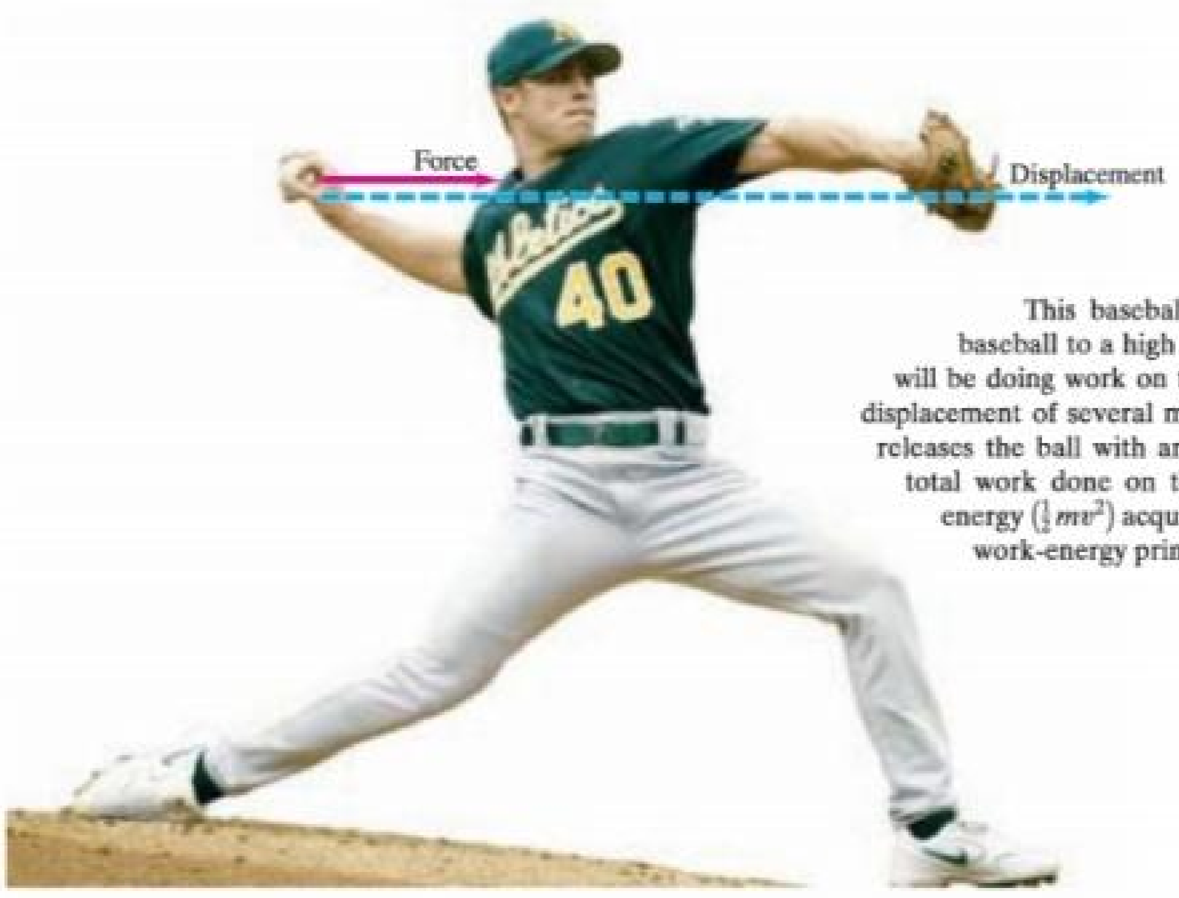


This baseball pitcher is about to accelerate the baseball to a high velocity by exerting a force on it. He will be doing work on the ball as he exerts the force over a displacement of several meters, from behind his head until he releases the ball with arm outstretched in front of him. The total work done on the ball will be equal to the kinetic energy ($\frac{1}{2}mv^2$) acquired by the ball, a result known as the work-energy principle.

CHAPTER 7

Work and Energy

CHAPTER-OPENING QUESTION—Guess now!
You push very hard on a heavy desk, trying to move it. You do work on the desk:
(a) Whether or not it moves, as long as you are exerting a force.
(b) Only if it starts moving.
(c) Only if it doesn't move.
(d) Never—it does work on you.
(e) None of the above.

Until now we have been studying the translational motion of an object in terms of Newton's three laws of motion. In that analysis, *force* has played a central role as the quantity determining the motion. In this Chapter and the two that follow, we discuss an alternative analysis of the translational motion of objects in terms of the quantities *energy* and *momentum*. The significance of energy and momentum is that they are *conserved*. In quite general circumstances they remain constant. That conserved quantities exist gives us not only a deeper insight into the nature of the world but also gives us another way to approach solving practical problems.

The conservation laws of energy and momentum are especially valuable in dealing with systems of many objects, in which a detailed consideration of the forces involved would be difficult or impossible. These laws are applicable to a wide range of phenomena, including the atomic and subatomic worlds, where Newton's laws cannot be applied.

This Chapter is devoted to the very important concept of *energy* and the closely related concept of *work*. These two quantities are scalars and so have no direction associated with them, which often makes them easier to work with than vector quantities such as acceleration and force.

CONTENTS

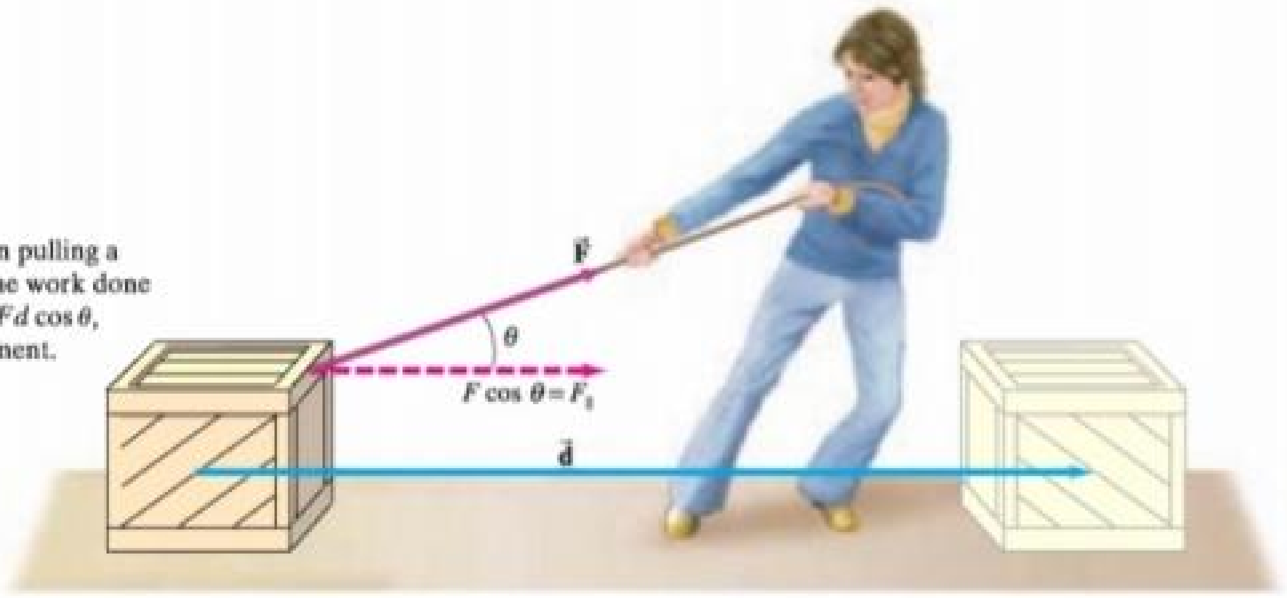


FIGURE 7–1 A person pulling a crate along the floor. The work done by the force $\vec{\mathbf{F}}$ is $W = Fd\cos\theta$, where $\vec{\mathbf{d}}$ is the displacement.

