Interpolation

Interpolation is the problem of fitting a smooth curve through a given set of points.

Polynomial Interpolation:

consider points $x_0, x_1, \ldots, x_N$ (N+1 points)

and values $f(x_0), f(x_1), \ldots, f(x_N)$

$P_N(x)$: Nth degree polynomial

$$y_j = P_N(x_j) \qquad j = 0, \ldots, N$$

$$P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$
$$= \sum_{n=0}^{N} a_n x^n$$

we have to find $a_0, a_1, \ldots, a_N$.

- N+1 coefficients → N+1 degree of freedom.
- If number of data points is larger than degree of polynomial, we may not find a curve passing through all the points. We will find "best approximate."

Ex: $P_2(x) = a_0 + a_1 x + a_2 x^2$

Let's say we have points

$y_0 = f(x_0) \qquad y_1 = f(x_1) \qquad y_2 = f(x_2)$

Putting these in the polynomial $P_2(x)$, we get

$y_0 = a_0 + a_1 x_0 + a_2 x_0^2$

$y_1 = a_0 + a_1 x_1 + a_2 x_1^2$

$y_2 = a_0 + a_1 x_2 + a_2 x_2^2$

we have to find a_0, a_1 and a_2.

Writing in the matrix form,

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

In general form

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ & & \vdots & & \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}}_{A}$$

$Y = VA$

$A = V^{-1} Y$

This is called Vandermonde matrix

Def: Let $L_k(x)$ be a polynomial of degree N,

such that

$$L_k(x_j) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \longrightarrow (1)$$

This $L_k(x)$ is called basis function or

Langrange elementary polynomial.

Consider

$$L_k(x) = C \prod_{\substack{j=0 \\ j \neq k}}^{N} (x - x_j)$$

Note x_j are roots of L_k.

Also note: If I put value of any root in $L_k(x)$,

then $L_k(x) = 0$. And If I put any value other than

a root, say x_k, I want to get $L_k(x) = 1$. This is easy,

if I set

$$C = \frac{1}{\prod_{j \neq k} (x_k - x_j)}$$

ensures x_k is not a root

putting this value of C in $L_k(x)$,

$$L_k(x) = \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)} \longrightarrow (2)$$

Lagrange Interpolation:

Given: data points $(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)$

so

$$P_N(x_0) = y_0$$
$$P_N(x_1) = y_1$$
$$\vdots$$
$$P_N(x_N) = y_N$$

So we can write:

$$P_N(x) = \sum_{k=0}^{N} y_k L_k(x) \longrightarrow (3)$$

@ $x = x_0$,

$$P_N(x_0) = \sum_{k=0}^{N} y_k L_k(x_0)$$

$$= y_0 L_0(x_0) + y_1 L_1(x_0) + \cdots + y_N L_N(x_0)$$

$$= y_0 \quad (\text{not } L_k(x_j) = 1 \text{ for } j = k)$$

Similarly, @ $x = x_1$

$$P_N(x_1) = y_0 \underbrace{L_0(x_1)}_{0} + y_1 \underbrace{L_1(x_1)}_{1} + y_2 \underbrace{L_2(x_1)}_{0} + \cdots + y_N \underbrace{L_N(x_1)}_{0}$$

$$= y_1$$

So we rewrite (3) as:

$$P_N(x) = \sum_{k=0}^{N} f(x_k) L_k(x) \quad \text{where } f(x_k) = y_k$$

Ex: Given (x_1, y_1) and (x_2, y_2)

$$L_1(x) = \frac{x - x_2}{x_1 - x_2}$$

$$L_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Note

$$L_k(x) = \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$

$$P_1(x) = y_1 L_1(x_1) + y_2 L_2(x_2)$$

$$= y_1 \cdot \frac{x - x_2}{x_1 - x_2} + y_2 \cdot \frac{x - x_1}{x_2 - x_1} = \frac{-y_1(x - x_2) + y_2(x - x_1)}{x_2 - x_1}$$

$$= \frac{-x y_1 + x_2 y_1 + x y_2 - x_1 y_2}{x_2 - x_1}$$

$$= \frac{-x y_1 + x_2 y_1 + x y_2 - x_1 y_2 - x_1 y_1 + x_1 y_1}{x_2 - x_1}$$

$$= \frac{y_1(x_2 - x_1) + x(y_2 - y_1) - x_1(y_2 - y_1)}{x_2 - x_1}$$

$$= y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Ex: $f(x) = e^{x}$, $N=2$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$

$$y_0 = f(x_0) = e^{-1},\; y_1 = f(x_1) = 1,\; y_2 = f(x_2) = e$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}x(x-1)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-1)}{(0+1)(0-1)} = \frac{x^2-1}{-1} = 1-x^2$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x-0)}{(1+1)(1-0)} = \frac{1}{2}x(x+1)$$

$$P_2(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

$$= e^{-1}\cdot\frac{1}{2}x(x-1) + 1\cdot(1-x^2) + e\cdot\frac{1}{2}x(x+1)$$

$$= \frac{1}{2}x\left[(x-1)e^{-1} + (x+1)e\right] + (1-x^2)$$

$$= \frac{1}{2}x\left[xe^{-1} - e^{-1} + xe + e\right] + (1-x^2)$$

$$= \frac{1}{2}x\left[x(e+e^{-1}) + (e-e^{-1})\right] + (1-x^2)$$

$$= \frac{1}{2}\times 3.0862x^2 + \frac{1}{2}\times 2.3504x + 1 - x^2$$

$$= 1 + 1.1752x + 0.5431x^2$$

$e = 2.7183$

$e^{-1} = 0.3679$

$e + e^{-1} = 3.0862$

$e - e^{-1} = 2.3504$

compare with $e^{x} = 1 + x + \frac{x^2}{2} + \cdots$