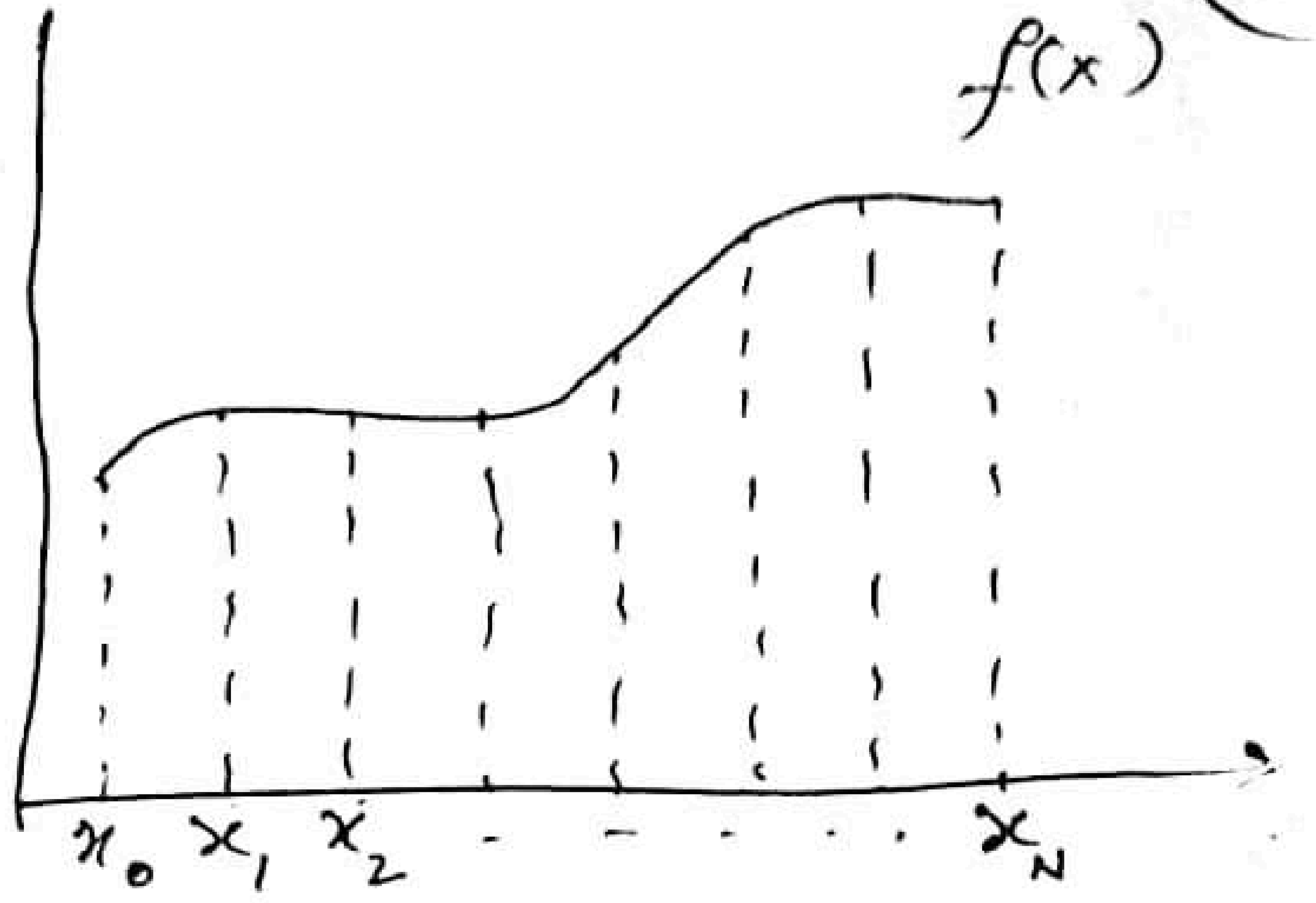
Numerical Integration

$$\int_0^1 f(x)\,dx$$



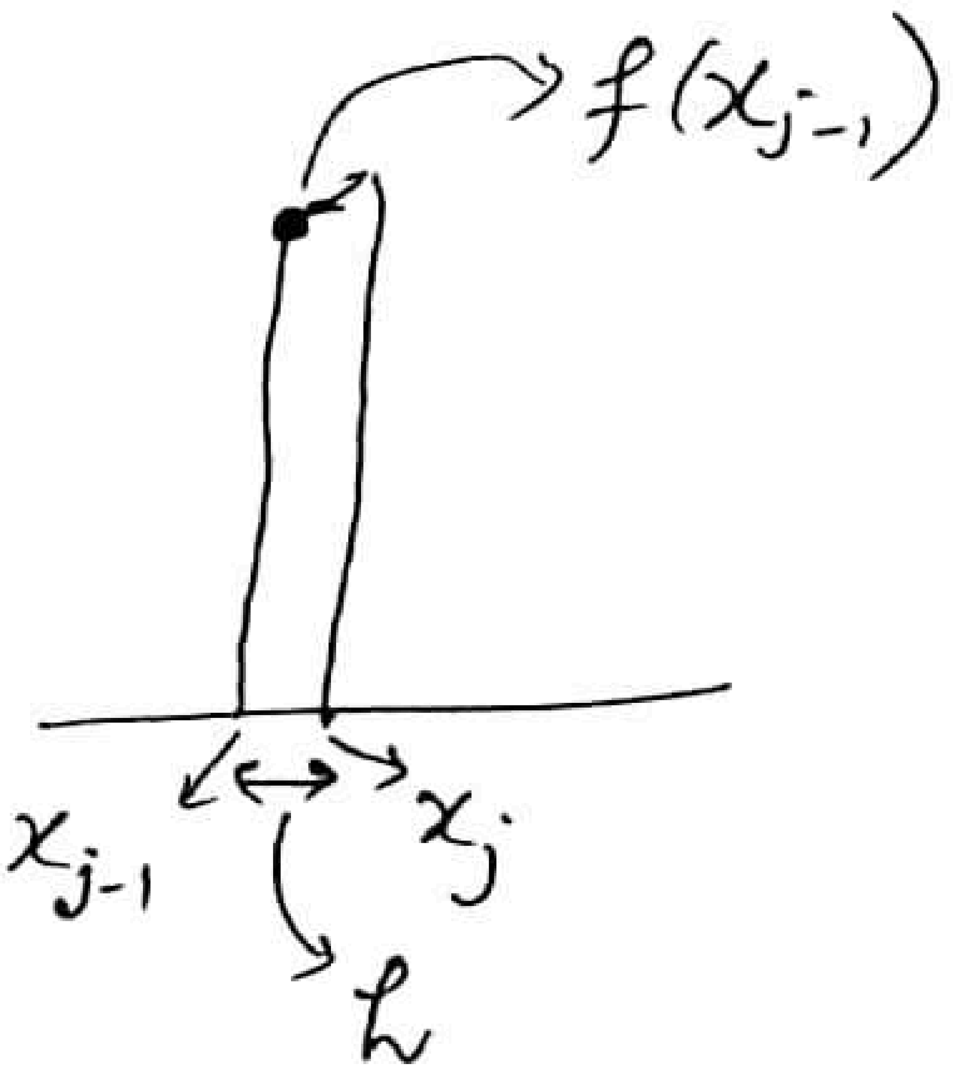
break up interval into subintervals

$$0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$$

(x_j) : Cartesian grid

$$\int_0^1 f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{N-1}}^{x_N} f(x)\,dx$$

$$= \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} f(x)\,dx$$



Now let $x_j - x_{j-1} = h$

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \simeq h\, f(x_{j-1})$$

So

$$\int_0^1 f(x)\,dx \simeq \sum_{j=1}^{N} h\, f(x_{j-1}) = h \sum_{j=1}^{N} f(x_{j-1}).$$

Now consider Taylor series expansion of $f(x)$ about $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{3!} f'''(a)(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