7–1 Work Done by a Constant Force

The word *work* has a variety of meanings in everyday language. But in physics, work is given a very specific meaning to describe what is accomplished when a force acts on an object, and the object moves through a distance. We consider only translational motion for now and, unless otherwise explained, objects are assumed to be rigid with no complicating internal motion, and can be treated like particles. Then the **work** done on an object by a constant force (constant in both magnitude and direction) is defined to be *the product of the magnitude of the displacement times the component of the force parallel to the displacement*. In equation form, we can write

$$W = F_{\parallel} d,$$

where $F_{\parallel}$ is the component of the constant force $\vec{\mathbf{F}}$ parallel to the displacement $\vec{\mathbf{d}}$. We can also write

$$W = Fd\cos\theta, \qquad (7\text{–}1)$$

where F is the magnitude of the constant force, d is the magnitude of the displacement of the object, and θ is the angle between the directions of the force and the displacement (Fig. 7–1). The $\cos\theta$ factor appears in Eq. 7–1 because $F\cos\theta\,(=F_{\parallel})$ is the component of $\vec{\mathbf{F}}$ that is parallel to $\vec{\mathbf{d}}$. Work is a scalar quantity—it has only magnitude, which can be positive or negative.

Let us consider the case in which the motion and the force are in the same direction, so $\theta = 0$ and $\cos\theta = 1$; in this case, $W = Fd$. For example, if you push a loaded grocery cart a distance of 50 m by exerting a horizontal force of 30 N on the cart, you do $30\,\text{N} \times 50\,\text{m} = 1500\,\text{N}\cdot\text{m}$ of work on the cart.

As this example shows, in SI units work is measured in newton-meters (N·m). A special name is given to this unit, the **joule** (J): $1\,\text{J} = 1\,\text{N}\cdot\text{m}$.

[In the cgs system, the unit of work is called the *erg* and is defined as $1\,\text{erg} = 1\,\text{dyne}\cdot\text{cm}$. In British units, work is measured in foot-pounds. It is easy to show that $1\,\text{J} = 10^7\,\text{erg} = 0.7376\,\text{ft}\cdot\text{lb}$.]

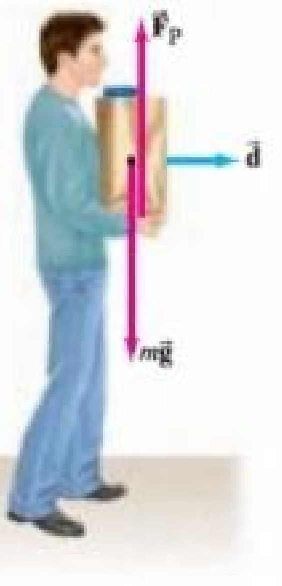


FIGURE 7–2 The person does no work on the bag of groceries since $\vec{\mathbf{F}}_P$ is perpendicular to the displacement $\vec{\mathbf{d}}$.

A force can be exerted on an object and yet do no work. If you hold a heavy bag of groceries in your hands at rest, you do no work on it. You do exert a force on the bag, but the displacement of the bag is zero, so the work done by you on the bag is $W = 0$. You need both a force and a displacement to do work. You also do no work on the bag of groceries if you carry it as you walk horizontally across the floor at constant velocity, as shown in Fig. 7–2. No horizontal force is required to move the bag at a constant velocity. The person shown in Fig. 7–2 does exert an upward force $\vec{\mathbf{F}}_P$ on the bag equal to its weight. But this upward force is perpendicular to the horizontal displacement of the bag and thus is doing no work. This conclusion comes from our definition of work, Eq. 7–1: $W = 0$, because $\theta = 90°$ and $\cos 90° = 0$.

Thus, when a particular force is perpendicular to the displacement, no work is done by that force. When you start or stop walking, there is a horizontal acceleration and you do briefly exert a horizontal force, and thus do work on the bag.

When we deal with work, as with force, it is necessary to specify whether you are talking about work done *by* a specific object or done *on* a specific object. It is also important to specify whether the work done is due to one particular force (and which one), or the total (net) work done by the *net force* on the object.

⚠ CAUTION
State that work is done on or by an object

EXAMPLE 7–1 **Work done on a crate.** A person pulls a 50-kg crate 40 m along a horizontal floor by a constant force $F_P = 100\,\text{N}$, which acts at a 37° angle as shown in Fig. 7–3. The floor is smooth and exerts no friction force. Determine (*a*) the work done by each force acting on the crate, and (*b*) the net work done on the crate.

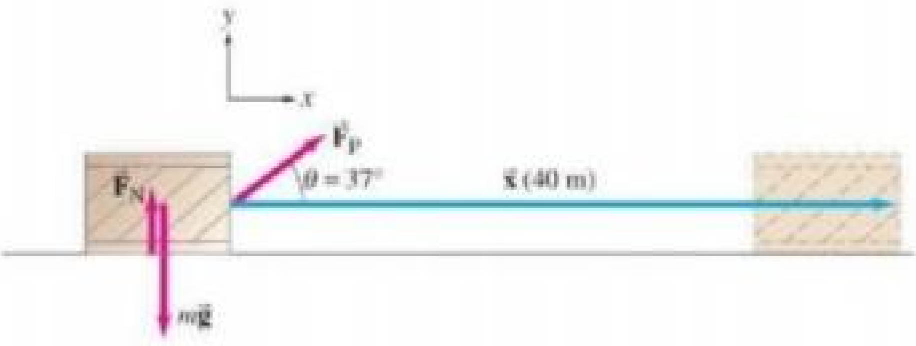


FIGURE 7–3 Example 7–1. A 50-kg crate is pulled along a smooth floor.

APPROACH We choose our coordinate system so that $\vec{x}$ can be the vector that represents the 40-m displacement (that is, along the *x* axis). Three forces act on the crate, as shown in Fig. 7–3: the force exerted by the person $\vec{F}_P$; the gravitational force exerted by the Earth, $m\vec{g}$; and the normal force $\vec{F}_N$ exerted upward by the floor. The net force on the crate is the vector sum of these three forces.

SOLUTION (*a*) The work done by the gravitational and normal forces is zero, since they are perpendicular to the displacement $\vec{x}$ ($\theta = 90°$ in Eq. 7–1):

$$W_G = mgx\cos 90° = 0$$
$$W_N = F_N x\cos 90° = 0.$$

The work done by $\vec{F}_P$ is

$$W_P = F_P x\cos\theta = (100\,\text{N})(40\,\text{m})\cos 37° = 3200\,\text{J}.$$

(*b*) The net work can be calculated in two equivalent ways:

(1) The net work done on an object is the algebraic sum of the work done by each force, since work is a scalar:

$$W_{net} = W_G + W_N + W_P = 0 + 0 + 3200\,\text{J} = 3200\,\text{J}.$$

(2) The net work can also be calculated by first determining the net force on the object and then taking its component along the displacement: $(F_{net})_x = F_P\cos\theta$. Then the net work is

$$W_{net} = (F_{net})_x x = (F_P\cos\theta)x = (100\,\text{N})(\cos 37°)(40\,\text{m}) = 3200\,\text{J}.$$

In the vertical (*y*) direction, there is no displacement and no work done.

