i}{\partial u_j} = \boldsymbol{\gamma}_{ij} \tag{4.294}$$

$$\frac{\partial \dot{\boldsymbol{\omega}}_i}{\partial \dot{u}_j} = \frac{\partial \boldsymbol{\omega}_i}{\partial u_j} = \boldsymbol{\beta}_{ij} \tag{4.295}$$

Also,

$$\dot{\mathbf{v}}_i \cdot \frac{\partial \dot{\boldsymbol{\omega}}_i}{\partial \dot{u}_j} \times \boldsymbol{\rho}_{ci} = \boldsymbol{\rho}_{ci} \times \dot{\mathbf{v}}_i \cdot \boldsymbol{\beta}_{ij} \tag{4.296}$$

Then, from (4.282) and (4.292), we obtain

$$\frac{\partial S}{\partial \dot{u}_j} = \sum_{i=1}^{N} m_i(\dot{\mathbf{v}}_i + \ddot{\boldsymbol{\rho}}_{ci}) \cdot \boldsymbol{\gamma}_{ij} + \sum_{i=1}^{N}[\mathbf{I}_i \cdot \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\mathbf{I}_i \cdot \boldsymbol{\omega}_i) + m_i \boldsymbol{\rho}_{ci} \times \dot{\mathbf{v}}_i] \cdot \boldsymbol{\beta}_{ij} = Q_j \qquad (j = 1, \ldots, n-m) \tag{4.297}$$

which is the general dynamical equation for a system of rigid bodies. It results in $(n-m)$ first-order dynamical equations.

If one takes the reference point of each rigid body at its center of mass, then $\boldsymbol{\rho}_{ci} = 0$ and the Gibbs–Appell function becomes

$$S = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\mathbf{v}}_i^2 + \sum_{i=1}^{N}\left[\frac{1}{2}\dot{\boldsymbol{\omega}}_i \cdot \mathbf{I}_{ci} \cdot \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times (\mathbf{I}_{ci} \cdot \boldsymbol{\omega}_i) \cdot \dot{\boldsymbol{\omega}}_i\right] \tag{4.298}$$

where $\mathbf{I}_{ci}$ is the inertia dyadic about the center of mass. An equivalent form is

$$S = \frac{1}{2}\sum_{i=1}^{N}\left(m_i \dot{\mathbf{v}}_i^2 + \dot{\mathbf{H}}_{ci} \cdot \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{H}_{ci} \cdot \dot{\boldsymbol{\omega}}_i\right) \tag{4.299}$$

where $\mathbf{H}_{ci}$ is the angular momentum of the ith body about its center of mass.

Example 4.11 Consider the rolling disk problem of Fig. 4.2. We again choose the independent quasi-velocities

$$u_1 = \dot{\theta}, \qquad u_2 = \omega_d, \qquad u_3 = \Omega \tag{4.300}$$

The angular velocity of the disk is

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{e}_\theta + \omega_d\mathbf{e}_d + \Omega\mathbf{e}_\psi \tag{4.301}$$

and the angular velocity of the $\mathbf{e}_\theta\mathbf{e}_d\mathbf{e}_\psi$ unit vector triad is equal to $\dot{\boldsymbol{\phi}} + \dot{\boldsymbol{\theta}}$ or

$$\boldsymbol{\omega}_c = \dot{\theta}\mathbf{e}_\theta + \omega_d\mathbf{e}_d + \omega_d \cot\theta \ \mathbf{e}_\psi \tag{4.302}$$

and we note that $\mathbf{e}_\theta$ remains horizontal. Thus, we find that

$$\dot{\mathbf{e}}_\theta = \boldsymbol{\omega}_c \times \mathbf{e}_\theta = \omega_d \cot\theta \ \mathbf{e}_d - \omega_d\mathbf{e}_\psi \tag{4.303}$$

$$\dot{\mathbf{e}}_d = \boldsymbol{\omega}_c \times \mathbf{e}_d = -\omega_d \cot\theta \ \mathbf{e}_\theta + \dot{\theta}\mathbf{e}_\psi \tag{4.304}$$

$$\dot{\mathbf{e}}_\psi = \omega_d\mathbf{e}_\theta - \dot{\theta}\mathbf{e}_d \tag{4.305}$$

and

$$\dot{\boldsymbol{\omega}} = \left(\ddot{\theta} - \omega_d^2 \cot\theta + \omega_d\Omega\right)\mathbf{e}_\theta + (\dot{\boldsymbol{\omega}}_d + \dot{\theta}\omega_d \cot\theta - \dot{\theta}\Omega)\,\mathbf{e}_d + \dot{\Omega}\mathbf{e}_\psi \tag{4.306}$$

The velocity of the center of the disk is

$$\mathbf{v} = -r\Omega\mathbf{e}_\theta + r\dot{\theta}\mathbf{e}_\psi \tag{4.307}$$

and the corresponding acceleration is

$$\dot{\mathbf{v}} = (-r\dot{\Omega} + r\dot{\theta}\omega_d)\,\mathbf{e}_\theta - (r\omega_d\Omega \cot\theta + r\dot{\theta}^2)\,\mathbf{e}_d + (r\ddot{\theta} + r\omega_d\Omega)\,\mathbf{e}_\psi \tag{4.308}$$

Hence, we see that

$$\dot{\mathbf{v}}^2 = (r\dot{\Omega} - r\dot{\theta}\omega_d)^2 + (r\omega_d\Omega \cot\theta + r\dot{\theta}^2)^2 + (r\ddot{\theta} + r\omega_d\Omega)^2 \tag{4.309}$$

The disk has moments of inertia

$$I_a = \frac{1}{2}mr^2, \qquad I_t = \frac{1}{4}mr^2 \tag{4.310}$$

Thus, the angular momentum about the center is

$$\mathbf{H}_c = \frac{1}{4}mr^2\dot{\theta}\mathbf{e}_\theta + \frac{1}{4}mr^2\omega_d\mathbf{e}_d + \frac{1}{2}mr^2\Omega\mathbf{e}_\psi \tag{4.311}$$

Upon differentiating with respect to time, we find that

$$\begin{aligned}\dot{\mathbf{H}}_c = \frac{1}{4}mr^2 \Big[&\left(\ddot{\theta} - \omega_d^2 \cot\theta + 2\omega_d\Omega\right)\mathbf{e}_\theta \\ &+ (\dot{\omega}_d + \dot{\theta}\omega_d \cot\theta - 2\dot{\theta}\Omega)\mathbf{e}_d + 2\dot{\Omega}\mathbf{e}_\psi\Big]\end{aligned} \tag{4.312}$$

In addition,

$$\boldsymbol{\omega} \times \mathbf{H}_c = \frac{1}{4}mr^2(\omega_d\Omega\mathbf{e}_\theta - \dot{\theta}\Omega\mathbf{e}_d) \tag{4.313}$$

Now we can use (4.299) to obtain the Gibbs–Appell function. If we omit terms not containing $\dot{u}$s, it is

$$
\begin{aligned}
S = {} & \frac{1}{2}mr^2[(\dot{\Omega} - \dot{\theta}\omega_d)^2 + (\ddot{\theta} + \omega_d\Omega)^2] \\
& + \frac{1}{8}mr^2\left[\left(\ddot{\theta} - \omega_d^2\cot\theta + 2\omega_d\Omega\right)\left(\ddot{\theta} - \omega_d^2\cot\theta + \omega_d\Omega\right)\right. \\
& \left. + (\dot{\omega}_d + \dot{\theta}\omega_d\cot\theta - 2\dot{\theta}\Omega)(\dot{\omega}_d + \dot{\theta}\omega_d\cot\theta - \dot{\theta}\Omega)\right] \\
& + \frac{1}{4}mr^2\dot{\Omega}^2 + \frac{1}{8}mr^2(\omega_d\Omega\ddot{\theta} - \dot{\theta}\Omega\dot{\omega}_d)
\end{aligned}
\tag{4.314}
$$