[Note: f is infinitely differentiable]

Now truncate the Taylor series after N terms.

$$f(x) \simeq \sum_{n=0}^{N} \frac{1}{n!} f^{(n)} (x-a)^n$$

$$= \sum_{n=0}$$

For $\int_{x_{j-1}}^{x_j} f(x)\,dx$, let us expand f in a Taylor Series near x

so $f(x) = f(x_{j-1}) + f'(y(x))\,h$

where $y(x) \in [x_{j-1}, x]$

where $y(x)$ is number between x_{j-1} and x_j

$$\int_{x_{j-1}}^{x_j} f(x)\,dx = \int_{x_{j-1}}^{x_j} f(x_{j-1})\,dx + \int_{x_{j-1}}^{x_j} f'(y(x))\,h\,dx$$

$$= f(x_{j-1}).\,x\Big|_{x_{j-1}}^{x_j} + h\int_{x_{j-1}}^{x_j} f'(y(x))\,dx$$

$$\left|\int_{x_{j-1}}^{x_j} f'(y(x))\,dx\right| \le \max_{y\in[x_{j-1},x_j]} |f'(y)| \int_{x_{j-1}}^{x_j} 1.\,dx = h \max_{y\in[x_{j-1},x_j]} |f'(y)|$$

so $$\int_{x_{j-1}}^{x_j} f(x)\,dx = h\,f(x_{j-1}) + h^2 \max_{y\in[x_{j-1},x_j]} |f'(y)|$$