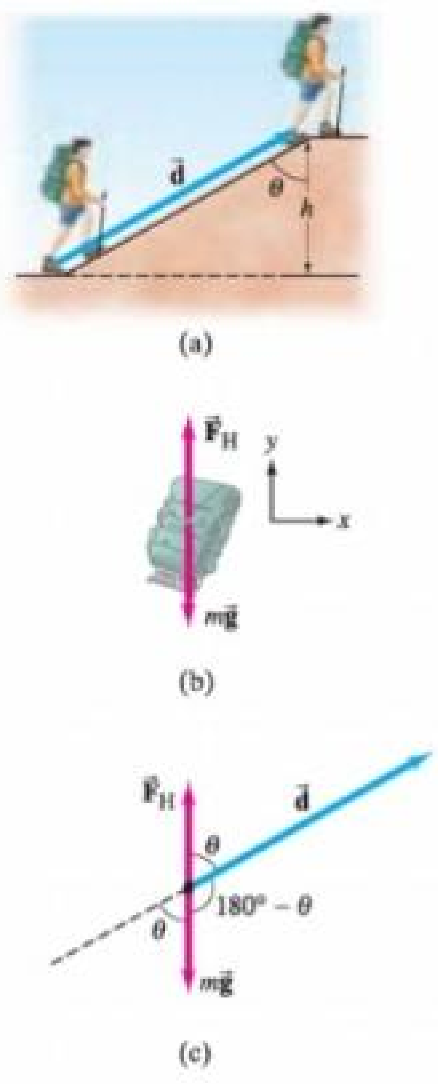
EXERCISE A A box is dragged a distance *d* across a floor by a force $\vec{F}_P$ which makes an angle θ with the horizontal as in Fig. 7–1 or 7–3. If the magnitude of $\vec{F}_P$ is held constant but the angle θ is increased, the work done by $\vec{F}_P$ (*a*) remains the same; (*b*) increases; (*c*) decreases; (*d*) first increases, then decreases.

EXERCISE B Return to the Chapter-Opening Question, page 163, and answer it again now. Try to explain why you may have answered differently the first time.

PROBLEM SOLVING
Work

1. **Draw a free-body diagram** showing all the forces acting on the object you choose to study.
2. **Choose an *xy* coordinate system.** If the object is in motion, it may be convenient to choose one of the coordinate directions as the direction of one of the forces, or as the direction of motion. [Thus, for an object on an incline, you might choose one coordinate axis to be parallel to the incline.]
3. **Apply Newton's laws** to determine any unknown forces.
4. Find the **work done *by* a specific force *on*** the object by using $W = Fd\cos\theta$ for a constant force. Note that the work done is negative when a force tends to oppose the displacement.
5. To find the **net work** done on the object, either (*a*) find the work done by each force and add the results algebraically; or (*b*) find the net force on the object, F_{net}, and then use it to find the net work done, which for constant net force is:
$$W_{net} = F_{net}\,d\cos\theta.$$

FIGURE 7–4 Example 7–2.



EXAMPLE 7–2 **Work on a backpack.** (*a*) Determine the work a hiker must do on a 15.0-kg backpack to carry it up a hill of height $h = 10.0\,\text{m}$, as shown in Fig. 7–4a. Determine also (*b*) the work done by gravity on the backpack, and (*c*) the net work done on the backpack. For simplicity, assume the motion is smooth and at constant velocity (i.e., acceleration is zero).

APPROACH We explicitly follow the steps of the Problem Solving Strategy above.

SOLUTION

1. **Draw a free-body diagram.** The forces on the backpack are shown in Fig. 7–4b: the force of gravity, $m\vec{g}$, acting downward; and $\vec{F}_H$, the force the hiker must exert upward to support the backpack. The acceleration is zero, so horizontal forces on the backpack are negligible.
2. **Choose a coordinate system.** We are interested in the vertical motion of the backpack, so we choose the *y* coordinate as positive vertically upward.
3. **Apply Newton's laws.** Newton's second law applied in the vertical direction to the backpack gives
$$\Sigma F_y = ma_y$$
$$F_H - mg = 0$$
since $a_y = 0$. Hence,
$$F_H = mg = (15.0\,\text{kg})(9.80\,\text{m/s}^2) = 147\,\text{N}.$$
4. **Work done by a specific force.** (*a*) To calculate the work done by the hiker on the backpack, we write Eq. 7–1 as
$$W_H = F_H(d\cos\theta),$$
and we note from Fig. 7–4a that $d\cos\theta = h$. So the work done by the hiker is
$$W_H = F_H(d\cos\theta) = F_H h = mgh = (147\,\text{N})(10.0\,\text{m}) = 1470\,\text{J}.$$
Note that the work done depends only on the change in elevation and not on the angle of the hill, θ. The hiker would do the same work to lift the pack vertically the same height *h*.
(*b*) The work done by gravity on the backpack is (from Eq. 7–1 and Fig. 7–4c)
$$W_G = F_G\,d\cos(180° - \theta).$$
Since $\cos(180° - \theta) = -\cos\theta$, we have
$$W_G = F_G\,d(-\cos\theta) = mg(-d\cos\theta) = -mgh = -(15.0\,\text{kg})(9.80\,\text{m/s}^2)(10.0\,\text{m}) = -1470\,\text{J}.$$

PROBLEM SOLVING
Work done by gravity depends on the height of the hill and not on the angle of incline

NOTE The work done by gravity (which is negative here) doesn't depend on the angle of the incline, only on the vertical height *h* of the hill. This is because gravity acts vertically, so only the vertical component of displacement contributes to work done.

5. **Net work done.** (*c*) The *net* work done on the backpack is $W_{net} = 0$, since the net force on the backpack is zero (it is assumed not to accelerate significantly). We can also determine the net work done by adding the work done by each force:
$$W_{net} = W_G + W_H = -1470\,\text{J} + 1470\,\text{J} = 0.$$

NOTE Even though the *net* work done by all the forces on the backpack is zero, the hiker *does do* work on the backpack equal to 1470 J.

CONCEPTUAL EXAMPLE 7–3 **Does the Earth do work on the Moon?** The Moon revolves around the Earth in a nearly circular orbit, with approximately constant tangential speed, kept there by the gravitational force exerted by the Earth. Does gravity do (*a*) positive work, (*b*) negative work, or (*c*) no work at all on the Moon?

RESPONSE The gravitational force $\vec{\mathbf{F}}_G$ on the Moon (Fig. 7–5) acts toward the Earth and provides its centripetal force, inward along the radius of the Moon's orbit. The Moon's displacement at any moment is tangent to the circle, in the direction of its velocity, perpendicular to the radius and perpendicular to the force of gravity. Hence the angle θ between the force $\vec{\mathbf{F}}_G$ and the instantaneous displacement of the Moon is 90°, and the work done by gravity is therefore zero ($\cos 90° = 0$). This is why the Moon, as well as artificial satellites, can stay in orbit without expenditure of fuel: no work needs to be done against the force of gravity.

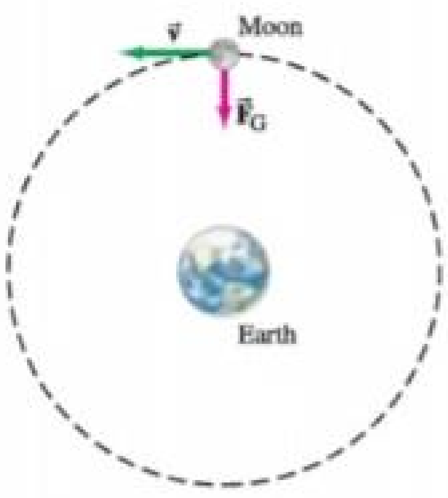


FIGURE 7–5 Example 7–3.

7–2 Scalar Product of Two Vectors

Although work is a scalar, it involves the product of two quantities, force and displacement, both of which are vectors. Therefore, we now investigate the multiplication of vectors, which will be useful throughout the book, and apply it to work.