The generalized applied forces are

$$
Q_1 = -mgr\cos\theta, \qquad Q_2 = 0, \qquad Q_3 = 0 \tag{4.315}
$$

The differential equations of motion are obtained from

$$
\frac{\partial S}{\partial \dot{u}_j} = Q_j \qquad (j = 1, 2, 3) \tag{4.316}
$$

The θ equation is

$$
\begin{aligned}
\frac{\partial S}{\partial \ddot{\theta}} &= mr^2(\ddot{\theta} + \omega_d\Omega) + \frac{1}{8}mr^2\left(2\ddot{\theta} - 2\omega_d^2\cot\theta + 4\omega_d\Omega\right) \\
&= \frac{5}{4}mr^2\ddot{\theta} - \frac{1}{4}mr^2\omega_d^2\cot\theta + \frac{3}{2}mr^2\omega_d\Omega = -mgr\cos\theta
\end{aligned}
\tag{4.317}
$$

The ω_d equation is

$$
\frac{\partial S}{\partial \dot{\omega}_d} = \frac{1}{4}mr^2(\dot{\omega}_d + \dot{\theta}\omega_d\cot\theta - 2\dot{\theta}\Omega) = 0 \tag{4.318}
$$

Finally, the Ω equation is

$$
\frac{\partial S}{\partial \dot{\Omega}} = \frac{3}{2}mr^2\dot{\Omega} - mr^2\dot{\theta}\omega_d = 0 \tag{4.319}
$$

These three dynamical equations constitute a minimum set for this system. The effort required in their derivation is about the same as for the general dynamical equation.

Principle of least constraint

The principle of least constraint was discovered by Gauss in 1829, and thereby preceded the Gibbs–Appell equations, to which it is related, by half a century. The principle of least constraint is an algebraic minimization principle which leads to the differential equations of motion. Briefly, it states that a certain function of the constraint forces is minimized by the actual motion, as compared with other motions which, at any given time, have the same configuration and velocities, but have small variations in the accelerations.

Consider a system of N particles whose configuration is given by $3N$ Cartesian coordinates relative to an inertial frame. Suppose there are m equations of constraint, holonomic or nonholonomic, which can be written in the form

$$\sum_{k=1}^{3N} a_{jk}(x,t)\dot{x}_k + a_{jt}(x,t) = 0 \qquad (j = 1,\ldots,m) \tag{4.320}$$

Newton's law, applied to individual particles, results in

$$m_k \ddot{x}_k = F_k + R_k \qquad (k = 1,\ldots,3N) \tag{4.321}$$

where F_k is an applied force component and R_k is the corresponding constraint force component. From (4.321),

$$R_k = m_k \left(\ddot{x}_k - \frac{F_k}{m_k} \right) \tag{4.322}$$

Now define the function

$$C = \frac{1}{2} \sum_{k=1}^{3N} \frac{R_k^2}{m_k} = \frac{1}{2} \sum_{k=1}^{3N} m_k \left(\ddot{x}_k - \frac{F_k}{m_k} \right)^2 \tag{4.323}$$

which represents the weighted sum of the squares of the constraint force magnitudes. The principle of least constraint states that C is minimized with respect to variations in the $\ddot{x}$s by the actual motion at each instant of time. It is assumed that the $\ddot{x}$s and $\delta\ddot{x}$s satisfy the constraints.

Thus, noting that F_k is not varied, we obtain

$$\delta C = \sum_{k=1}^{3N} m_k \left(\ddot{x}_k - \frac{F_k}{m_k} \right) \delta\ddot{x}_k = 0 \tag{4.324}$$

where

$$\sum_{k=1}^{3N} a_{jk} \delta\ddot{x}_k = 0 \qquad (j = 1,\ldots,m) \tag{4.325}$$

The constraints are incorporated into the analysis by using Lagrange multipliers. Multiply (4.325) by λ_j and sum over j. Then, upon adding this result to (4.324), we obtain

$$\sum_{k=1}^{3N} \left(m_k \ddot{x}_k - F_k + \sum_{j=1}^{m} \lambda_j a_{jk} \right) \delta\ddot{x}_k = 0 \tag{4.326}$$

where the $\delta\ddot{x}$s are now regarded as independent. Hence, each coefficient must be zero, or

$$m_k \ddot{x}_k = F_k - \sum_{j=1}^{m} \lambda_j a_{jk} \qquad (k = 1,\ldots,3N) \tag{4.327}$$

These are the equations of motion in terms of Cartesian coordinates.

Now let us broaden the analysis by transforming to quasi-velocities and generalized coordinates. Write (4.323) in the form

$$C = \frac{1}{2} \sum_{k=1}^{3N} m_k \ddot{x}_k^2 - \sum_{k=1}^{3N} F_k \ddot{x}_k + \sum_{k=1}^{3N} \frac{F_k^2}{2m_k} \tag{4.328}$$

The Gibbs–Appell function is, from (4.279),

$$S = \frac{1}{2}\sum_{k=1}^{3N} m_k \ddot{x}_k^2 \tag{4.329}$$

Then, noting that F_k is not varied, we can write

$$C = S - \sum_{k=1}^{3N} F_k \ddot{x}_k + \text{const} \tag{4.330}$$

The transformation equations are

$$x_k = x_k(q, t) \qquad (k = 1, \ldots, 3N) \tag{4.331}$$

$$\dot{x}_k = \dot{x}_k(q, u, t) \qquad (k = 1, \ldots, 3N) \tag{4.332}$$

and therefore

$$\ddot{x}_k = \sum_{j=1}^{n-m} \frac{\partial \dot{x}_k}{\partial u_j}\dot{u}_j + f_k(q, u, t) \qquad (k = 1, \ldots, 3N) \tag{4.333}$$

where the $(n - m)$ us are independent and consistent with any constraints on the xs. Note that $\dot{x}_k$ is a linear function of the us, and $\ddot{x}_k$ is a linear function of the $\dot{u}$s. Also, the generalized force associated with u_j is

$$Q_j = \sum_{k=1}^{3N} F_k \frac{\partial \dot{x}_k}{\partial u_j} \qquad (j = 1, \ldots, n - m) \tag{4.334}$$

Then we obtain

$$C(q, u, \dot{u}, t) = S(q, u, \dot{u}, t) - \sum_{j=1}^{n-m} Q_j \dot{u}_j - \sum_{k=1}^{3N} F_k f_k + \text{const} \tag{4.335}$$

Now consider a variation δC due to small variations in the $\dot{u}$s, with the qs and us held fixed.

$$\delta C = \sum_{j=1}^{n-m} \frac{\partial S}{\partial \dot{u}_j}\delta \dot{u}_j - \sum_{j=1}^{n-m} Q_j \delta \dot{u}_j \tag{4.336}$$

Thus, for a stationary value of C, we have

$$\delta C = \sum_{j=1}^{n-m} \left(\frac{\partial S}{\partial \dot{u}_j} - Q_j \right) \delta \dot{u}_j = 0 \tag{4.337}$$

for arbitrary $\delta \dot{u}$s. This requires that each coefficient be zero, and we obtain

$$\frac{\partial S}{\partial \dot{u}_j} = Q_j \qquad (j = 1, \ldots, n - m) \tag{4.338}$$

Thus, the principle of least constraint applied to a system of particles results in the Gibbs–Appell equation.