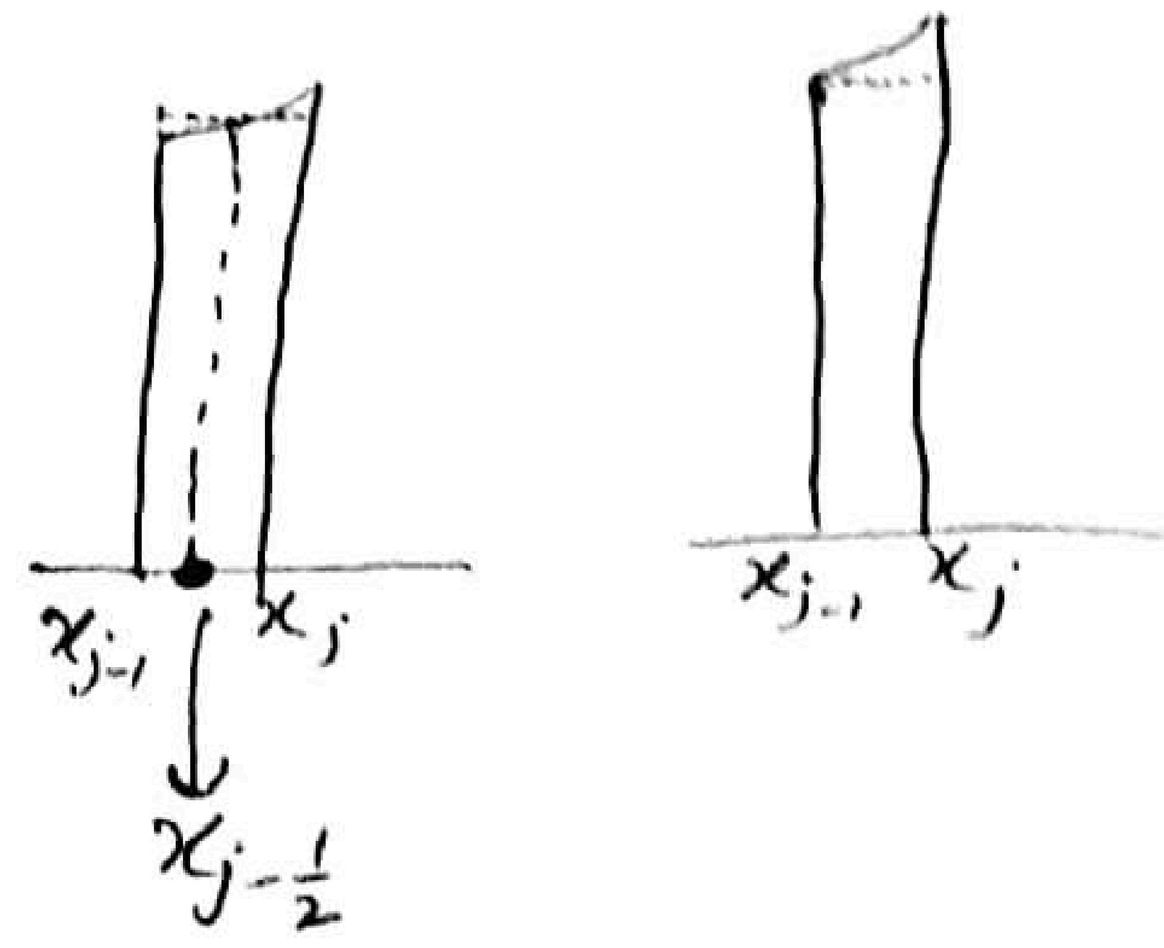
$$= h\,f(x_{j-1}) + O(h^2)$$

So $$\boxed{\int^{1} f(x)\,dx \simeq h \sum_{j=1}^{N} f(x_{j-1}) + O(h)}$$

$$\sum_{j=0}^{N} O(h^2) = N.O(h^2) = O(h)$$

As $h = \frac{1}{N}$

Better if we take mid of x_{j-1} and x_j, that is at $x_{j-\frac{1}{2}}$



This is called mid-point method.

