Ordinary differential equations:

Let solution is known at, say $t=0$.

$$y'(t) = f(t, y(t)) \qquad y(0) = y_0$$

Ex: $\quad y' = \sqrt{y} \qquad y(0) = 0$

$$\int \frac{dy}{\sqrt{y}} = \int dt$$

$$\int y^{-1/2}\,dy = \int dt$$

$$\frac{y^{1/2}}{1/2} = t + c \implies y^{1/2} = \frac{t+c}{2}$$

$$y = \frac{(t+c)^2}{4}.$$

@ $t=0$, $y=0$

$$0 = \frac{(0+c)^2}{4} \implies c = 0$$

so $\quad y = \frac{t^2}{4}$

[Note $y=0$ is also a solution].

Ex: $\quad y' = y^2 \qquad y(0) = 1$

$$\frac{dy}{dt} = y^2$$

$$\int y^{-2}\,dy = \int dt \implies \frac{y^{-1}}{-1} = t + c$$

$$\implies y^{-1} = -(t+c) \implies y = -\frac{1}{t+c}$$

at $t=0$, $y=1$

$$y = -\frac{1}{t+C} \Rightarrow 1 = -\frac{1}{0+C} \Rightarrow 1 = -\frac{1}{C}$$

$\Rightarrow C = -1$

so solution is

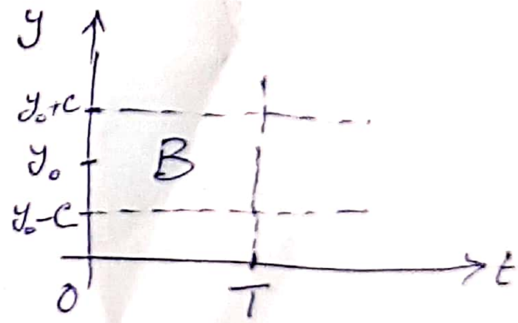
$$y = -\frac{1}{t-1} = \frac{1}{1-t}$$

[Note y does not exist at $t=1$].

Rather solution exists to the left of $t=1$

Given T, C, y_0, consider the box B in (t, y) space

$$B = [0, T] \times [y_0 - C, y_0 + C]$$



Assume:

1. $f(t,y)$ is continuous over B.

2. $f(t,y) \leq K$ when $(t,y) \in B$. (boundedness)

3. $|f(t,u) - f(t,v)| \leq L|u-v|$ whenever $(t,u), (t,v) \in B$.
(Lipschitz continuity)

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Numerical methods for IVP

Let $t_n = nh$, (grid on time)

y_n is the approximation of $y(t_n)$

1. Forward Euler (Explicit Euler)

$$y_{n+1} = y_n + h \cancel{f} f(t_n, y_n)$$

Proof:

[We can use forward difference formula for y'].

$$y' = f(t, y)$$

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n)$$

$$y_{n+1} = y_n + h f(t_n, y_n)$$

2. Backward Euler (Implicit Euler).

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$$

Proof:

$$\frac{y_n - y_{n-1}}{h} = f(t_n, y_n)$$

$$y_n = y_{n-1} + h f(t_n, y_n).$$

or

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1})$$

3. Trapezoidal (midpoint) rule (implicit).

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

4. Improved Euler, Runge-Kutta 2 (explicit).

$$\tilde{y}_{n+1} = y_n + h\, f(t_n, y_n) \qquad \text{(method 1)}$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, \tilde{y}_{n+1})\right]$$

(method 3)

5. Runge-Kutta 4 (Explicit).

$$k_1 = f(t_n, y_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right)$$

$$k_4 = f(t_n + h, y_n + h k_3)$$

$$y_{n+1} = y_n + h\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

6. In these methods, we used one past value y_n. There are other methods which use more past values like y_{n-1}, y_{n-2}, etc.

You can choose any method but the method must be convergent (true solution as $h \to 0$) and stable ($|y_n - \tilde{y}_n| \le c|y_0 - \tilde{y}_0|$)

Error: Let $y_{n+1} = \psi(t_n, y_n, h)$ [numerical solution]

$y(t_{n+1})$ is the actual solution

$$e_{n+1}(h) = \psi(t_n, y_n, h) - y(t_{n+1})$$

This is local error.

Let

$$E_n(h) = y_n - y(t_n)$$

This is the global error.