It has been shown that the Gibbs–Appell equation follows from the requirement that C be stationary with respect to variations of the $\dot{u}$s. That this stationary point is also a minimum can be shown by using (4.323) to evaluate

$$\frac{\partial^2 C}{\partial \ddot{x}_i \partial \ddot{x}_j} = \begin{cases} m_i & i = j \\ 0 & i \neq j \end{cases} \tag{4.339}$$

The corresponding matrix is positive definite, that is,

$$\delta^2 C = \sum_{i=1}^{3N} \sum_{j=1}^{3N} \left(\frac{\partial^2 C}{\partial \ddot{x}_i \partial \ddot{x}_j} \right) \delta \ddot{x}_i \delta \ddot{x}_j \geq 0 \tag{4.340}$$

and the value zero occurs only when all the $\delta\ddot{x}$s are zero. Since the second variation of C is a positive-definite function of the $\delta\ddot{x}$s, the stationary point is also a minimum.

4.7 Constraints and energy rates

Ideal and conservative constraints

Consider a dynamical system having m constraints of the general nonholonomic form

$$f_j(q, \dot{q}, t) = 0 \qquad (j = 1, \ldots, m) \tag{4.341}$$

This general form includes the usual nonholonomic constraints which are linear in the velocities, that is,

$$\sum_{i=1}^{n} a_{ji}(q, t)\, \dot{q}_i + a_{jt}(q, t) = 0 \qquad (j = 1, \ldots, m) \tag{4.342}$$

Furthermore, holonomic constraints of the form

$$\phi_j(q, t) = 0 \qquad (j = 1, \ldots, m) \tag{4.343}$$

can be expressed in the linear form of (4.342) after differentiation with respect to time.

$$\dot{\phi}_j(q, \dot{q}, t) = \sum_{i=1}^{n} \frac{\partial \phi_j}{\partial q_i} \dot{q}_i + \frac{\partial \phi_j}{\partial t} = 0 \qquad (j = 1, \ldots, m) \tag{4.344}$$

Of course, this linear form is integrable.

Let us define an *ideal constraint* as a workless kinematic constraint which may be either scleronomic or rheonomic. By *workless*, we mean that no work is done by the constraint forces in an arbitrary reversible virtual displacement that is consistent with the instantaneous constraints. For example, an ideal constraint might be a frictionless surface on which sliding occurs, or it might involve rolling contact without slipping. Another example is a knife-edge constraint with no frictional resistance for motion along the knife edge, but with no slipping allowed perpendicular to it.

Let $\mathbf{C}_j$ be the generalized ideal constraint force corresponding to the jth constraint. The virtual work of $\mathbf{C}_j$ in an arbitrary virtual displacement consistent with the instantaneous constraints is

$$\delta W = \mathbf{C}_j \cdot \delta\mathbf{q} = \sum_{i=1}^{n} C_{ji}\delta q_i = 0 \qquad (j = 1, \ldots, m) \tag{4.345}$$

where, for the general nonholonomic case,

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial \dot{q}_i}\delta q_i = 0 \qquad (j = 1, \ldots, m) \tag{4.346}$$

Assuming the usual case of nonholonomic constraints which are linear in the $\dot{q}$s, we have

$$\sum_{i=1}^{n} a_{ji}(q, t)\,\delta q_i = 0 \qquad (j = 1, \ldots, m) \tag{4.347}$$

If the constraints are holonomic, the δqs satisfy

$$\sum_{i=1}^{n} \frac{\partial \phi_j}{\partial q_i}\delta q_i = 0 \qquad (j = 1, \ldots, m) \tag{4.348}$$

Equation (4.345) states that the ideal constraint force vector $\mathbf{C}_j$ and an allowable virtual displacement $\delta\mathbf{q}$ are orthogonal in n-dimensional configuration space. A comparison of (4.345) and (4.348) shows that C_{ji} and $\partial\phi_j/\partial q_i$ are proportional for any given j, so $\mathbf{C}_j$ is directed normal to the constraint surface; that is, in the direction of the gradient of $\phi_j(q, t)$ in q-space. Similarly, a comparison of (4.345) and (4.346) indicates that $\mathbf{C}_j$ is directed normal to the constraint surface in velocity space.

A virtual velocity $\delta\mathbf{w}$ is subject to instantaneous constraint equations of the form

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial \dot{q}_i}\delta w_i = 0 \qquad (j = 1, \ldots, m) \tag{4.349}$$

or

$$\sum_{i=1}^{n} a_{ji}\delta w_i = 0 \qquad (j = 1, \ldots, m) \tag{4.350}$$

A comparison of (4.346) and (4.349) or (4.347) and (4.350) shows that the permitted directions of $\delta\mathbf{q}$ and $\delta\mathbf{w}$ are the same, namely, in the tangent plane at the operating point P in velocity space (Fig. 4.7). The direction of an ideal constraint force $\mathbf{C}_j$ is perpendicular to this tangent plane. Note that, for a holonomic constraint, the tangent plane at the operating point in configuration space has the same orientation as the corresponding constraint plane in velocity space.

Now let us define a *conservative constraint* to be an ideal constraint which meets the additional condition that

$$\mathbf{C}_j \cdot \dot{\mathbf{q}} = 0 \tag{4.351}$$

that is, the generalized constraint force $\mathbf{C}_j$ does no work in any possible actual motion of the system. We found earlier from (4.345) and (4.346) that the components C_{ji} and $\partial f_j/\partial \dot{q}_i$

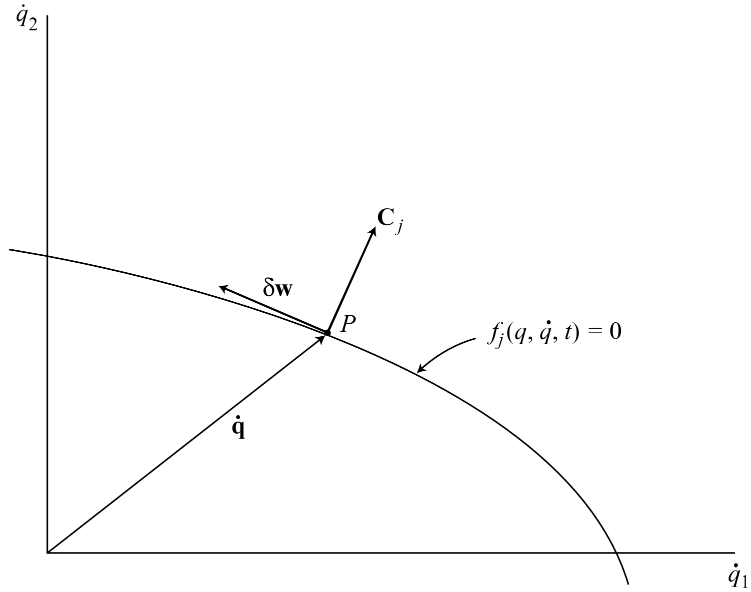

Figure 4.7.

are proportional. Hence,

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial \dot{q}_i} \dot{q}_i = 0 \tag{4.352}$$

This is just the condition that $f_j(q, \dot{q}, t)$ is a *homogeneous function* of the $\dot{q}$s. In other words, if a generalized velocity $\dot{\mathbf{q}}$ satisfies the constraint, then that velocity multiplied by an arbitrary scalar constant will also satisfy the constraint. This implies that the corresponding constraint surface in velocity space can be generated by sweeping a straight line passing through the origin. It is clear that the common case of a plane passing through the origin is included, but other possibilities exist, such as, for example, a conical surface with its vertex at the origin.

The homogeneity condition for a conservative constraint requires that a_{jt} be zero for the common case of a linear nonholonomic constraint, that is, it must be *catastatic*. For a *holonomic constraint* to be conservative, it must be *scleronomic*. Thus, coefficients of the form $a_{ji}(q, t)$ are acceptable in the nonholonomic case, but any holonomic constraint function must be of the form $\phi_j(q)$.

