Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(xn)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx$$

Ex:

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{0} (-k) \cos(nx)\,dx + \frac{1}{\pi} \int_{0}^{\pi} (k) \cos(nx)\,dx$$

$$= -\frac{k}{\pi} \sin(nx)\Big|_{-\pi}^{0} + \frac{k}{\pi} \sin(nx)\Big|_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{0} (-k) \sin(nx)\,dx + \frac{1}{\pi} \int_{0}^{\pi} k \sin(nx)\,dx$$

$$= -\frac{k}{\pi}\left[-\cos(nx)\right]\Big|_{-\pi}^{0} + \frac{k}{\pi}\left[-\cos(nx)\right]\Big|_{0}^{\pi}$$

$$= \frac{k}{\pi} \underbrace{\left[\cos 0 - \cos(-n\pi)\right]}_{1-\cos(n\pi)} - \frac{k}{\pi} \underbrace{\left[\cos(n\pi) - \cos 0\right]}_{-(1-\cos(n\pi))}$$

$$b_n = \frac{2k}{\pi} \underbrace{\left[1 - \cos(n\pi)\right]}_{2 \text{ or } 0}$$

@ $n = 1$ $\cos\pi = -1$

$n = 2$ $\cos 2\pi = 1$

$\vdots$

$$b_n = \begin{cases} \frac{4k}{\pi} & \text{for } n \text{ is odd.} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$$f(x) = \frac{4\pi}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin \pi x + \cdots\right)$$

<u>orthogonal trignometry.</u>

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0 \qquad n \neq m$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0 \qquad n \neq m$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$$

Fundamental period is smallest +ve period.

$\cos x$, $\sin x$, $\cos 2x$, $\sin 2\pi$, $\cos \pi x$, $\sin 2\pi x$.

$\cos \frac{2\pi n x}{k}$.

Find Fourier Series of:

1. $f(x) = |x|$

2. $f(x) = \begin{cases} x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$

3. $f(x) = x^2 \qquad -\pi < x < \pi$

4. $f(x) = x^2 \qquad 0 < x < 2\pi$

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When period is not 2π. ②

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right]$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Ex:
$$f(x) = \begin{cases} -k & \text{if } -2 < x < 0 \\ k & \text{if } 0 < x < 2 \end{cases}$$

$p = 2L = 4 \Rightarrow L = 2.$

$$a_n = \frac{1}{L} \int_{-2}^{0} (-k) \cos\left(\frac{n\pi x}{2}\right) dx + \frac{1}{L} \int_{0}^{2} (k) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= -\frac{k}{L} \sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0} \cdot \frac{2}{n\pi} + \frac{k}{L} \sin\left(\frac{n\pi x}{2}\right)\Big|_{0}^{2} \cdot \frac{2}{n\pi} = 0$$

$$b_n = \frac{1}{L} \int_{-2}^{0} (-k) \sin\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2} \int_{0}^{2} (k) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{2k}{L} \int_{0}^{2} \sin \frac{n\pi x}{2}\, dx = \frac{2k}{L}\left(-\cos \frac{n\pi x}{2}\right)\Big|_{0}^{2} \frac{2}{n\pi}$$

$$= \frac{2k}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \text{ for } n \text{ is odd.}$$

Ex:

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

$$p = 2L = 4 \Rightarrow L = 2.$$

$$a_n = \frac{1}{2} \int_{-2}^{2} f(x) \cos\left(\frac{n\pi x}{2}\right) dx = \frac{1}{2} \int_{-1}^{1} k \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{k}{n\pi} \int_{-1}^{1} \cos\left(\frac{n\pi x}{2}\right)\left(\frac{n\pi}{2} dx\right) = \frac{k}{n\pi} \sin \frac{n\pi x}{2} \Big|_{-1}^{1}$$

Fourier Transform:

Def 1: A function f is called integrable or absolutely integrable, when

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

x is a physical variable, but we can use t if function is defined on time.

Frequency variable is k (alternates are ω, p or ξ)

Def 2: The ~~f~~ Fourier transform (FT) of an integrable function $f(x)$ is

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx.$$

$\hat{f}(k)$ is also integrable.

We can get $f(x)$ from $\hat{f}(k)$ as:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\, dk \qquad \text{(IFT)}$$

$\hat{f}(k)$ is the amplitude density of f at freq k.