Def (Consistency):

ψ is consistent if, for any $n \ge 0$,

$$\lim_{h \to 0} \frac{e_n(h)}{h} = 0$$

Def (Order) ψ is of order p if $e_n(h) = O(h^{p+1})$

Note: If local error is $O(h^{p+1})$, then global error is $O(h^p)$

Stability:

Criterion $|y_n - \tilde{y}_n| \le C|y_0 - \tilde{y}_0|$ is too complex.

Taylor series expansion of $y' = f(t, y)$ about y_0 is

$$\frac{d}{dt}(y(t) - y_0) = f(t, y(t)) + \left.\frac{\partial f}{\partial y}\right|_{(t, y_0)} (y(t) - y_0) + o(|y(t) - y_0|).$$

It is sufficient to check the stability for

$$y' = \lambda y \qquad \because \lambda \text{ is } \left.\frac{\partial f}{\partial y}\right|_{(t, y_0)}$$

Def (Linear stability): Suppose $y' = \lambda y$ for some $\lambda \in \mathbb{C}$. Then the numerical method φ is linearly stable if $y_n \to 0$ as $n \to \infty$.

$y' = \lambda y$ is stable if $Re(\lambda) < 0$.

Ex: For forward Euler method, applied to $y' = \lambda y$,

$$y_{n+1} = y_n + h\lambda y_n = (1 + h\lambda) y_n$$

$y_n \to 0$ if $\cancel{(1+h\lambda)}$ $|1 + h\lambda| < 1$. $\quad |\cdot|$ magnitude of complex #.

Let

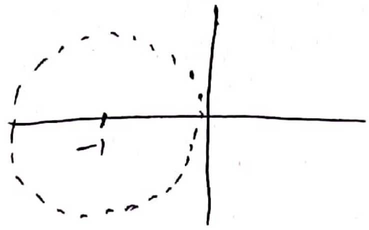
$$h\lambda = x + iy$$
$$1 + h\lambda = (1 + x) + iy$$

$$|1 + h\lambda| = \sqrt{(1+x)^2 + y^2} < 1$$
$$(1+x)^2 + y^2 < 1$$

$$(x-(-1))^2 + (y-0)^2 < 1$$

Inside a circle with center $(-1, 0)$ and radius 1.

$Re(\lambda) = 1 + x < 0$
holds.



So backward Euler method is unconditionally stable.
↓
"independent of h"

<u>Ex:</u> For midpoint method.

$$y_{n+1} = y_n + \frac{h}{2}\lambda\left[y_n + y_{n+1}\right]$$

hence
$$y_{n+1} = y_n + \frac{h}{2}\lambda y_n + \frac{h}{2}\lambda y_{n+1}.$$

$$\left[1 - \frac{h}{2}\lambda\right] y_{n+1} = \left(1 + \frac{h}{2}\lambda\right) y_n$$

$$y_{n+1} = \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} y_n$$

For stability

$$\frac{\left|1 + \frac{h}{2}\lambda\right|}{\left|1 - \frac{h}{2}\lambda\right|} < 1 \Rightarrow \left|1 + \frac{h\lambda}{2}\right| < \left|1 - \frac{h\lambda}{2}\right|$$

Let $h\lambda = x + iy \Rightarrow 1 + \frac{h\lambda}{2} = 1 + \frac{x}{2} + i\frac{y}{2}$

$$1 - \frac{h\lambda}{2} = 1 - \frac{x}{2} - i\frac{y}{2}$$

$$\left(1+\frac{x}{2}\right)^2 + \cancel{\left(\frac{y}{2}\right)^2} < \left(1-\frac{x}{2}\right)^2 + \cancel{\left(\frac{y}{2}\right)^2}$$

$$\left(1+\frac{x}{2}\right)^2 < \left(1-\frac{x}{2}\right)^2$$

$$1+\frac{x^2}{4}+x < 1+\frac{x^2}{4}-x.$$

$$x < -x$$

holds if $\boxed{x<0}$.

For a system of ODE's $y'(t) = f(t, y(t))$.

$$A = \nabla_y f \Big|_{(t, y_0)}.$$

$$y'(t) = A\, y(t).$$

and this has the solution $y(t) = y(0)\, e^{tA}$

where e^{tA} is a matrix exponential.

Stability is acheived when eigen values λ of A all obey $Re(\lambda) < 0$.