It is possible that a constraint which does not meet the homogeneity condition for a set of generalized coordinates may be homogeneous in form or may disappear entirely for another choice of generalized coordinates. Thus, whether a constraint is classed as conservative or not may depend upon the choice of coordinates.

Conservative system

A conservative system can be defined as a dynamical system for which an energy integral can be found. As an example, let us consider a dynamical system having m nonholonomic constraints of the general form

$$f_j(q, \dot{q}, t) = 0 \qquad (j = 1, \ldots, m) \tag{4.353}$$

Assume that the system can be described by Lagrange's equation in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial \dot{q}_i} \qquad (i = 1, \ldots, n) \tag{4.354}$$

Now let us use the same procedure that we employed previously in (2.149)–(2.155). The result in this more general case is

$$\dot{E} = \dot{T}_2 - \dot{T}_0 + \dot{V} = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial \dot{q}_i} \dot{q}_i - \frac{\partial L}{\partial t} \tag{4.355}$$

where the λs are usually nonzero and not easily evaluated. The first term on the right will be zero, however, if $f_j(q, \dot{q}, t)$ is a homogeneous function of the $\dot{q}$s, that is, if each constraint is conservative. The term $\partial L/\partial t$ will equal zero if neither T nor V is an explicit function of time.

In summary, a system having holonomic or nonholonomic constraints will be *conservative* if it meets the following conditions:

1. The standard form of Lagrange's equation, as given by (4.354), applies.
2. All constraints are conservative.
3. The Lagrangian function $L = T - V$ is not an explicit function of time.

These are *sufficient conditions* for a conservative system. Note that the conserved integral of the motion

$$E(q, \dot{q}) = T_2 - T_0 + V \tag{4.356}$$

is equal in value to the Hamiltonian function $H(q, p)$.

Work and energy rates

Consider a system of N rigid bodies. The forces acting on the ith body are equivalent to a force $\mathbf{F}_i$ acting at a reference point P_i plus a couple $\mathbf{M}_i$. Let $\mathbf{v}_i$ be the velocity of point P_i, fixed in the ith body, and let $\boldsymbol{\omega}_i$ be the angular velocity of the body. Assume linear nonholonomic constraints.

From the principle of work and kinetic energy, and summing over the N bodies,

$$\dot{T} = \dot{W} = \sum_{i=1}^{N} (\mathbf{F}_i \cdot \mathbf{v}_i + \mathbf{M}_i \cdot \boldsymbol{\omega}_i) \tag{4.357}$$

where $\mathbf{F}_i$ and $\mathbf{M}_i$ may include constraint forces as well as applied forces.

Now let us assume that a portion of $\mathbf{F}_i$ and $\mathbf{M}_i$ arise from a potential energy function $V(q, t)$, but the remaining primed quantities $\mathbf{F}'_i$ and $\mathbf{M}'_i$ do not. Thus, we have

$$\mathbf{F}_i = \mathbf{F}'_i - \frac{\partial V}{\partial \mathbf{r}_i} \qquad (i = 1, \ldots, N) \tag{4.358}$$

$$\mathbf{M}_i = \mathbf{M}'_i - \frac{\partial V}{\partial \boldsymbol{\theta}_i} \qquad (i = 1, \ldots, N) \tag{4.359}$$

where $\mathbf{r}_i$ is the position vector of P_i and $\boldsymbol{\omega}_i = d\boldsymbol{\theta}_i/dt$. In terms of Euler angles, we use the notation

$$\frac{\partial V}{\partial \boldsymbol{\theta}} = \frac{\partial V}{\partial \psi}\mathbf{e}_\psi + \frac{\partial V}{\partial \theta}\mathbf{e}_\theta + \frac{\partial V}{\partial \phi}\mathbf{e}_\phi \tag{4.360}$$

$$\boldsymbol{\omega} = \dot{\psi}\mathbf{e}_\psi + \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\mathbf{e}_\phi \tag{4.361}$$

A similar notation is used for $\partial V/\partial \mathbf{r}$. From (4.357)–(4.359), we obtain

$$\dot{T} = \sum_{i=1}^{N}(\mathbf{F}'_i \cdot \mathbf{v}_i + \mathbf{M}'_i \cdot \boldsymbol{\omega}_i) - \sum_{i=1}^{N}\left(\frac{\partial V}{\partial \mathbf{r}_i} \cdot \mathbf{v}_i + \frac{\partial V}{\partial \boldsymbol{\theta}_i} \cdot \boldsymbol{\omega}_i\right) \tag{4.362}$$

But,

$$\dot{V} = \sum_{i=1}^{N}\left(\frac{\partial V}{\partial \mathbf{r}_i} \cdot \mathbf{v}_i + \frac{\partial V}{\partial \boldsymbol{\theta}_i} \cdot \boldsymbol{\omega}_i\right) + \frac{\partial V}{\partial t} \tag{4.363}$$

Hence, we find that the rate of change of the total energy is

$$\dot{T} + \dot{V} = \sum_{i=1}^{N}(\mathbf{F}'_i \cdot \mathbf{v}_i + \mathbf{M}'_i \cdot \boldsymbol{\omega}_i) + \frac{\partial V}{\partial t} \tag{4.364}$$

Note that constraint forces are included in $\mathbf{F}'_i$ and $\mathbf{M}'_i$, but forces derived from potential energy are not included.

Now let us consider a system whose motion is described in terms of independent quasi-velocities. Assume a system of N rigid bodies and start with the general dynamical equation in the form

$$\sum_{i=1}^{N}[m_i\dot{\mathbf{v}}_i \cdot \boldsymbol{\gamma}_{ij} + (\mathbf{I}_{ci} \cdot \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_{ci} \cdot \boldsymbol{\omega}_i) \cdot \boldsymbol{\beta}_{ij}] = Q_j \qquad (j = 1, \ldots, n-m) \tag{4.365}$$

where the *center of mass* is chosen as the reference point for each body.

Multiplying (4.365) by u_j and summing over j, we obtain

$$\sum_{i=1}^{N}\sum_{j=1}^{n-m}[m_i\dot{\mathbf{v}}_i \cdot \boldsymbol{\gamma}_{ij}u_j + (\mathbf{I}_{ci} \cdot \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_{ci} \cdot \boldsymbol{\omega}_i) \cdot \boldsymbol{\beta}_{ij}u_j] = \sum_{j=1}^{n-m} Q_j u_j \tag{4.366}$$

Recall that $\boldsymbol{\gamma}_{ij} = \partial \mathbf{v}_i/\partial u_j$ and $\boldsymbol{\beta}_{ij} = \partial \boldsymbol{\omega}_i/\partial u_j$. The kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \mathbf{v}_i + \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{\omega}_i \cdot \mathbf{I}_{ci} \cdot \boldsymbol{\omega}_i \tag{4.367}$$

Then, using Euler's theorem on homogeneous functions, we see that

$$\begin{aligned}\sum_{j=1}^{n-m}\frac{\partial T}{\partial u_j}u_j &= \sum_{i=1}^{N}\sum_{j=1}^{n-m}\left(m_i\mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial u_j}u_j + \boldsymbol{\omega}_i \cdot \mathbf{I}_{ci} \cdot \frac{\partial \boldsymbol{\omega}_i}{\partial u_j}u_j\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{n-m}(m_i\mathbf{v}_i \cdot \boldsymbol{\gamma}_{ij}u_j + \boldsymbol{\omega}_i \cdot \mathbf{I}_{ci} \cdot \boldsymbol{\beta}_{ij}u_j) \\ &= 2T_2 + T_1\end{aligned} \tag{4.368}$$

where T_2 is quadratic and T_1 is linear in the us.

