$$a_1 x_1 + a_2 x_2 = c_1$$
$$b_1 x_1 + b_2 x_2 = c_2$$

To find line of intersection.

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \frac{1}{a_1 b_2 - b_1 a_2}\begin{bmatrix} b_2 & -a_2 \\ -b_1 & a_1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

<u>Ex</u>:
$$x_1 - 4x_2 = -10$$
$$0.5x_1 - x_2 = -2$$

$\Rightarrow a_1 = 1, \quad a_2 = -4, \quad c_1 = -10$

$b_1 = 0.5, \quad b_2 = -1, \quad c_2 = -2$

$$a_1 b_2 - b_1 a_2 = (1)(-1) - (0.5)(-4) = -1 + 2 = 1$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -0.5 & 1 \end{bmatrix}\begin{bmatrix} -10 \\ -2 \end{bmatrix} = \begin{bmatrix} 10-8 \\ 5-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so $(2, 3)$ is the point of intersection.

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Ex:

$$x_1 + x_2 = 5$$
$$3x_1 + 3x_2 = 15$$

$a_1 = 1, \quad a_2 = 1, \quad c_1 = 5$

$b_1 = 3, \quad b_2 = 3, \quad c_2 = 15$

$a_1 b_2 - b_1 a_2 = (1)(3) - (1)(3) = 0$

$\Rightarrow$ line are parallel and do not intersect.

Eigen Values and Vectors

$A\underline{x} = \lambda \underline{x}$ and $\underline{x} \neq \underline{0}$, $\underline{x}$ is called eigen vector
λ is called eigen value of $\underline{x}$.

$(\lambda, \underline{x})$ is called eigen pair.

Set of all λ values forms spectrum of A.

Consider n eigen pairs $(\lambda_j, \underline{x}_j)$ for $j = 1, \cdots, n$.

Then

$$\underline{y} = \sum_{j=1}^{n} \alpha_j \underline{x}_j$$ $\underline{y}$ is an arbitrary vector

$$A\underline{y} = \sum_{j=1}^{n} \alpha_j A\underline{x}_j = \sum_{j=1}^{n} \alpha_j \lambda_j \underline{x}_j$$

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Let A be an $n \times n$ matrix

$$A\underline{x} = \lambda \underline{x}$$

$$(\lambda I - A)\underline{x} = \underline{0}$$

as $\underline{x} \neq \underline{0}$

$$\Rightarrow \quad |\lambda I - A| = 0$$

This is called characteristic polynomial. Its roots shall give all values of λ's

Note: You will find λ's mostly as complex numbers.

If z is a complex number, then

In cartesian coordinates: $z = x + iy \qquad i = \sqrt{-1}$

$$\text{Real}(z) = x, \qquad \text{Imag}(z) = y$$

Magnitude $|z| = \sqrt{x^2 + y^2}$

Conjugate $\bar{z} = x - iy$ (replace i with $-i$)

Euler identity: $e^{i\theta} = \cos\theta + i\sin\theta$

$z = (r, \theta)$ in polar coordinates

$r = |z| \qquad z = re^{i\theta}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

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Properties:

1. If $A\underline{x} = \lambda \underline{x}$

$$(A + \alpha I)\underline{x} = (\lambda + \alpha)\underline{x}$$

2. $k > 0$, $A^k \underline{x} = \lambda^k \underline{x}$

3. Let $\underline{x} = S\underline{y} \Rightarrow \underline{y} = S^{-1}\underline{x}$

and $B = S^{-1}AS$

So $B\underline{y} = S^{-1}A\underbrace{S\underline{y}}_{x}$

$$= S^{-1}A\underline{x} = S^{-1}\lambda\underline{x} = \lambda \underbrace{S^{-1}\underline{x}}_{y}$$

$$B\underline{y} = \lambda\underline{y}$$

B has the same eigenvalues as A.

$\Rightarrow$ B is similar to A

So $A \rightarrow S^{-1}AS$ is similarity transformation of A

4. Spectral decomposition:

Matrix of eigen vectors $\rightarrow X = [\underline{x}_1, \underline{x}_2, \cdots \underline{x}_n]$

$$AX = A[\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n]$$

$$= [A\underline{x}_1, A\underline{x}_2, \cdots A\underline{x}_n]$$

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$$A\cancel{X} = [\lambda_1 \underline{x}_1, \lambda_2 \underline{x}_2, \ldots, \lambda_n \underline{x}_n]$$

$$= [\underline{x}_1\ \underline{x}_2 \cdots \underline{x}_n] \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \lambda_n \end{bmatrix}}_{\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)} \cancel{[\underline{x}_1\ \underline{x}_2 \cdots \underline{x}_n]}$$

$$= X\Lambda$$

As $AX = X\Lambda$

$A = X\Lambda X^{-1}$

we say A is diagonalizable.

If this does not hold, A is not diagonalizable and A is called Defective matrix.

Algebraic multiplicity: how many times a λ exists

geometric " : " , different $\underline{x}$ exists

Ex. $A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ has $\lambda = 4, 4$

eigen value 4 has algebraic multiplicity 2,

and eigen vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

eigen value 4 has geometric multiplicity 1.

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Ex $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ $\lambda = 4, 4$

$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

geometric multiplicity of $4 = 2$.

Vector Norms:

1. $\|\underline{x}\|_2 = \sqrt{x^T x}$ $\quad l_2$- norm

2. $\|\underline{x}\|_\infty = \max\limits_{1 \le i \le n} |x_i|$ $\quad l_\infty$- norm

3. $\|\underline{x}\|_1 = \sum\limits_{i=1}^{n} |x_i|$ $\quad l_1$- norm

You can define your own norm but that must satisfy the followings.