Because vectors have direction as well as magnitude, they cannot be multiplied in the same way that scalars are. Instead we must *define* what the operation of vector multiplication means. Among the possible ways to define how to multiply vectors, there are three ways that we find useful in physics: (1) multiplication of a vector by a scalar, which was discussed in Section 3–3; (2) multiplication of one vector by a second vector to produce a scalar; (3) multiplication of one vector by a second vector to produce another vector. The third type, called the *vector product*, will be discussed later, in Section 11–2.

We now discuss the second type, called the *scalar product*, or *dot product* (because a dot is used to indicate the multiplication). If we have two vectors, $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, then their **scalar** (or **dot**) **product** is defined to be

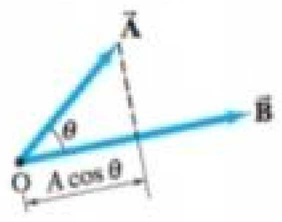
$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta, \qquad (7\text{–}2)$$

where A and B are the magnitudes of the vectors and θ is the angle ($< 180°$) between them when their tails touch, Fig. 7–6. Since A, B, and $\cos\theta$ are scalars, then so is the scalar product $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ (read "A dot B").

This definition, Eq. 7–2, fits perfectly with our definition of the work done by a constant force, Eq. 7–1. That is, we can write the work done by a constant force as the scalar product of force and displacement:

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = Fd\cos\theta. \qquad (7\text{–}3)$$

FIGURE 7–6 The scalar product, or dot product, of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\theta$. The scalar product can be interpreted as the magnitude of one vector (B in this case) times the projection of the other vector, $A\cos\theta$, onto $\vec{\mathbf{B}}$.



Indeed, the definition of scalar product, Eq. 7–2, is so chosen because many physically important quantities, such as work (and others we will meet later), can be described as the scalar product of two vectors.

An equivalent definition of the scalar product is that it is the product of the magnitude of one vector (say B) and the component (or projection) of the other vector along the direction of the first ($A\cos\theta$). See Fig. 7–6.

Since A, B, and $\cos\theta$ are scalars, it doesn't matter in what order they are multiplied. Hence the scalar product is **commutative**:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}. \qquad \text{[commutative property]}$$

It is also easy to show that it is **distributive** (see Problem 33 for the proof):

$$\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \vec{\mathbf{C}}. \qquad \text{[distributive property]}$$

Let us write our vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ in terms of their rectangular components using unit vectors (Section 3–5, Eq. 3–5) as

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$
$$\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}.$$

We will take the scalar product, $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$, of these two vectors, remembering that the unit vectors, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, are perpendicular to each other

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$$
$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0.$$

Thus the scalar product equals

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \cdot (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}})$$
$$= A_xB_x + A_yB_y + A_zB_z. \qquad (7\text{–}4)$$

Equation 7–4 is very useful.

If $\vec{\mathbf{A}}$ is perpendicular to $\vec{\mathbf{B}}$, then Eq. 7–2 tells us $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos 90° = 0$. But the converse, given that $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = 0$, can come about in three different ways: $\vec{\mathbf{A}} = 0$, $\vec{\mathbf{B}} = 0$, or $\vec{\mathbf{A}} \perp \vec{\mathbf{B}}$.

FIGURE 7–7 Example 7–4. Work done by a force $\vec{\mathbf{F}}_P$ acting at an angle θ to the ground is $W = \vec{\mathbf{F}}_P \cdot \vec{\mathbf{d}}$.

EXAMPLE 7–4 **Using the dot product.** The force shown in Fig. 7–7 has magnitude $F_P = 20\,\text{N}$ and makes an angle of 30° to the ground. Calculate the work done by this force using Eq. 7–4 when the wagon is dragged 100 m along the ground.

APPROACH We choose the x axis horizontal to the right and the y axis vertically upward, and write $\vec{\mathbf{F}}_P$ and $\vec{\mathbf{d}}$ in terms of unit vectors.

SOLUTION

$$\vec{\mathbf{F}}_P = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} = (F_P\cos 30°)\hat{\mathbf{i}} + (F_P\sin 30°)\hat{\mathbf{j}} = (17\,\text{N})\hat{\mathbf{i}} + (10\,\text{N})\hat{\mathbf{j}},$$

whereas $\vec{\mathbf{d}} = (100\,\text{m})\hat{\mathbf{i}}$. Then, using Eq. 7–4,

$$W = \vec{\mathbf{F}}_P \cdot \vec{\mathbf{d}} = (17\,\text{N})(100\,\text{m}) + (10\,\text{N})(0) + (0)(0) = 1700\,\text{J}.$$

Note that by choosing the x axis along $\vec{\mathbf{d}}$ we simplified the calculation because $\vec{\mathbf{d}}$ then has only one component.

7–3 Work Done by a Varying Force

If the force acting on an object is constant, the work done by that force can be calculated using Eq. 7–1. In many cases, however, the force varies in magnitude or direction during a process. For example, as a rocket moves away from Earth, work is done to overcome the force of gravity, which varies as the inverse square of the distance from the Earth's center. Other examples are the force exerted by a spring, which increases with the amount of stretch, or the work done by a varying force exerted to pull a box or cart up an uneven hill.

FIGURE 7–8 A particle acted on by a variable force, $\vec{F}$, moves along the path shown from point a to point b.

Figure 7–8 shows the path of an object in the xy plane as it moves from point a to point b. The path has been divided into short intervals each of length $\Delta\ell_1, \Delta\ell_2, \ldots, \Delta\ell_7$. A force $\vec{F}$ acts at each point on the path, and is indicated at two points as $\vec{F}_1$ and $\vec{F}_5$. During each small interval $\Delta\ell$, the force is approximately constant. For the first interval, the force does work ΔW of approximately (see Eq. 7–1)

$$\Delta W \approx F_1 \cos\theta_1 \Delta\ell_1.$$

In the second interval the work done is approximately $F_2 \cos\theta_2 \Delta\ell_2$, and so on. The total work done in moving the particle the total distance $\ell = \Delta\ell_1 + \Delta\ell_2 + \ldots + \Delta\ell_7$ is the sum of all these terms:

$$W \approx \sum_{i=1}^{7} F_i \cos\theta_i \Delta\ell_i. \qquad (7\text{–}5)$$

We can examine this graphically by plotting $F\cos\theta$ versus the distance ℓ along the path as shown in Fig. 7–9a. The distance ℓ has been subdivided into the same seven intervals (see the vertical dashed lines). The value of $F\cos\theta$ at the center of each interval is indicated by the *horizontal* dashed lines. Each of the shaded rectangles has an area $(F_i \cos\theta)(\Delta\ell_i)$, which is a good estimate of the work done during the interval. The estimate of the work done along the entire path given by Eq. 7–5, equals the sum of the areas of all the rectangles. If we subdivide the distance into a greater number of intervals, so that each $\Delta\ell_i$ is smaller, the estimate of the work done becomes more accurate (the assumption that F is constant over each interval is more accurate). Letting each $\Delta\ell_i$ approach zero (so we approach an infinite number of intervals), we obtain an exact result for the work done:

$$W = \lim_{\Delta\ell_i \to 0} \Sigma F_i \cos\theta_i \Delta\ell_i = \int_a^b F\cos\theta\, d\ell. \qquad (7\text{–}6)$$

This limit as $\Delta\ell_i \to 0$ is the *integral* of $(F\cos\theta\, d\ell)$ from point a to point b. The symbol for the integral, $\int$, is an elongated S to indicate an infinite sum; and $\Delta\ell$ has been replaced by $d\ell$, meaning an infinitesimal distance. [We also discussed this in the optional Section 2–9.]