Now let us differentiate (4.368) with respect to time and recall that

$$\sum_{j=1}^{n-m} \gamma_{ij} u_j = \mathbf{v}_i - \gamma_{it} \tag{4.369}$$

$$\sum_{j=1}^{n-m} \boldsymbol{\beta}_{ij} u_j = \boldsymbol{\omega}_i - \boldsymbol{\beta}_{it} \tag{4.370}$$

We obtain

$$\begin{aligned}\frac{d}{dt}\left[\sum_{i=1}^{N}\sum_{j=1}^{n-m}(m_i \mathbf{v}_i \cdot \gamma_{ij} u_j + \boldsymbol{\omega}_i \cdot \mathbf{I}_{ci} \cdot \boldsymbol{\beta}_{ij} u_j)\right] &= 2\dot{T}_2 + \dot{T}_1 \\ = \sum_{i=1}^{N}\sum_{j=1}^{n-m}[m_i \dot{\mathbf{v}}_i \cdot \gamma_{ij} u_j + (\mathbf{I}_{ci} \cdot \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_{ci} \cdot \boldsymbol{\omega}_i) \cdot \boldsymbol{\beta}_{ij} u_j] \\ + \sum_{i=1}^{N}[m_i \mathbf{v}_i \cdot (\dot{\mathbf{v}}_i - \dot{\gamma}_{it}) + \boldsymbol{\omega}_i \cdot \mathbf{I}_{ci} \cdot (\dot{\boldsymbol{\omega}}_i - \dot{\boldsymbol{\beta}}_{it})]\end{aligned} \tag{4.371}$$

Differentiating (4.367) with respect to time, we obtain

$$\dot{T} = \sum_{i=1}^{N}(m_i \mathbf{v}_i \cdot \dot{\mathbf{v}}_i + \boldsymbol{\omega}_i \cdot \mathbf{I}_{ci} \cdot \dot{\boldsymbol{\omega}}_i) = \dot{T}_2 + \dot{T}_1 + \dot{T}_0 \tag{4.372}$$

where we note that

$$\boldsymbol{\omega}_i \cdot \dot{\mathbf{I}}_{ci} \cdot \boldsymbol{\omega}_i = 0 \tag{4.373}$$

Next, subtract (4.372) from (4.371). The result is

$$\begin{aligned}\dot{T}_2 - \dot{T}_0 = &\sum_{i=1}^{N}\sum_{j=1}^{n-m}[m_i \dot{\mathbf{v}}_i \cdot \gamma_{ij} u_j + (\mathbf{I}_{ci} \cdot \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_{ci} \cdot \boldsymbol{\omega}_i) \cdot \boldsymbol{\beta}_{ij} u_j] \\ &- \sum_{i=1}^{N}(m_i \mathbf{v}_i \cdot \dot{\gamma}_{it} + \boldsymbol{\omega}_i \cdot \mathbf{I}_{ci} \cdot \dot{\boldsymbol{\beta}}_{it})\end{aligned} \tag{4.374}$$

Then, using (4.366), we obtain

$$\dot{T}_2 - \dot{T}_0 = \sum_{j=1}^{n-m} Q_j u_j - \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \dot{\gamma}_{it} - \sum_{i=1}^{N} \boldsymbol{\omega}_i \cdot \mathbf{I}_{ci} \cdot \dot{\boldsymbol{\beta}}_{it} \tag{4.375}$$

Let us assume that a portion of Q_j is obtained from a potential function $V(q, t)$. Thus, we can write

$$Q_j = Q'_j - \sum_{k=1}^{n} \frac{\partial V}{\partial q_k}\frac{\partial \dot{q}_k}{\partial u_j} = Q'_j - \sum_{k=1}^{n} \frac{\partial V}{\partial q_k}\Phi_{kj} \tag{4.376}$$

where Q'_j is that portion which is not obtainable from a potential energy function. Also note that

$$\dot{q}_k = \sum_{j=1}^{n-m} \Phi_{kj} u_j + \Phi_{kt} \tag{4.377}$$

and therefore

$$\dot{V} - \frac{\partial V}{\partial t} = \sum_{j=1}^{n-m} \sum_{k=1}^{n} \frac{\partial V}{\partial q_k} \Phi_{kj} u_j + \sum_{k=1}^{n} \frac{\partial V}{\partial q_k} \Phi_{kt} \tag{4.378}$$

Finally, adding (4.375) and (4.378), and using (4.376), we obtain

$$\begin{aligned}\dot{E} = \dot{T}_2 - \dot{T}_0 + \dot{V} = &\sum_{j=1}^{n-m} Q'_j u_j + \frac{\partial V}{\partial t} + \sum_{k=1}^{n} \frac{\partial V}{\partial q_k} \Phi_{kt} \\ &- \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \dot{\boldsymbol{\gamma}}_{it} - \sum_{i=1}^{N} \boldsymbol{\omega}_i \cdot \mathbf{I}_{ci} \cdot \dot{\boldsymbol{\beta}}_{it}\end{aligned} \tag{4.379}$$

This is the general energy rate equation for a system of rigid bodies. An alternate form is

$$\dot{E} = \dot{T}_2 - \dot{T}_0 + \dot{V} = \sum_{j=1}^{n-m} Q'_j u_j + \frac{\partial V}{\partial t} + \sum_{k=1}^{n} \frac{\partial V}{\partial q_k} \Phi_{kt} - \sum_{i=1}^{N} \mathbf{p}_i \cdot \dot{\boldsymbol{\gamma}}_{it} - \sum_{i=1}^{N} \mathbf{H}_{ci} \cdot \dot{\boldsymbol{\beta}}_{it} \tag{4.380}$$

where

$$\mathbf{p}_i = m_i \mathbf{v}_i \tag{4.381}$$

$$\mathbf{H}_{ci} = \mathbf{I}_{ci} \cdot \boldsymbol{\omega}_i \tag{4.382}$$

and we use a center of mass reference point on each body.

The meaning of these energy rate equations can be clarified by noting that $\boldsymbol{\gamma}_{it}$ represents the velocity of the ith reference point when all the us are set equal to zero. Similarly, $\boldsymbol{\beta}_{it}$ is equal to the angular velocity of the ith body if all us equal zero. Additionally, note that Φ_{kt} is equal to the value of $\dot{q}_k$ when all the us are set equal to zero.

Another approach to energy rate calculations is to begin with the Boltzmann–Hamel equation in the general form of (4.85)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right) - \sum_{i=1}^{n} \frac{\partial T}{\partial q_i} \Phi_{ir} + \sum_{j=1}^{m} \sum_{l=1}^{n-m} \frac{\partial T}{\partial u_j} \gamma_{rl}^j u_l + \sum_{j=1}^{n} \frac{\partial T}{\partial u_j} \gamma_r^j = Q_r \qquad (r = 1, \ldots, n-m) \tag{4.383}$$

Multiply by u_r and sum over r, using (4.376). The result is

$$\begin{aligned}&\sum_{r=1}^{n-m} \frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right) u_r - \sum_{i=1}^{n} \sum_{r=1}^{n-m} \frac{\partial T}{\partial q_i} \Phi_{ir} u_r + \sum_{j=1}^{n} \sum_{l=1}^{n-m} \sum_{r=1}^{n-m} \frac{\partial T}{\partial u_j} \gamma_{rl}^j u_l u_r \\ &+ \sum_{j=1}^{n} \sum_{r=1}^{n-m} \frac{\partial T}{\partial u_j} \gamma_r^j u_r = \sum_{r=1}^{n-m} Q'_r u_r - \sum_{i=1}^{n} \sum_{r=1}^{n-m} \frac{\partial V}{\partial q_i} \Phi_{ir} u_r\end{aligned} \tag{4.384}$$