IFT gives decomposition of f into constituent waves.

Properties of FT

1. Differentiation $\hat{f'}(k) = ik \hat{f}(k)$
2. Translation: if $g(x) = f(x+a)$

$$\hat{g}(k) = e^{ika} \hat{f}(k)$$

__Ex:__

$$f(x) = \begin{cases} \frac{1}{2a} & \text{if } x \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{f}(k) = \int_{-a}^{a} \frac{1}{2a} \cdot e^{-ikx}\, dx = \frac{1}{2a} \cdot \frac{1}{-ik} \int_{-a}^{a} e^{-ikx}(-ik\, dx)$$

$$= -\frac{1}{2aik} \, e^{-ikx} \Big|_{-a}^{a}$$

$$= -\frac{1}{2aik} \left[e^{-ika} - e^{+ika} \right] = \frac{1}{ka} \left[\frac{e^{ika} - e^{-ika}}{2i} \right]$$

$$= \frac{\sin(ka)}{ka} = \operatorname{sinc}(ka).$$

$$\operatorname{sinc}(k) = \frac{\sin k}{k}$$

$\operatorname{sinc}(0) = 1$ (using L'Hospital's rule).

$$\lim_{k \to \infty} \operatorname{sinc}(k) = 0$$

__Ex:__ $f(x) = e^{-x^2/2} \longleftrightarrow \hat{f}(k) = \sqrt{2\pi}\, e^{-k^2/2}$

__Ex:__ The Dirac Delta $\delta(x)$

$$\hat{\delta}(k) = \int_{-\infty}^{\infty} \delta(x)\, e^{-ikx}\, dx = e^{-ik(0)} = 1.$$

__Theorem:__

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy.$$

$$\widehat{(f * g)}(k) = \hat{f}(k)\, \hat{g}(k)$$

Let $h = f * g = \int_{-\infty}^{\infty} f(y) g(x-y) dy$

$$\hat{h}(k) = \int e^{-ikx} \left[\int f(y) g(x-y) dy \right] dx$$

$$= \int\int e^{-iky} f(y) \, e^{-ik(x-y)} g(x-y) \, dy \, dx$$

$$= \left[\int e^{-iky} f(y) dy \right] \left[\int e^{-ikx'} g(x') dx' \right] \quad \left(\begin{array}{l} x' = x-y \\ dx' = dx \end{array} \right)$$

$$= \hat{f}(k) \cdot \hat{g}(k).$$

One of the applicability is linear differential equations.

$$a_n \frac{d^n u}{dx^n} + a_{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_1 \frac{du}{dx} + a_0 = f(x).$$

$$\underbrace{\left[a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \frac{d}{dx} + a_0 \right]}_{} u = f(x).$$

$$P\left(\frac{d}{dx}\right) u = f(x).$$

$$\left(a_n (ik)^n + a_{n-1} (ik)^{n-1} + \cdots + a_1 (ik) + a_0 \right) \hat{u}(k) = \hat{f}(k).$$

$P(ik)$

so $$\hat{u}(k) = \frac{\hat{f}(k)}{a_n (ik)^n + \cdots + a_1 (ik) + a_0}$$

Taking IFT

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \frac{\hat{f}(k)}{a_n (ik)^n + \cdots + a_1 (ik) + a_0} dk.$$

Consider $x_j = jh$ for $j \in \mathbb{Z}$

The important consequence of sampling is that some complex exponential waves e^{ikx} for different k will appear to be the same on grid x_j. We call this "aliases".

<u>Def</u>: The functions e^{ik_1x} and e^{ik_2x} are aliases on the grid $x_j = jh$ if $e^{ik_1x_j} = e^{ik_2x_j}$ $\forall j \in \mathbb{Z}$

Condition for aliases:

$$k_1jh = k_2jh + 2\pi n$$

for $j=1$

$$k_1h - k_2h = 2\pi n \quad n \in \mathbb{Z}.$$

$$k_1 - k_2 = \frac{2\pi}{h} \cdot n$$

$\Rightarrow$ ~~waves~~ frequencies k_1 and k_2 are indistinguishable in the grid if they differ by an integer multiple to $2\pi/h$.

So we restrict frequency to be: $k \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$.