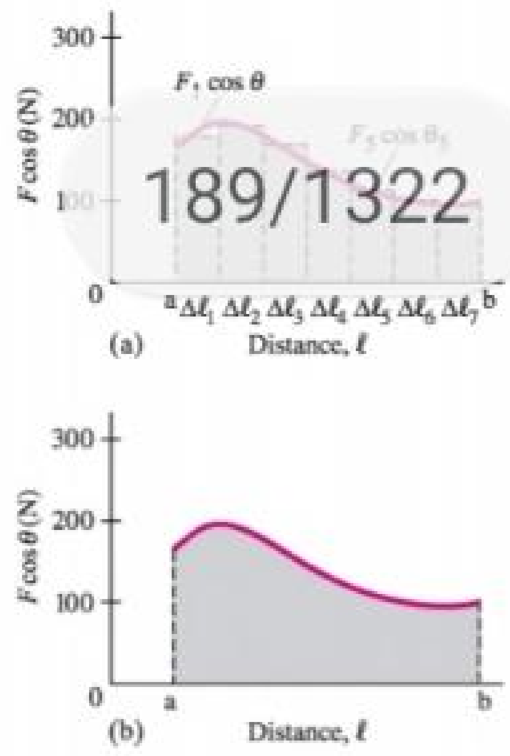
In this limit as $\Delta\ell$ approaches zero, the total area of the rectangles (Fig. 7–9a) approaches the area between the $(F\cos\theta)$ curve and the ℓ axis from a to b as shown shaded in Fig. 7–9b. That is, *the work done by a variable force in moving an object between two points is equal to the area under the $(F\cos\theta)$ versus (ℓ) curve between those two points.*

In the limit as $\Delta\ell$ approaches zero, the infinitesimal distance $d\ell$ equals the magnitude of the infinitesimal displacement vector $d\vec{\ell}$. The direction of the vector $d\vec{\ell}$ is along the tangent to the path at that point, so θ is the angle between $\vec{F}$ and $d\vec{\ell}$ at any point. Thus we can rewrite Eq. 7–6, using dot-product notation:

$$W = \int_a^b \vec{F}\cdot d\vec{\ell}. \qquad (7\text{–}7)$$

This is a *general definition of work*. In this equation, a and b represent two points in space, (x_a, y_a, z_a) and (x_b, y_b, z_b). The integral in Eq. 7–7 is called a *line integral* since it is the integral of $F\cos\theta$ along the line that represents the path of the object. (Equation 7–1 for a constant force is a special case of Eq. 7–7.)

FIGURE 7–9 Work done by a force F is (a) approximately equal to the sum of the areas of the rectangles, (b) exactly equal to the area under the curve of $F\cos\theta$ vs. ℓ.



In rectangular coordinates, any force can be written

$$\vec{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$$

and the displacement $d\vec{\ell}$ is

$$d\vec{\ell} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}.$$

Then the work done can be written

$$W = \int_{x_a}^{x_b} F_x\, dx + \int_{y_a}^{y_b} F_y\, dy + \int_{z_a}^{z_b} F_z\, dz.$$

To actually use Eq. 7–6 or 7–7 to calculate the work, there are several options: (1) If $F\cos\theta$ is known as a function of position, a graph like that of Fig. 7–9b can be made and the area determined graphically. (2) Another possibility is to use numerical integration (numerical summing), perhaps with the aid of a computer or calculator. (3) A third possibility is to use the analytical methods of integral calculus, when it is doable. To do so, we must be able to write $\vec{F}$ as a function of position, $F(x, y, z)$, and we must know the path. Let's look at some specific examples.

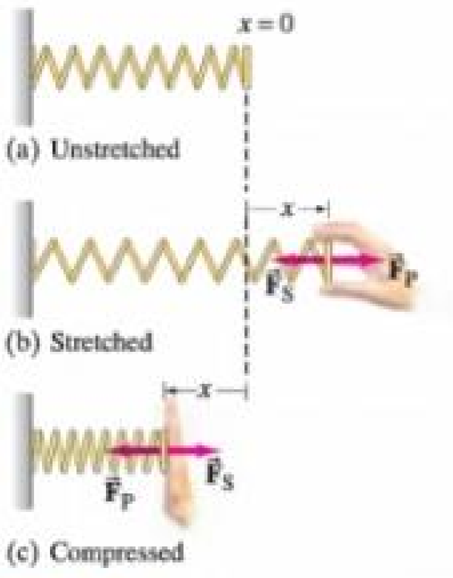


FIGURE 7–10 (a) Spring in normal (unstretched) position. (b) Spring is stretched by a person exerting a force $\vec{F}_P$ to the right (positive direction). The spring pulls back with a force $\vec{F}_S$ where $F_S = -kx$. (c) Person compresses the spring $(x < 0)$, and the spring pushes back with a force $F_S = -kx$ where $F_S > 0$ because $x < 0$.

Work Done by a Spring Force

Let us determine the work needed to stretch or compress a coiled spring, such as that shown in Fig. 7–10. For a person to hold a spring either stretched or compressed an amount x from its normal (relaxed) length requires a force F_P that is directly proportional to x. That is,

$$F_P = kx,$$

where k is a constant, called the *spring constant* (or *spring stiffness constant*), and is a measure of the stiffness of the particular spring. The spring itself exerts a force in the opposite direction (Fig. 7–10b or c):

$$F_S = -kx. \qquad (7\text{–}8)$$

This force is sometimes called a "restoring force" because the spring exerts its force in the direction opposite the displacement (hence the minus sign), and thus acts to return the spring to its normal length. Equation 7–8 is known as the **spring equation** or **Hooke's law**, and is accurate for springs as long as x is not too great (see Section 12–4) and no permanent deformation occurs.

Let us calculate the work a person does to stretch (or compress) a spring from its normal (unstretched) length, $x_a = 0$, to an extra length, $x_b = x$. We assume the stretching is done slowly, so that the acceleration is essentially zero. The force $\vec{F}_P$ is exerted parallel to the axis of the spring, along the x axis, so $\vec{F}_P$ and $d\vec{\ell}$ are parallel. Hence, since $d\vec{\ell} = dx\hat{\mathbf{i}}$ in this case, the work done by the person is[†]

$$W_P = \int_{x_a=0}^{x_b=x} [F_P(x)\hat{\mathbf{i}}]\cdot[dx\hat{\mathbf{i}}] = \int_0^x F_P(x)\,dx = \int_0^x kx\,dx = \tfrac{1}{2}kx^2\Big|_0^x = \tfrac{1}{2}kx^2.$$

(As is frequently done, we have used x to represent both the variable of integration, and the particular value of x at the end of the interval $x_a = 0$ to $x_b = x$.) Thus we see that the work needed is proportional to the square of the distance stretched (or compressed), x.

This same result can be obtained by computing the area under the graph of F vs. x (with $\cos\theta = 1$ in this case) as shown in Fig. 7–11. Since the area is a triangle of altitude kx and base x, the work a person does to stretch or compress a spring an amount x is

$$W = \tfrac{1}{2}(x)(kx) = \tfrac{1}{2}kx^2,$$

which is the same result as before. Because $W \propto x^2$, it takes the same amount of work to stretch a spring or compress it the same amount x.

FIGURE 7–11 Work done to stretch a spring a distance x equals the triangular area under the curve $F = kx$. The area of a triangle is $\frac{1}{2} \times$ base $\times$ altitude, so $W = \frac{1}{2}(x)(kx) = \frac{1}{2}kx^2$.

[†]See the Table of Integrals, Appendix B.

EXAMPLE 7–5 **Work done on a spring.** (*a*) A person pulls on the spring in Fig. 7–10, stretching it 3.0 cm, which requires a maximum force of 75 N. How much work does the person do? (*b*) If, instead, the person compresses the spring 3.0 cm, how much work does the person do?