Using Euler's theorem, as in (4.368), we find that

$$\frac{d}{dt}\left(\sum_{r=1}^{n-m}\frac{\partial T}{\partial u_r}u_r\right)=\sum_{r=1}^{n-m}\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right)u_r+\sum_{r=1}^{n-m}\frac{\partial T}{\partial u_r}\dot{u}_r=2\dot{T}_2+\dot{T}_1 \tag{4.385}$$

Now

$$\dot{T}=\sum_{i=1}^{n}\frac{\partial T}{\partial q_i}\dot{q}_i+\sum_{r=1}^{n-m}\frac{\partial T}{\partial u_r}\dot{u}_r+\frac{\partial T}{\partial t}=\dot{T}_2+\dot{T}_1+\dot{T}_0 \tag{4.386}$$

so, from (4.385) and (4.386), we obtain

$$\sum_{r=1}^{n-m}\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right)u_r=\dot{T}_2-\dot{T}_0+\sum_{i=1}^{n}\frac{\partial T}{\partial q_i}\dot{q}_i+\frac{\partial T}{\partial t} \tag{4.387}$$

Furthermore, we see that

$$\dot{V}=\sum_{i=1}^{n}\frac{\partial V}{\partial q_i}\dot{q}_i+\frac{\partial V}{\partial t} \tag{4.388}$$

and

$$\sum_{r=1}^{n-m}\Phi_{ir}u_r=\dot{q}_i-\Phi_{it} \tag{4.389}$$

From (4.387) and (4.388), we see that

$$\dot{T}_2-\dot{T}_0+\dot{V}=\sum_{r=1}^{n-m}\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right)u_r-\sum_{i=1}^{n}\frac{\partial L}{\partial q_i}\dot{q}_i-\frac{\partial L}{\partial t} \tag{4.390}$$

where $L(q,u,t)=T(q,u,t)-V(q,t)$. Because of the skew symmetry of γ_{rl}^{j} with respect to r and l, we note that

$$\sum_{l=1}^{n-m}\sum_{r=1}^{n-m}\gamma_{rl}^{j}u_r u_l=0 \tag{4.391}$$

Then, from (4.384) and (4.389), we obtain

$$\sum_{r=1}^{n-m}\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right)u_r=\sum_{r=1}^{n-m}Q_r'u_r+\sum_{i=1}^{n}\frac{\partial L}{\partial q_i}(\dot{q}_i-\Phi_{it})-\sum_{j=1}^{n}\sum_{r=1}^{n-m}\frac{\partial T}{\partial u_j}\gamma_r^j u_r \tag{4.392}$$

Finally, using (4.390) and (4.392), we have the energy rate expression

$$\dot{E}=\dot{T}_2-\dot{T}_0+\dot{V}=\sum_{r=1}^{n-m}Q_r'u_r-\frac{\partial L}{\partial t}-\sum_{i=1}^{n}\frac{\partial L}{\partial q_i}\Phi_{it}-\sum_{j=1}^{n}\sum_{r=1}^{n-m}\frac{\partial T}{\partial u_j}\gamma_r^j u_r \tag{4.393}$$

Let us compare the energy rate equations (4.380) and (4.393). For a general system of N rigid bodies, we have the corresponding terms

$$\sum_{i=1}^{N}(\mathbf{p}_i\cdot\dot{\boldsymbol{\gamma}}_{it}+\mathbf{H}_{ci}\cdot\dot{\boldsymbol{\beta}}_{it})=\frac{\partial T}{\partial t}+\sum_{k=1}^{n}\frac{\partial T}{\partial q_k}\Phi_{kt}+\sum_{j=1}^{n}\sum_{r=1}^{n-m}\frac{\partial T}{\partial u_j}\gamma_r^j u_r \tag{4.394}$$

where

$$\gamma_r^j = \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial \Psi_{ji}}{\partial q_k} - \frac{\partial \Psi_{jk}}{\partial q_i}\right)\Phi_{kt}\Phi_{ir} + \sum_{i=1}^{n}\left(\frac{\partial \Psi_{ji}}{\partial t} - \frac{\partial \Psi_{jt}}{\partial q_i}\right)\Phi_{ir} \tag{4.395}$$

It appears that, in the general nonholonomic case, the left-hand side of (4.394) is easier to evaluate that is its right-hand side. Thus, (4.380) is more direct than (4.393) in the general case.

From (4.380), we see that sufficient conditions for a *conservative system*, implying a constant value of E, are:

1. $Q'_j = 0$ for all j, that is, all the generalized forces Q_j are derivable from a potential energy function of the form $V(q)$.
2. The functions Φ_{kt}, $\mathbf{p}_i \cdot \dot{\gamma}_{it}$, and $\mathbf{H}_{ci} \cdot \dot{\boldsymbol{\beta}}_{it}$ are all continuously equal to zero.

Example 4.12 A particle of mass m can slide on a wire in the form of a circle of radius r which rotates about a vertical diameter with a variable angular velocity $\Omega(t)$ (Fig. 4.8). We wish to determine the energy rate $\dot{E}$.

This is a rheonomic holonomic system with one generalized coordinate and no constraints. Lagrange's equation applies and (4.355) reduces to

$$\dot{E} = \dot{T}_2 - \dot{T}_0 + \dot{V} = -\frac{\partial L}{\partial t} \tag{4.396}$$

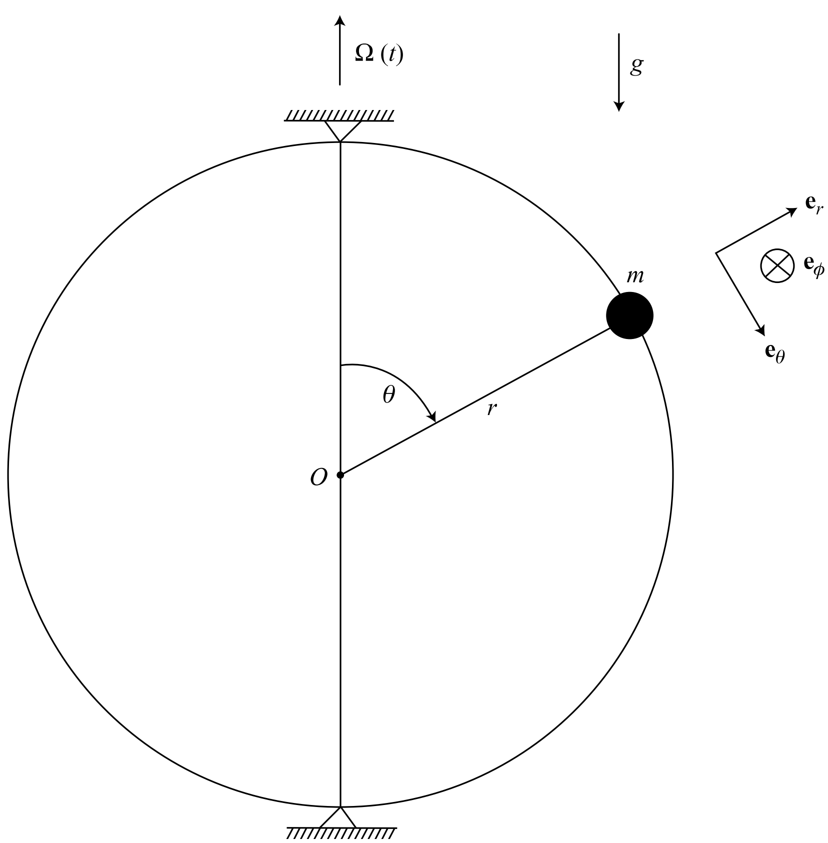

Figure 4.8.

