$$\hat{f}(n) = \sum_{k=0}^{N-1} f_k \, e^{-i\frac{2\pi}{N}nk} \qquad n = 0, 1, \dots, N-1$$

Complexity for one value of n is

- N complex multiplications
- $N-1$ " adds.

} for each n.

Complexity : $O(N^2)$ for DFT

: $O(N \log_2 N)$ for FFT

N	10^3	10^6	10^9
N^2	10^6	10^{12}	10^{18}
$N \log_2 N$	10^4	20×10^6	30×10^9

1 ns/operation

10^{18} ns ~ 31.2 years

30×10^9 ns ~ 30 sec.

FFT exploits some properties of $e^{-i\frac{2\pi}{N}kn}$ ~~W (say)~~.

say $W_N = e^{-i\frac{2\pi}{N}}$ (last lecture we said it as F_N).

1. Complex conjugate symmetry

$$W_N^{k(N-n)} = W_N^{kN} W_N^{-kn} = W_N^{-kn} = \left(W_N^{kn}\right)^*$$

$$W_N^{kN} = e^{-i\frac{2\pi}{N}\cdot kN} = \underbrace{\cos(2\pi k)}_{1} - i\underbrace{\sin(2k\pi)}_{0}$$

2. Periodicity in n, k

$$W_N^{kn} = W_N^{k(N+n)} = W_N^{(k+N)n}$$

Assume: $N = 2^m$

Separate f_k into even and odd indexed sub sequences.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

0	2	4	6	8	10	12	14

1	3	5	7	9	11	13	15

$$\hat{f}_n = \sum_{k=0}^{N-1} f_k W_N^{kn} = \sum_{k\,\text{even}} f_k W_N^{kn} + \sum_{k\,\text{odd}} f_k W_N^{kn}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} f_{2r} W_N^{2rn} + \sum_{r=0}^{\frac{N}{2}-1} f_{2r+1} W_N^{(2r+1)n}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} f_{2r} \left(W_N^2\right)^{rn} + \sum_{r=0}^{\frac{N}{2}-1} f_{2r+1} W_N^{2rn} \cdot W_N^{n}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} f_{2r} \left(W_N^2\right)^{rn} + W_N^{n} \sum_{r=0}^{\frac{N}{2}-1} f_{2r+1} \left(W_N^2\right)^{rn}$$

$$W_N^2 = e^{-i\frac{2\pi}{N}\cdot 2} = e^{-i\frac{2\pi}{N/2}} = W_{N/2}$$

$$\hat{f}_n = \underbrace{\sum_{r=0}^{\frac{N}{2}-1} f_{2r} W_{N/2}^{rn}}_{\frac{N}{2}\text{ DFT of even samples } \hat{f}_n^{even}} + W_N^{n} \underbrace{\sum_{r=0}^{\frac{N}{2}-1} f_{2r+1} W_{N/2}^{rn}}_{N/2\text{ DFN of odd samples } \hat{f}_n^{odd}}$$

$$\hat{f}_n = \hat{f}_n^{even} + W_N^{n} \hat{f}_n^{odd}$$

This is sum of two $\frac{N}{2}$ point DFT's

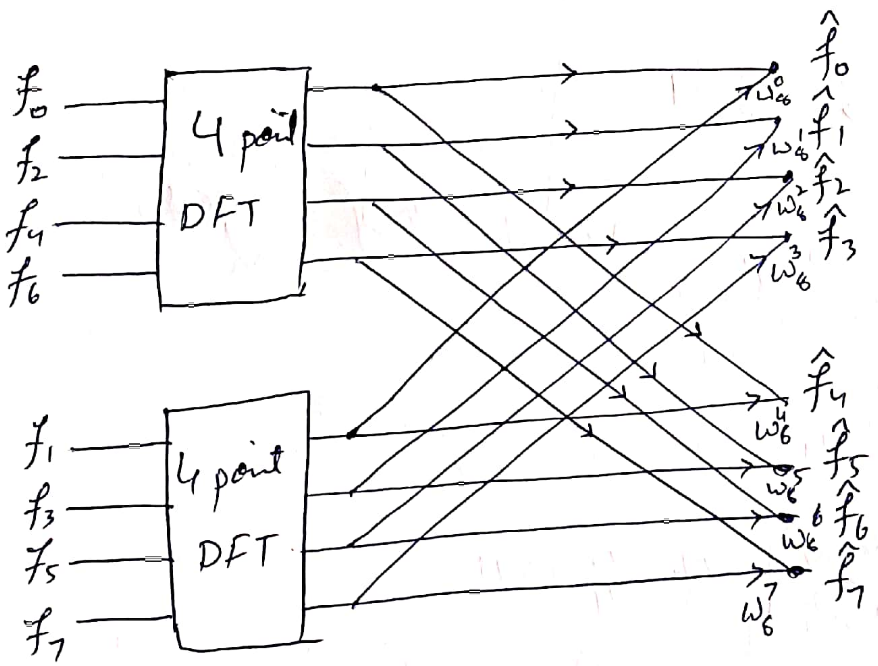
Ex: $N=8$

$$W_8^n = e^{-i\frac{2\pi}{8}\cdot n} = e^{-i\frac{\pi}{4}n} \qquad n=0,1,--,7.$$

n	0	1	2	3	4	5	6	7
$e^{-i\frac{\pi}{4}n}$								

$\downarrow$

W_8^n



$\left(\frac{N}{2}\right)^2\cdot 2$ $\qquad + N$ $\qquad \approx \underbrace{\frac{N^2}{2} + N}$ Mult.