1. $\|\underline{x}\| \geq 0$

2. $\|\alpha \underline{x}\| = |\alpha| \|\underline{x}\|$

3. $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

General form of l_1, l_2 and l_∞ is

$\|\underline{x}\|_p = \left(\sum\limits_{i=1}^{n} |x_i|^p \right)^{1/p}$ $\quad 1 \le p \le \infty$

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Ex: We want to find distance b/w $\underbrace{\begin{pmatrix} 11 \\ 12 \\ 13 \end{pmatrix}}_{\underline{x}}$ and $\underbrace{\begin{pmatrix} 12 \\ 14 \\ 16 \end{pmatrix}}_{\underline{y}}$.

Let $\underline{z} = \underline{y} - \underline{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$\|z\|_1 = |1| + |2| + |3| = 6$$

$$\|z\|_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} = 3.7417$$

$$\|z\|_\infty = \max\{|1|, |2|, |3|\} = 3$$

Note that

$$\|z\|_1 \geq \|z\|_2 \geq \|z\|_\infty$$

This is true in general

Matrix Norms: Let A be a matrix

1. $\|A\|_\infty = \max(\text{Sum} \cancel{\text{maximas}}$ of each row with absolute values$)$.

2. $\|A\|_1 = \max(\text{Sum} \cancel{\text{maximas}}$ of each col " " $)$.

3. $\|A\|_2 = \sqrt{\max \text{ of } \lambda \text{ of } A^T A}$

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E_1: $A = \begin{bmatrix} 1 & 3 & 7 \\ -4 & 1.27 & -2 \end{bmatrix}$

$$\|A\|_\infty = \max\{1+3+7,\ 4+1.27+2\}$$
$$= \max\{11, 7.27\} = 11$$

$$\|A\|_1 = \max\{1+4, 3+1.27, 7+2\}$$
$$= \max\{5, 4.27, 9\} = 9$$

$$A^T A = \begin{bmatrix} 1 & -4 \\ 3 & 1.27 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ -4 & 1.27 & -2 \end{bmatrix} = \begin{bmatrix} 17 & -2.08 & 15 \\ -2.08 & 10.6129 & 18.46 \\ 15 & 18.46 & 53 \end{bmatrix}$$

Eigen values: $\lambda = 0,\ 16.8\quad 63.7$ (approximately)

$$\|A\|_2 = \sqrt{\max\{0, 16.8, 63.7\}} = \sqrt{63.7} \approx 7.98$$

Some Definitions.

1. A is symmetric if $A^T = A$
2. A is positive definite if $\underline{x}^T A \underline{x} > 0 \quad \forall \underline{x} \neq 0$
3. If A is symmetric, A has n real eigenpairs
4. Orthogonal vectors u and v: $\underline{u}^T \underline{v} = 0$
5. A real square matrix Q is orthogonal if $Q^T Q = I$ (columns are orthogonal)

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Singular Value Decomposition.

$A : m \times n$

SVD is: $A = U \Sigma V^T$

$U : m \times m, \qquad V : n \times n \qquad \Sigma : m \times n$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \\ & & & \end{bmatrix}$$

$\sigma_1 \geq \sigma_2 \geq \cdots \cdots \geq \sigma_r > 0$

1. Power Method to find eigen values/vectors

We know $\quad A\underline{x} = \underbrace{\lambda \underline{x}}_{\underline{y}}$

Note $\cancel{y}$ $\underline{x}$ is a unit vector.

$$\underline{y}_k = A\underline{x}_k$$

$$\underline{x}_{k+1} = \frac{\underline{y}_k}{\|\underline{y}_k\|}$$

Corresponding $\quad \lambda = \cancel{y} \, \underline{x}_{k+1}^T A \underline{x}_{k+1}$

Algo: $\quad x_0$: initial guess

for $k = 1, 2, \ldots$

$\quad \underline{y}_k = A\underline{x}_{k-1}$

$\quad \underline{x}_k = \dfrac{\underline{y}_k}{\|\underline{y}_k\|}$

$\quad \lambda^{(k)} = \underline{x}_k A \underline{x}_k$

end

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2. Inverse Iteration Method.

Let A have λ_j as largest eigen value

$A - \alpha I$ has eigen values $\lambda_j - \alpha$

$(A - \alpha I)^{-1}$ " " " $\frac{1}{\lambda_j - \alpha}$

If $\alpha \to \lambda_j$, $\frac{1}{\lambda_j - \alpha}$ is very large

so $\frac{\lambda_1 - \alpha}{\lambda_2 - \alpha} \longrightarrow 0$ as $\alpha \to \lambda_1$

Algorithm:

Given initial guess $\underline{x}_0$ and α,

for $k = 1, 2, \ldots$

$$\underline{y} = (A - \alpha I)^{-1} \underline{x}_{k-1}$$

$$x_k = \frac{\underline{y}}{\|\underline{y}\|}$$

$$\lambda^{(k)} = \underline{x}_k^T A \underline{x}_k$$

end

Eigen values obtained

Rayleigh Quotient Iteration:

Initial guess $\underline{x}_0$,

Set $\lambda^{(0)} = \underline{x}_0^T A \underline{x}_0$

for $k = 1, 2, \ldots$

$$\underline{y} = \left(A - \lambda^{k-1} I\right)^{-1} \underline{x}_{k-1}$$

$$x_k = \frac{\underline{y}}{\|\underline{y}\|}$$

$$\lambda^{(k)} = \underline{x}_k^T A \underline{x}_k$$

end

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