EXAMPLE 7–5 Work done on a spring. (*a*) A person pulls on the spring in Fig. 7–10, stretching it 3.0 cm, which requires a maximum force of 75 N. How much work does the person do? (*b*) If, instead, the person compresses the spring 3.0 cm, how much work does the person do?

APPROACH The force $F = kx$ holds at each point, including x_{max}. Hence F_{max} occurs at $x = x_{max}$.

SOLUTION (*a*) First we need to calculate the spring constant k:

$$k = \frac{F_{max}}{x_{max}} = \frac{75\text{ N}}{0.030\text{ m}} = 2.5 \times 10^3\text{ N/m}.$$

Then the work done by the person on the spring is

$$W = \tfrac{1}{2}kx_{max}^2 = \tfrac{1}{2}(2.5 \times 10^3\text{ N/m})(0.030\text{ m})^2 = 1.1\text{ J}.$$

(*b*) The force that the person exerts is still $F_P = kx$, though now both x and F_P are negative (x is positive to the right). The work done is

$$W_P = \int_{x=0}^{x=-0.030\text{ m}} F_P(x)\,dx = \int_0^{x=-0.030\text{ m}} kx\,dx = \tfrac{1}{2}kx^2\Big|_0^{-0.030\text{ m}}$$

$$= \tfrac{1}{2}(2.5 \times 10^3\text{ N/m})(-0.030\text{ m})^2 = 1.1\text{ J},$$

which is the same as for stretching it.

NOTE We cannot use $W = Fd$ (Eq. 7–1) for a spring because the force is not constant.

A More Complex Force Law—Robot Arm

EXAMPLE 7–6 Force as function of *x*. A robot arm that controls the position of a video camera (Fig. 7–12) in an automated surveillance system is manipulated by a motor that exerts a force on the arm. The force is given by

$$F(x) = F_0\left(1 + \frac{1}{6}\frac{x^2}{x_0^2}\right),$$

where $F_0 = 2.0\text{ N}$, $x_0 = 0.0070\text{ m}$, and x is the position of the end of the arm. If the arm moves from $x_1 = 0.010\text{ m}$ to $x_2 = 0.050\text{ m}$, how much work did the motor do?

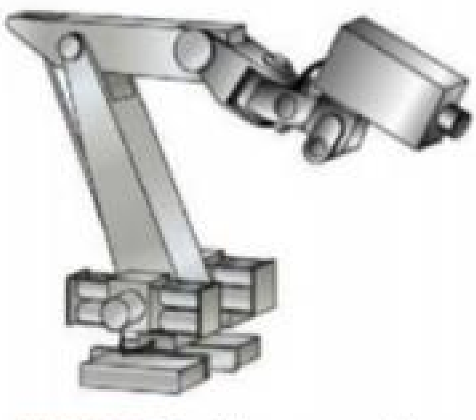
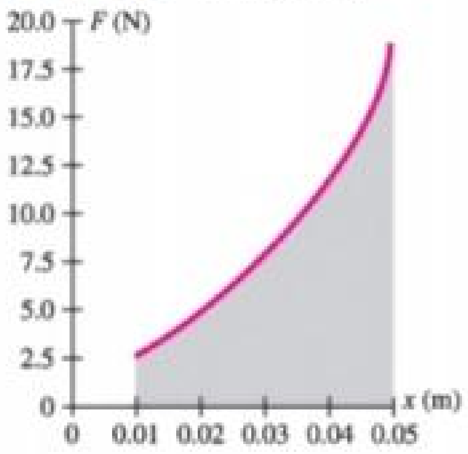


FIGURE 7–12 Robot arm positions a video camera.

APPROACH The force applied by the motor is not a linear function of x. We can determine the integral $\int F(x)\,dx$, or the area under the $F(x)$ curve (shown in Fig. 7–13).

FIGURE 7–13 Example 7–6.



SOLUTION We integrate to find the work done by the motor:

$$W_M = F_0\int_{x_1}^{x_2}\left(1 + \frac{x^2}{6x_0^2}\right)dx = F_0\int_{x_1}^{x_2}dx + \frac{F_0}{6x_0^2}\int_{x_1}^{x_2}x^2\,dx$$

$$= F_0\left(x + \frac{1}{6x_0^2}\frac{x^3}{3}\right)\Bigg|_{x_1}^{x_2}.$$

We put in the values given and obtain

$$W_M = 2.0\text{ N}\left[(0.050\text{ m} - 0.010\text{ m}) + \frac{(0.050\text{ m})^3 - (0.010\text{ m})^3}{(3)(6)(0.0070\text{ m})^2}\right] = 0.36\text{ J}.$$

7–4 Kinetic Energy and the Work-Energy Principle

Energy is one of the most important concepts in science. Yet we cannot give a simple general definition of energy in only a few words. Nonetheless, each specific type of energy can be defined fairly simply. In this Chapter we define translational kinetic energy; in the next Chapter, we take up potential energy. In later Chapters we will examine other types of energy, such as that related to heat (Chapters 19 and 20). The crucial aspect of all the types of energy is that the sum of all types, the *total energy*, is the same after any process as it was before: that is, energy is a conserved quantity.

For the purposes of this Chapter, we can define energy in the traditional way as "the ability to do work." This simple definition is not very precise, nor is it really valid for all types of energy.[†] It works, however, for mechanical energy which we discuss in this Chapter and the next. We now define and discuss one of the basic types of energy, kinetic energy.

A moving object can do work on another object it strikes. A flying cannonball does work on a brick wall it knocks down; a moving hammer does work on a nail it drives into wood. In either case, a moving object exerts a force on a second object which undergoes a displacement. An object in motion has the ability to do work and thus can be said to have energy. The energy of motion is called **kinetic energy**, from the Greek word *kinetikos*, meaning "motion."

To obtain a quantitative definition for kinetic energy, let us consider a simple rigid object of mass m (treated as a particle) that is moving in a straight line with an initial speed v_1. To accelerate it uniformly to a speed v_2, a constant net force F_{net} is exerted on it parallel to its motion over a displacement d, Fig. 7–14.

FIGURE 7–14 A constant net force F_{net} accelerates a car from speed v_1 to speed v_2 over a displacement d. The net work done is $W_{net} = F_{net}d$.

Then the net work done on the object is $W_{net} = F_{net}d$. We apply Newton's second law, $F_{net} = ma$, and use Eq. 2–12c ($v_2^2 = v_1^2 + 2ad$), which we rewrite as

$$a = \frac{v_2^2 - v_1^2}{2d},$$

where v_1 is the initial speed and v_2 the final speed. Substituting this into $F_{net} = ma$, we determine the work done:

$$W_{net} = F_{net}d = mad = m\left(\frac{v_2^2 - v_1^2}{2d}\right)d = m\left(\frac{v_2^2 - v_1^2}{2}\right)$$

or

$$W_{net} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2. \tag{7–9}$$

We *define* the quantity $\tfrac{1}{2}mv^2$ to be the **translational kinetic energy**, K, of the object:

Kinetic energy (defined)

$$K = \tfrac{1}{2}mv^2. \tag{7–10}$$

(We call this "translational" kinetic energy to distinguish it from rotational kinetic energy, which we discuss in Chapter 10.) Equation 7–9, derived here for one-dimensional motion with a constant force, is valid in general for translational motion of an object in three dimensions and even if the force varies, as we will show at the end of this Section.

[†]Energy associated with heat is often not available to do work, as we will discuss in Chapter 20.