#Mult is cut by 2.

Scanned with CamScanner

keep splitting

[0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15]

[0 2 4 6 8 10 12 14]   [1 3 5 7 9 11 13 15]

[0 4 8 12]   [2 6 10 14]   [1 5 9 13]   [3 7 11 15]

[0 8]   [4 12]   [2 10]   [6 14]   [1 9]   [5 13]   [3 11]   [7 15]

0   8   4   12   2   10   6   14   1   9   5   13   3   11   7   15

As we assumed N is a power of 2 ($N = 2^p$)

so $p = \log_2 N$.

$$\frac{N}{2}, \frac{N}{4}, \frac{N}{8}, \ldots, \frac{N}{2^p} = 1$$

$$\frac{N}{2^1}, \frac{N}{2^2}, \frac{N}{2^3}, \ldots, \frac{N}{2^p} \Rightarrow p = \text{\# of stages}$$

1. $\frac{N}{2} \longrightarrow 2\left(\frac{N}{2}\right)^2 + N$

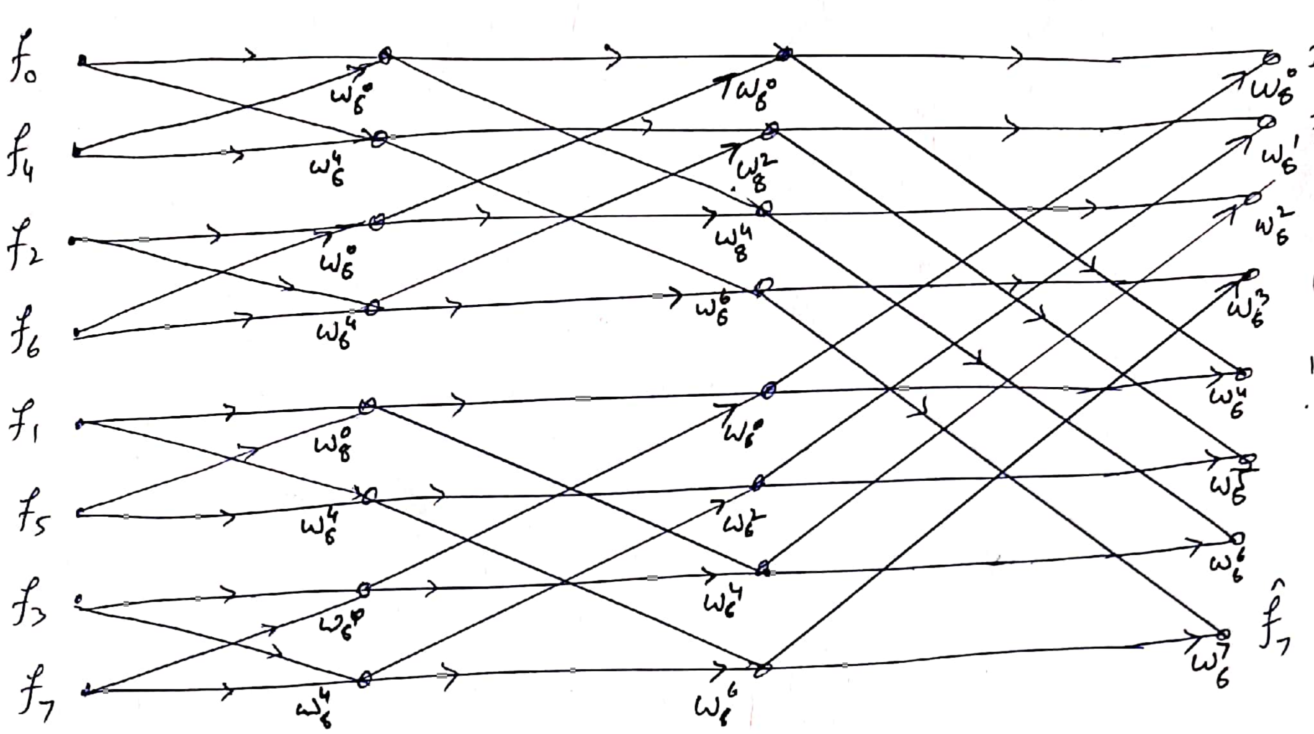
2. $\frac{N}{4} \longrightarrow 2\left[2\left(\frac{N}{4}\right)^2 + \frac{N}{2}\right] + N = \frac{N^2}{4} + 2N$

3. $\frac{N}{8} \longrightarrow 2\left[2\left(2\cdot\left(\frac{N}{16}\right)^2 + \frac{N}{4}\right) + \frac{N}{2}\right] + N = \frac{N^2}{8} + 3N$

$\vdots$

$p: \frac{N}{2^p} = 1 \longrightarrow \frac{N^2}{2^p} + pN = \frac{N^2}{N} + N\log_2 N = O(N\log_2 N)$

for large N.



↳ Bit reversal

$0 = 000 \longrightarrow 000 = 0$

$1 = 001 \longrightarrow 100 = 4$

$2 = 010 \longrightarrow 010 = 2$

$3 = 011 \longrightarrow 110 = 6$

$4 = 100 \longrightarrow 001 = 1$

$5 = 101 \longrightarrow 101 = 5$

$6 = 110 \longrightarrow 011 = 3$

$7 = 111 \longrightarrow 111 = 7$