We see that

$$T = \frac{1}{2}mr^2(\dot{\theta}^2 + \Omega^2 \sin^2\theta) \tag{4.397}$$

$$V = mgr\cos\theta \tag{4.398}$$

Thus, we obtain

$$\dot{E} = -\frac{\partial T}{\partial t} = -mr^2\Omega\dot{\Omega}\sin^2\theta \tag{4.399}$$

For the particular case in which Ω is constant, we see that $E = T_2 - T_0 + V$ has a constant value and the system is conservative. The total energy $T + V$ is not constant, however, because there is a torque about the vertical axis that is required to keep the angular velocity Ω constant even though θ is varying.

If we use the Boltzmann–Hamel approach, as given in (4.393), the same result occurs. Here we can take $u_1 = \dot{\theta}$ and note that both Φ_{it} and γ_r^j vanish.

Finally, if we use the general energy rate expression in (4.379), we find that it reduces to

$$\dot{E} = -m\mathbf{v}_1 \cdot \dot{\boldsymbol{\gamma}}_{1t} \tag{4.400}$$

where

$$\mathbf{v}_1 = r\dot{\theta}\mathbf{e}_\theta + r\Omega\sin\theta\,\mathbf{e}_\phi \tag{4.401}$$

$$\boldsymbol{\gamma}_{1t} = r\Omega\sin\theta\,\mathbf{e}_\phi \tag{4.402}$$

Now

$$\dot{\mathbf{e}}_\phi = \boldsymbol{\Omega} \times \mathbf{e}_\phi = -\Omega\sin\theta\,\mathbf{e}_r - \Omega\cos\theta\,\mathbf{e}_\theta \tag{4.403}$$

so

$$\dot{\boldsymbol{\gamma}}_{1t} = -r\Omega^2\sin^2\theta\,\mathbf{e}_r - r\Omega^2\sin\theta\cos\theta\,\mathbf{e}_\theta + r(\dot{\Omega}\sin\theta + \Omega\dot{\theta}\cos\theta)\mathbf{e}_\phi \tag{4.404}$$

Thus, (4.400) results in

$$\dot{E} = -mr^2\Omega\dot{\Omega}\sin^2\theta \tag{4.405}$$

in agreement with our earlier results.

One can check the energy rate by first noting that

$$\dot{T}_2 = mr^2\dot{\theta}\ddot{\theta} \tag{4.406}$$

$$\dot{T}_0 = mr^2\Omega^2\dot{\theta}\sin\theta\cos\theta + mr^2\Omega\dot{\Omega}\sin^2\theta \tag{4.407}$$

$$\dot{V} = -mgr\dot{\theta}\sin\theta \tag{4.408}$$

The equation of motion, obtained from Lagrange's equation, is

$$mr^2\ddot{\theta} - mr^2\Omega^2\sin\theta\cos\theta - mgr\sin\theta = 0 \tag{4.409}$$

Substitute the expression for $mr^2\ddot{\theta}$ from (4.409) into (4.406). Then we find that

$$\dot{E} = \dot{T}_2 - \dot{T}_0 + \dot{V} = -mr^2\Omega\dot{\Omega}\sin^2\theta \tag{4.410}$$

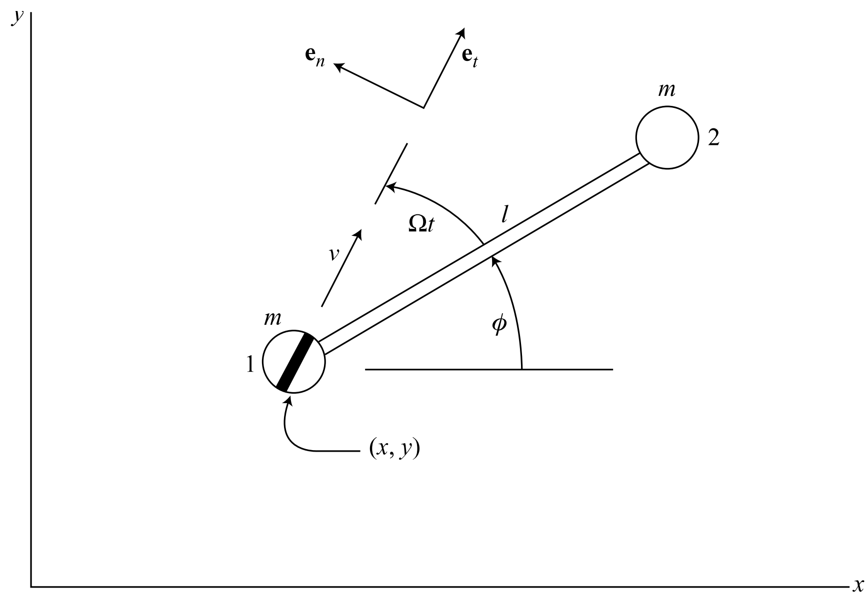

Figure 4.9.

which checks with (4.405).

Example 4.13 A rheonomic nonholonomic system consists of two particles, each of mass m which are connected by a massless rod of length l, as shown in Fig. 4.9. Particle 1 has a nonholonomic constraint in the form of a knife-edge which rotates at a constant rate Ω relative to the rod. Let us choose (x, y, ϕ) as qs, and let

$$u_1 = v, \qquad u_2 = \dot{\phi} \tag{4.411}$$

where v is a quasi-velocity. We wish to find the differential equations of motion and to evaluate the energy rate $\dot{E}$.

First, the differential equations of motion are obtained from the fundamental equation for a system of particles, namely,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_j}\right) - \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \dot{\boldsymbol{\gamma}}_{ij} = Q_j \tag{4.412}$$

Assume that the xy-plane is horizontal. The only constraint force is perpendicular to the knife edge and does no work in an arbitrary virtual displacement. Therefore, each generalized applied force Q_j is zero. The velocities of the two particles are

$$\mathbf{v}_1 = v\mathbf{e}_t, \qquad \mathbf{v}_2 = (v + l\dot{\phi}\sin\Omega t)\mathbf{e}_t + l\dot{\phi}\cos\Omega t\;\mathbf{e}_n \tag{4.413}$$

Thus, the constrained kinetic energy is

$$T = \frac{1}{2}m\left(\mathbf{v}_1^2 + \mathbf{v}_2^2\right) = \frac{1}{2}m(2v^2 + l^2\dot{\phi}^2 + 2lv\dot{\phi}\sin\Omega t) \tag{4.414}$$

The velocity coefficients are

$$\boldsymbol{\gamma}_{ij} = \frac{\partial \mathbf{v}_i}{\partial u_j} \tag{4.415}$$

resulting in

$$\gamma_{11} = \mathbf{e}_t, \qquad \gamma_{12} = 0, \qquad \gamma_{21} = \mathbf{e}_t, \qquad \gamma_{22} = l(\sin\Omega t\ \mathbf{e}_t + \cos\Omega t\ \mathbf{e}_n) \tag{4.416}$$

Also, we note that the γ_{it} coefficients are equal to zero.

A differentiation of (4.416) with respect to time results in

$$\dot{\gamma}_{12} = 0, \qquad \dot{\gamma}_{11} = \dot{\gamma}_{21} = (\dot{\phi} + \Omega)\mathbf{e}_n, \qquad \dot{\gamma}_{22} = l\dot{\phi}(-\cos\Omega t\ \mathbf{e}_t + \sin\Omega t\ \mathbf{e}_n) \tag{4.417}$$

where we note that $\mathbf{e}_t$ and $\mathbf{e}_n$ rotate counterclockwise with an angular velocity $(\dot{\phi} + \Omega)$.