We can rewrite Eq. 7–9 as:

$$W_{net} = K_2 - K_1$$

or

$$W_{net} = \Delta K = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2. \quad (7\text{–}11)$$

WORK-ENERGY PRINCIPLE

Equation 7–11 (or Eq. 7–9) is a useful result known as the **work-energy principle.** It can be stated in words:

The net work done on an object is equal to the change in the object's kinetic energy.

WORK-ENERGY PRINCIPLE

Notice that we made use of Newton's second law, $F_{net} = ma$, where F_{net} is the *net* force—the sum of all forces acting on the object. Thus, the work-energy principle is valid only if W is the *net work* done on the object—that is, the work done by all forces acting on the object.

⚠ CAUTION
Work-energy valid only for net work

The work-energy principle is a very useful reformulation of Newton's laws. It tells us that if (positive) net work W is done on an object, the object's kinetic energy increases by an amount W. The principle also holds true for the reverse situation: if the net work W done on an object is negative, the object's kinetic energy decreases by an amount W. That is, a net force exerted on an object opposite to the object's direction of motion decreases its speed and its kinetic energy. An example is a moving hammer (Fig. 7–15) striking a nail. The net force on the hammer ($-\vec{\mathbf{F}}$ in Fig. 7–15, where $\vec{\mathbf{F}}$ is assumed constant for simplicity) acts toward the left, whereas the displacement $\vec{\mathbf{d}}$ of the hammer is toward the right. So the net work done on the hammer, $W_h = (F)(d)(\cos 180°) = -Fd$, is negative and the hammer's kinetic energy decreases (usually to zero).

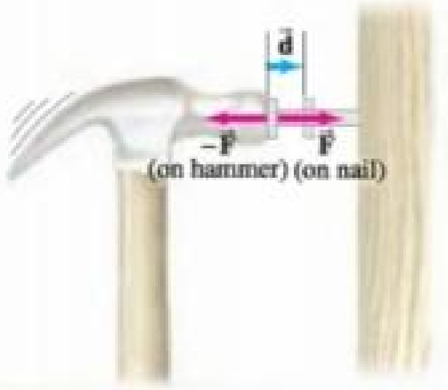


FIGURE 7–15 A moving hammer strikes a nail and comes to rest. The hammer exerts a force F on the nail; the nail exerts a force $-F$ on the hammer (Newton's third law). The work done on the nail by the hammer is positive ($W_n = Fd > 0$). The work done on the hammer by the nail is negative ($W_h = -Fd$).

Figure 7–15 also illustrates how energy can be considered the ability to do work. The hammer, as it slows down, does positive work on the nail: $W_n = (+F)(+d)(\cos 0°) = Fd$ and is positive. The decrease in kinetic energy of the hammer ($= Fd$ by Eq. 7–11) is equal to the work the hammer can do on another object, the nail in this case.

The translational kinetic energy ($= \frac{1}{2}mv^2$) is directly proportional to the mass of the object, and it is also proportional to the *square* of the speed. Thus, if the mass is doubled, the kinetic energy is doubled. But if the speed is doubled, the object has four times as much kinetic energy and is therefore capable of doing four times as much work.

Because of the direct connection between work and kinetic energy, energy is measured in the same units as work: joules in SI units. [The energy unit is ergs in the cgs, and foot-pounds in the British system.] Like work, kinetic energy is a scalar quantity. The kinetic energy of a group of objects is the sum of the kinetic energies of the individual objects.

The work-energy principle can be applied to a particle, and also to an object that can be approximated as a particle, such as an object that is rigid or whose internal motions are insignificant. It is very useful in simple situations, as we will see in the Examples below. The work-energy principle is not as powerful and encompassing as the law of conservation of energy which we treat in the next Chapter, and should not itself be considered a statement of energy conservation.

EXAMPLE 7–7 **Kinetic energy and work done on a baseball.** A 145-g baseball is thrown so that it acquires a speed of 25 m/s. (*a*) What is its kinetic energy? (*b*) What was the net work done on the ball to make it reach this speed, if it started from rest?

APPROACH We use $K = \frac{1}{2}mv^2$, and the work-energy principle, Eq. 7–11.

SOLUTION (*a*) The kinetic energy of the ball after the throw is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(0.145\text{ kg})(25\text{ m/s})^2 = 45\text{ J}.$$

(*b*) Since the initial kinetic energy was zero, the net work done is just equal to the final kinetic energy, 45 J.

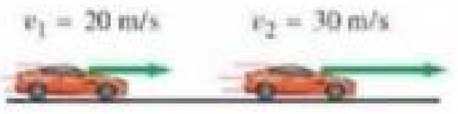


FIGURE 7–16 Example 7–8.

EXAMPLE 7–8 ESTIMATE **Work on a car, to increase its kinetic energy.** How much net work is required to accelerate a 1000-kg car from 20 m/s to 30 m/s (Fig. 7–16)?

APPROACH A car is a complex system. The engine turns the wheels and tires which push against the ground, and the ground pushes back (see Example 4–4). We aren't interested right now in those complications. Instead, we can get a useful result using the work-energy principle, but only if we model the car as a particle or simple rigid object.

SOLUTION The net work needed is equal to the increase in kinetic energy:

$$W = K_2 - K_1 = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$
$$= \tfrac{1}{2}(1000\text{ kg})(30\text{ m/s})^2 - \tfrac{1}{2}(1000\text{ kg})(20\text{ m/s})^2 = 2.5 \times 10^5\text{ J}.$$

EXERCISE C (*a*) Make a guess: will the work needed to accelerate the car in Example 7–8 from rest to 20 m/s be more than, less than, or equal to the work already calculated to accelerate it from 20 m/s to 30 m/s? (*b*) Make the calculation.

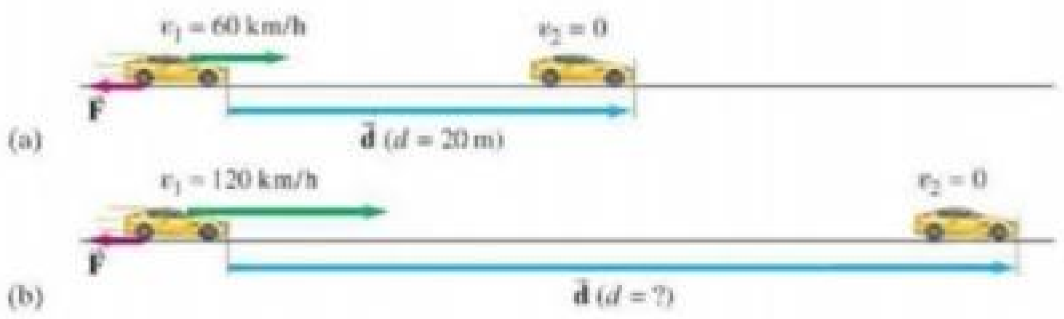


FIGURE 7–17 Example 7–9.

CONCEPTUAL EXAMPLE 7–9 **Work to stop a car.** A car traveling 60 km/h can brake to a stop within a distance d of 20 m (Fig. 7–17a). If the car is going twice as fast, 120 km/h, what is its stopping distance (Fig. 7–17b)? Assume the maximum braking force is approximately independent of speed.

RESPONSE Again we model the car as if it were a particle. Because the net stopping force F is approximately constant, the work needed to stop the car, Fd, is proportional to the distance traveled. We apply the work-energy principle, noting that $\vec{\mathbf{F}}$ and $\vec{\mathbf{d}}$ are in opposite directions and that the final speed of the car is zero:

$$W_{net} = Fd\cos 180° = -Fd.$$

Then

$$-Fd = \Delta K = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$
$$= 0 - \tfrac{1}{2}mv_1^2.$$

PHYSICS APPLIED
Car's stopping distance ∝ initial speed squared

Thus, since the force and mass are constant, we see that the stopping distance, d, increases with the square of the speed:

$$d \propto v^2.$$

If the car's initial speed is doubled, the stopping distance is $(2)^2 = 4$ times as great, or 80 m.

