Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(xn)\,dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx$$

Ex:

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{0} (-k) \cos(nx)\,dx + \frac{1}{\pi} \int_{0}^{\pi} (k) \cos(nx)\,dx$$

$$= -\frac{k}{\pi} \sin(nx) \Big|_{-\pi}^{0} + \frac{k}{\pi} \sin(nx) \Big|_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{0} (-k) \sin(nx)\,dx + \frac{1}{\pi} \int_{0}^{\pi} k \sin(nx)\,dx$$

$$= -\frac{k}{\pi} \left[-\cos(nx)\right] \Big|_{-\pi}^{0} + \frac{k}{\pi} \left[-\cos(nx)\right] \Big|_{0}^{\pi}$$

$$= \frac{k}{\pi} \underbrace{\left[\cos 0 - \cos(-n\pi)\right]}_{1-\cos(n\pi)} - \frac{k}{\pi} \underbrace{\left[\cos(n\pi) - \cos 0\right]}_{-(1-\cos(n\pi))}$$

$$b_n = \frac{2k}{\pi} \underbrace{\left[1 - \cos(n\pi)\right]}_{2 \text{ or } 0}$$

@ $n = 1$ $\cos \pi = -1$

$n = 2$ $\cos 2\pi = 1$

⋮

$$b_n = \begin{cases} \frac{4k}{\pi} & \text{for } n \text{ is odd.} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$$f(x) = \frac{4\pi}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin \pi x + \cdots\right)$$

<u>orthogonal trignometry.</u>

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = 0 \qquad n \neq m$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = 0 \qquad n \neq m$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$$

Fundamental period is smallest +ve period.

$\cos x$, $\sin x$, $\cos 2x$, $\sin 2\pi$, $\cos \pi x$, $\sin 2\pi x$.

$\cos \frac{2\pi n x}{k}$.

Find Fourier Series of:

1. $f(x) = |x|$

2. $f(x) = \begin{cases} x & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$

3. $f(x) = x^2$ $\quad -\pi < x < \pi$

4. $f(x) = x^2$ $\quad 0 < x < 2\pi$

when period is not 2π. ②

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Ex:

$$f(x) = \begin{cases} -k & \text{if } -2 < x < 0 \\ k & \text{if } 0 < x < 2 \end{cases}$$

$p = 2L = 4 \Rightarrow L = 2.$

$$a_n = \frac{1}{L} \int_{-2}^{0} (-k) \cos\left(\frac{n\pi x}{2}\right) dx + \frac{1}{L} \int_{0}^{2} (k) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= -\frac{k}{L} \sin\left(\frac{n\pi x}{2}\right)\Big|_{-2}^{0} \cdot \frac{2}{n\pi} + \frac{k}{L} \sin\left(\frac{n\pi x}{2}\right)\Big|_{0}^{2} \cdot \frac{2}{n\pi} = 0$$

$$b_n = \frac{1}{L} \int_{-2}^{0} (-k) \sin\left(\frac{n\pi x}{2}\right) dx + \frac{1}{2} \int_{0}^{2} (k) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{2k}{L} \int_{0}^{2} \sin \frac{n\pi x}{2}\, dx = \frac{2k}{L}\left(-\cos \frac{n\pi x}{2}\right)\Big|_{0}^{2} \frac{2}{n\pi}$$

$$= \frac{2k}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \text{ for } n \text{ is odd.}$$

Ex:

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

$$P = 2L = 4 \Rightarrow L = 2.$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{k}{n\pi}\int_{-1}^{1}\cos\left(\frac{n\pi x}{2}\right)\left(\frac{n\pi}{2}dx\right) = \frac{k}{n\pi}\sin\frac{n\pi x}{2}\Bigg|_{-1}^{1}$$