Boundary Value Problem.

Examples: (1) $-u''(x) = f(x)$ $\quad x \in [0,1]$ $\quad$ Dirichlet

$u(0) = a, \quad u(1) = b$

(2) $-u''(x) = f(x)$ $\quad x \in [0,1]$

$u'(0) = a \quad u'(1) = b$ $\quad$ Neumann

(3) $-u''(x) = f(x)$ $\quad x \in [0,1]$ and $\int_0^1 f(x)\,dx = 0$

$u(0) = u(1)$ $\quad$ periodic

General form:

$$-u''(x) + \alpha(x) u(x) = f(x) \qquad u(0) = a, \quad u(1) = b$$

Let us start with simple form:

$$-u'' = f \qquad u(0) = u(1) = 0$$

grid $x_j = jh$ $\quad \underbrace{j = 0, \cdots, N.}_{N+1 \text{ points}}$ and $h = \frac{1}{N}$

u: exact sol.

U: Numerical sol.

$$-\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} = f(x_j) \qquad U_0 = U_N = 0$$

$$\downarrow$$

$$\frac{-U_{j-1} + 2U_j - U_{j+1}}{h^2} = f(x_j)$$

U_j for $j = 1, \cdots, N-1$ are unknown

So equations are

$j=1$
$$-\frac{1}{h^2}\left[U_0 - 2U_1 + U_2\right] = f(x_1) \longleftarrow U_0 = 0$$

$$j=2 \quad -\frac{1}{h^2}\left[U_1 - 2U_2 + U_3\right] = f(x_2)$$

$$j=3 \quad -\frac{1}{h^2}\left[U_2 - 2U_3 + U_4\right] = f(x_3)$$

$$\vdots$$

$$j=N-1 \quad -\frac{1}{h^2}\left[U_{N-2} - 2U_{N-1} + U_N\right] = f(x_{N-1}) \longleftarrow U_N = 0$$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 & & & \\ -1 & 2 & -1 & 0 & \cdots & & \\ 0 & -1 & 2 & -1 & & & \\ & & \vdots & & & & \\ & & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & & 0 & -1 & 2 \\ 0 & \cdots & & & & & \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} U_1 \\ \vdots \\ \vdots \\ U_{N-1} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_{N-1}) \end{bmatrix}}_{F}$$

$$K = UF$$

$$U = K \backslash F$$

If we have $U_0 = a$, $U_N = b$,

chang the first and last eqns.

first is
$$-\frac{1}{h^2}\left[a - 2U_1 + U_2\right] = f(x_1)$$

$$-\frac{1}{h^2}\left[-2U_1 + U_2\right] = f(x_1) + \frac{a}{h^2}$$

Last $\Rightarrow$
$$-\frac{1}{h^2}\left[U_{N-2} - 2U_{N-1} + U_N\right] = f(x_{N-1})$$

$$-\frac{1}{h^2}\left[U_{N-2} - 2U_{N-1}\right] = f(x_{N-1}) + \frac{b}{h^2}$$

So F becomes
$$\begin{bmatrix} f(x_1) + \frac{a}{h^2} \\ f(x_2) \\ \vdots \\ f(x_{N-2}) \\ f(x_{N-1}) + \frac{b}{h^2} \end{bmatrix}$$

For Neumann:

$$U'(0) = a$$

or

$$\frac{U_1 - U_0}{h} = a$$

$$\frac{1}{h}\left[-U_0 + U_1\right] = a$$ shall be added to K as first row (Similarly one row at end).

Similarly you can use backward formula.

or

$$\frac{U_1 - U_{-1}}{2h} = a$$ U_{-1} is a ghost node.

due to U_{-1}, at $j=0$,

$$\frac{U_1 + 2U_0 - U_{-1}}{h^2} = f(x_0)$$

$\Rightarrow$ two additional rows and columns per boundary condition.
[as same treatment is applied at $x=1$].

First and last rows added can be rescaled to make it look better.

e.g., $\frac{U_1 - U_{-1}}{2h} = a$ by $\frac{2}{h}$

So the matrix will look like

$$\begin{bmatrix} -1 & 0 & 1 & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & & \end{bmatrix}$$

For Neumann, we have another condition,

$$\underline{1}^T f = \sum_i f(x_i) = 0$$

Periodic Problem

$$-u''(x) = f(x) \qquad x \in [0,1] \qquad u(0) = u(1)$$

$U_0 = U_N \implies$ put it in the first and last eqns. $\cancel{U_0 - U_N} = 0$

So K is

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & \cdots & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & 0 & -1 \\ -1 & \cdots & \cdots & -1 & 2 \end{bmatrix}$$