EXERCISE D Can kinetic energy ever be negative?

EXERCISE E (*a*) If the kinetic energy of an arrow is doubled, by what factor has its speed increased? (*b*) If its speed is doubled, by what factor does its kinetic energy increase?

EXAMPLE 7–10 **A compressed spring.** A horizontal spring has spring constant $k = 360\ \mathrm{N/m}$. (*a*) How much work is required to compress it from its uncompressed length ($x = 0$) to $x = 11.0\ \mathrm{cm}$? (*b*) If a 1.85-kg block is placed against the spring and the spring is released, what will be the speed of the block when it separates from the spring at $x = 0$? Ignore friction. (*c*) Repeat part (*b*) but assume that the block is moving on a table as in Fig. 7–18 and that some kind of constant drag force $F_D = 7.0\ \mathrm{N}$ is acting to slow it down, such as friction (or perhaps your finger).

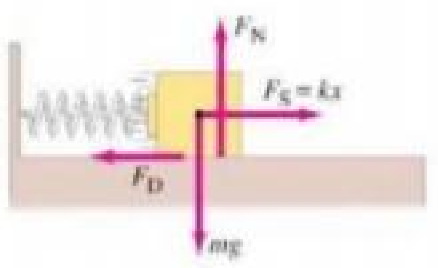


FIGURE 7–18 Example 7–10.

APPROACH We use our result from Section 7–3 that the net work, W, needed to stretch or compress a spring by a distance x is $W = \frac{1}{2}kx^2$. In (*b*) and (*c*) we use the work-energy principle.

SOLUTION (*a*) The work needed to compress the spring a distance $x = 0.110\ \mathrm{m}$ is

$$W = \tfrac{1}{2}(360\ \mathrm{N/m})(0.110\ \mathrm{m})^2 = 2.18\ \mathrm{J},$$

where we have converted all units to SI.

(*b*) In returning to its uncompressed length, the spring does 2.18 J of work on the block (same calculation as in part (*a*), only in reverse). According to the work-energy principle, the block acquires kinetic energy of 2.18 J. Since $K = \frac{1}{2}mv^2$, the block's speed must be

$$v = \sqrt{\frac{2K}{m}}$$

$$= \sqrt{\frac{2(2.18\ \mathrm{J})}{1.85\ \mathrm{kg}}} = 1.54\ \mathrm{m/s}.$$

(*c*) There are two forces on the block: that exerted by the spring and that exerted by the drag force, $\vec{F}_D$. Work done by a force such as friction is complicated. For one thing, heat (or, rather, "thermal energy") is produced—try rubbing your hands together. Nonetheless, the product $\vec{F}_D \cdot \vec{d}$ for the drag force, even when it is friction, can be used in the work-energy principle to give correct results for a particle-like object. The spring does 2.18 J of work on the block. The work done by the friction or drag force on the block, in the negative x direction, is

$$W_D = -F_D x = -(7.0\ \mathrm{N})(0.110\ \mathrm{m}) = -0.77\ \mathrm{J}.$$

This work is negative because the drag force acts in the direction opposite to the displacement x. The net work done on the block is $W_{net} = 2.18\ \mathrm{J} - 0.77\ \mathrm{J} = 1.41\ \mathrm{J}$. From the work-energy principle, Eq. 7–11 (with $v_2 = v$ and $v_1 = 0$), we have

$$v = \sqrt{\frac{2W_{net}}{m}}$$

$$= \sqrt{\frac{2(1.41\ \mathrm{J})}{1.85\ \mathrm{kg}}} = 1.23\ \mathrm{m/s}$$

for the block's speed at the moment it separates from the spring ($x = 0$).

FIGURE 7–8 (repeated) A particle acted on by a variable force $\vec{F}$, moves along the path shown from point a to point b.

General Derivation of the Work-Energy Principle

We derived the work-energy principle, Eq. 7–11, for motion in one dimension with a constant force. It is valid even if the force is variable and the motion is in two or three dimensions, as we now show. Suppose the net force $\vec{F}_{net}$ on a particle varies in both magnitude and direction, and the path of the particle is a curve as in Fig. 7–8. The net force may be considered to be a function of ℓ, the distance along the curve. The net work done is (Eq. 7–6):

$$W_{net} = \int \vec{F}_{net} \cdot d\vec{\ell} = \int F_{net} \cos\theta\, d\ell = \int F_{\parallel}\, d\ell,$$

where $F_{\parallel}$ represents the component of the net force parallel to the curve at any point. By Newton's second law,

$$F_{\parallel} = ma_{\parallel} = m\frac{dv}{dt},$$

where $a_{\parallel}$, the component of a parallel to the curve at any point, is equal to the rate of change of speed, dv/dt. We can think of v as a function of ℓ, and using the chain rule for derivatives, we have

$$\frac{dv}{dt} = \frac{dv}{d\ell}\frac{d\ell}{dt} = \frac{dv}{d\ell}v,$$

since $d\ell/dt$ is the speed v. Thus (letting 1 and 2 refer to the initial and final quantities, respectively):

$$W_{net} = \int_1^2 F_{\parallel}\, d\ell = \int_1^2 m\frac{dv}{dt}\, d\ell = \int_1^2 mv\frac{dv}{d\ell}\, d\ell = \int_1^2 mv\, dv,$$

which integrates to

$$W_{net} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = \Delta K.$$

This is again the work-energy principle, which we have now derived for motion in three dimensions with a variable net force, using the definitions of work and kinetic energy plus Newton's second law.

Notice in this derivation that only the component of $\vec{F}_{net}$ parallel to the motion, $F_{\parallel}$, contributes to the work. Indeed, a force (or component of a force) acting perpendicular to the velocity vector does no work. Such a force changes only the direction of the velocity. It does not affect the magnitude of the velocity. One example of this is uniform circular motion in which an object moving with constant speed in a circle has a ("centripetal") force acting on it toward the center of the circle. This force does no work on the object, because (as we saw in Example 7–3) it is always perpendicular to the object's displacement $d\vec{\ell}$.

Summary

Work is done on an object by a force when the object moves through a distance, d. The **work** W done by a constant force $\vec{F}$ on an object whose position changes by a displacement $\vec{d}$ is given by

$$W = Fd\cos\theta = \vec{F} \cdot \vec{d}, \qquad (7\text{–}1, 7\text{–}3)$$

where θ is the angle between $\vec{F}$ and $\vec{d}$.

The last expression is called the scalar product of $\vec{F}$ and $\vec{d}$. In general, the **scalar product** of any two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{A} \cdot \vec{B} = AB\cos\theta \qquad (7\text{–}2)$$

where θ is the angle between $\vec{A}$ and $\vec{B}$. In rectangular coordinates we can also write

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z. \qquad (7\text{–}4)$$

The work W done by a variable force $\vec{F}$ on an object that moves from point a to point b is

$$W = \int_a^b \vec{F} \cdot d\vec{\ell} = \int_a^b F\cos\theta\, d\ell, \qquad (7\text{–}7)$$

where $d\vec{\ell}$ represents an infinitesimal displacement along the path of the object and θ is the angle between $d\vec{\ell}$ and $\vec{F}$ at each point of the object's path.

The translational **kinetic energy**, K, of an object of mass m moving with speed v is defined to be

$$K = \tfrac{1}{2}mv^2. \qquad (7\text{–}10)$$

The **work-energy principle** states that the net work done on an object by the net resultant force is equal to the change in kinetic energy of the object:

$$W_{net} = \Delta K = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2. \qquad (7\text{–}11)$$