The u_1 equation is obtained by first evaluating

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v}\right) = 2m\dot{v} + ml\ddot{\phi}\sin\Omega t + ml\Omega\dot{\phi}\cos\Omega t \tag{4.418}$$

In addition, we find that

$$m(\mathbf{v}_1 \cdot \dot{\gamma}_{11} + \mathbf{v}_2 \cdot \dot{\gamma}_{21}) = ml\dot{\phi}(\dot{\phi} + \Omega)\cos\Omega t \tag{4.419}$$

Then, using (4.412), the first equation of motion is

$$2m\dot{v} + ml\ddot{\phi}\sin\Omega t - ml\dot{\phi}^2\cos\Omega t = 0 \tag{4.420}$$

In a similar manner, we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = ml^2\ddot{\phi} + ml\dot{v}\sin\Omega t + ml\Omega v\cos\Omega t \tag{4.421}$$

$$m(\mathbf{v}_1 \cdot \dot{\gamma}_{12} + \mathbf{v}_2 \cdot \dot{\gamma}_{22}) = -mlv\dot{\phi}\cos\Omega t \tag{4.422}$$

The second equation of motion is

$$ml^2\ddot{\phi} + ml\dot{v}\sin\Omega t + mlv(\dot{\phi} + \Omega)\cos\Omega t = 0 \tag{4.423}$$

These two equations of motion constitute a minimum set for this system which has two degrees of freedom.

Equation (4.420) can be interpreted as stating that the rate of change of linear momentum in the direction of the knife edge is equal to zero. The second equation of motion, (4.423), states that, if particle 1 is chosen as a noninertial reference point, the rate of change of angular momentum is equal to the inertial moment due to the acceleration of particle 1. Thus, it is convenient to think in terms of an accelerating but nonrotating reference frame in this instance.

The energy rate $\dot{E}$ for a system of N particles can be written in the form

$$\dot{E} = \sum_{r=1}^{n-m} Q'_r u_r + \frac{\partial V}{\partial t} + \sum_{k=1}^{n} \frac{\partial V}{\partial q_k}\Phi_{kt} - \sum_{i=1}^{N} m_i \mathbf{v}_i \cdot \dot{\gamma}_{it} \tag{4.424}$$

in accordance with (4.379). We find that Q'_r, V, Φ_{kt} and γ_{it} are all equal to zero, so $\dot{E} = 0$. Thus, the energy function E, which in this case is the total kinetic energy, is constant during the motion.

This is an example of a system for which the kinetic energy is an explicit function of time and yet it is conservative.

Second method If the Lagrangian method is used, we have three differential equations of motion involving Lagrange multipliers. Also, there is the nonholonomic constraint equation

$$-\dot{x}\sin(\phi + \Omega t) + \dot{y}\cos(\phi + \Omega t) = 0 \tag{4.425}$$

which states that the velocity of particle 1 normal to the knife edge is zero.

The kinetic energy of the unconstrained system is

$$T = \frac{1}{2}m[2\dot{x}^2 + 2\dot{y}^2 + l^2\dot{\phi}^2 + 2l\dot{\phi}(-\dot{x}\sin\phi + \dot{y}\cos\phi)] \tag{4.426}$$

We note that T is not an explicit function of time, and we can take the potential energy function V equal to zero; hence $\partial L/\partial t = 0$. Furthermore, the nonholonomic constraint is homogeneous and linear in the $\dot{q}$s, and is therefore conservative. Thus, the sufficient conditions for a conservative system are satisfied, implying, in this case, that the total kinetic energy is a constant of the motion.

Third method Let us use the Boltzmann–Hamel approach. The equations for a complete set of three us in terms of $\dot{q}$s are

$$u_1 = v = \dot{x}\cos(\phi + \Omega t) + \dot{y}\sin(\phi + \Omega t) \tag{4.427}$$

$$u_2 = \dot{\phi} \tag{4.428}$$

$$u_3 = -\dot{x}\sin(\phi + \Omega t) + \dot{y}\cos(\phi + \Omega t) \tag{4.429}$$

The nonholonomic constraint is applied by setting u_3 equal to zero.

The unconstrained kinetic energy is

$$T = m\left[u_1^2 + \frac{1}{2}l^2u_2^2 + u_3^2 + lu_2(u_1\sin\Omega t + u_3\cos\Omega t)\right] \tag{4.430}$$

and the general Boltzmann–Hamel equation (4.85) reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial u_r}\right) + \sum_{j=1}^{3}\sum_{l=1}^{2}\frac{\partial T}{\partial u_j}\gamma_{rl}^j u_l + \sum_{j=1}^{3}\frac{\partial T}{\partial u_j}\gamma_r^j = 0 \qquad (r = 1, 2) \tag{4.431}$$

In evaluating γ_{rl}^j and γ_r^j, we note that the Ψ_{ji} coefficients are explicit functions of time, in general, but all the Ψ_{jt} coefficients vanish, as well as the Φ_{kt}.

After a rather lengthy calculation, the equations of motion found earlier in (4.420) and (4.423) are obtained. The Boltzmann–Hamel energy rate expression in (4.393) reduces for this example to

$$\dot{E} = -\frac{\partial T}{\partial t} - \sum_{j=1}^{3}\sum_{r=1}^{2}\frac{\partial T}{\partial u_j}\gamma_r^j u_r \tag{4.432}$$

We find that

$$\frac{\partial T}{\partial t} = ml\Omega v\dot{\phi}\cos\Omega t \tag{4.433}$$

and

$$\sum_{j=1}^{3}\sum_{r=1}^{2}\frac{\partial T}{\partial u_j}\gamma_r^j u_r = -ml\Omega v\dot{\phi}\cos\Omega t \tag{4.434}$$

Hence, $\dot{E} = 0$ and $E = T$ is a constant of the motion even though T is an explicit function of time. Note that the kinetic energy is a homogeneous quadratic function of the us.

Comparing the three methods which were presented for analyzing this rheonomic nonholonomic system, the first method using the fundamental dynamical equation for a system of particles would seem to be preferable. It provides a minimum set of equations of motion without Lagrange multipliers. Furthermore, the energy rate is found to be zero by inspection.

4.8 Summary of differential methods

In the study of differential methods in the dynamics of systems of particles or rigid bodies, it is well to begin with Newton's law of motion. Angular momentum methods can also be employed, resulting in Euler's equations for the rotational motion of rigid bodies. These elementary approaches frequently require the introduction of constraint forces and moments as additional variables in the dynamical equations, thereby complicating the analysis.

Lagrange's equations, when applied to holonomic systems with independent qs, result in a minimum set of equations of motion without the necessity of solving for constraint forces. In other words, although one set of qs may be subject to holonomic constraints, another set of qs can be found which are independent and are consistent with the previous constraints. No generalized constraint forces enter into the Lagrange equations of motion for this system.

On the other hand, if there are nonholonomic constraints, then more qs are required than the number of degrees of freedom. The use of the Lagrangian procedure involves Lagrange multipliers which are associated with generalized constraint forces. If there are n qs and m constraint equations, one obtains n second-order dynamical equations in addition to the m constraint equations.

The use of Maggi's equation eliminates the Lagrange multipliers and results in $(n - m)$ second-order equations of motion plus the m constraint equations. The Lagrange and Maggi methods have the disadvantages, however, that the kinetic energy cannot be written in terms of quasi-velocities, and must be written for the unconstrained system.

In the efficient representation of dynamical systems, it is desirable to obtain a minimal set of $(n - m)$ first-order differential equations of motion. This is possible in the general nonholonomic case if one uses independent quasi-velocities as velocity variables. The remaining four differential methods discussed in this chapter all result in a minimal set of dynamical equations. The use of the Boltzmann–Hamel equation is the most complicated of these methods and requires that the kinetic energy be written for the unconstrained system having n degrees of freedom. The other three methods allow one to assume a constrained system with $(n - m)$ degrees of freedom from the beginning.