So
$$\boxed{\int_0^1 f(x)\,dx \simeq h \sum_{j=1}^{N} f(x_{j-\frac{1}{2}}) + O(h)}$$

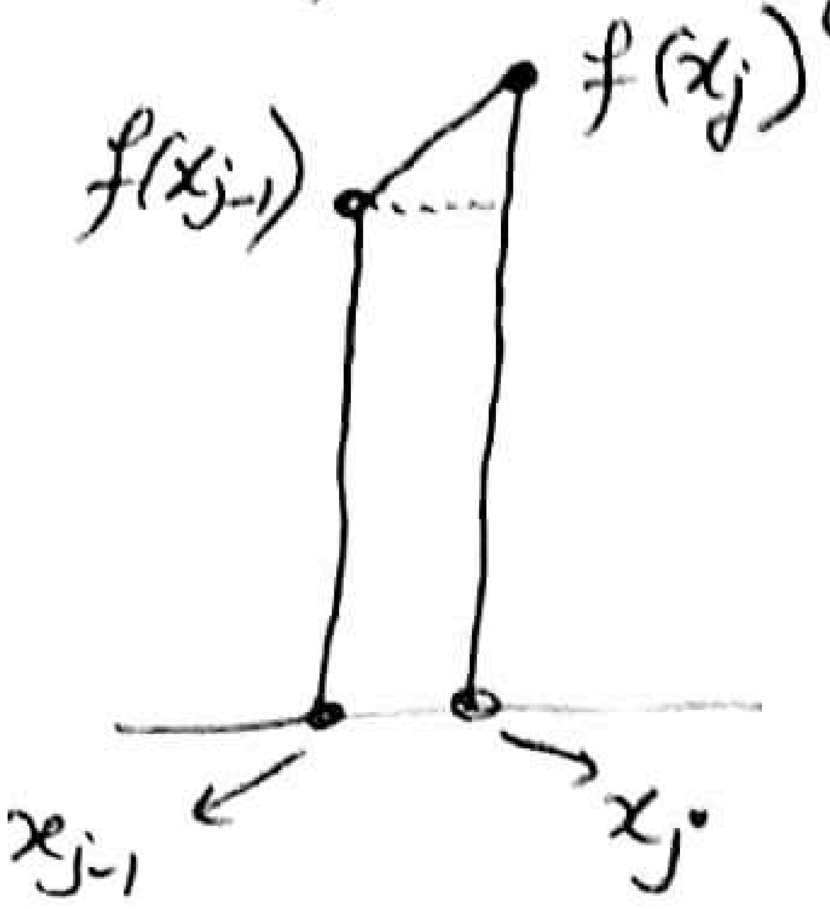
provided f'' exists, as f' is hidden in O.

Accuracy increases as $h \to 0$.

Another way:

In both the above methods, rectangles are used.

Another idea is to use trapezoid instead of a rectangle.



~~Four points of the~~

Area of Trapezid is:

= Area of Rectangle + Area of Triangle

$$= (x_j - x_{j-1})\, f(x_{j-1}) + \frac{1}{2}(x_j - x_{j-1})\left[f(x_j) - f(x_{j-1})\right]$$

$$= h\, f(x_{j-1}) + \frac{1}{2} h \left[f(x_j) - f(x_{j-1})\right]$$

$$= \frac{1}{2} h \left[2f(x_{j-1}) + f(x_j) - f(x_{j-1})\right] = \frac{h}{2}\left[f(x_{j-1}) + f(x_j)\right]$$

Truncated Taylor Series is (page 3 b)

$$f(x) = f(x_{j-1}) + f'(y(x))\,h + O(h^2).$$

at $x = x_j$

$$f(x_j) = f(x_{j-1}) + f'(y(x_j)) \cdot h + O(h^2).$$

[Note: $y(x_j)$ is a number between x_{j-1} and x_j].

Take $y(x_j) = x_{j-1}$

$$f(x_j) = f(x_{j-1}) + f'(x_{j-1}) \cdot h + O(h^2).$$

~~integrating from x_{j-1}~~ to x_j

$$\int_{x_{j-1}}^{x_j} f(x_j)\,dx \,-$$

We just saw (General form of Taylor series about x_{j-1}).

$$\boxed{f(x) = f(x_{j-1}) + f'(x_{j-1})(x - x_{j-1}) + O(h^2)}$$

Integrate it from x_{j-1} to x_j

$$\int_{x_{j-1}}^{x_j} f(x)\,dx = \int_{x_{j-1}}^{x_j} f(x_{j-1})\,dx + \int_{x_{j-1}}^{x_j} f'(x_{j-1})(x - x_{j-1})\,dx + \int_{x_{j-1}}^{x_j} O(h^2)\,dx$$

$$= f(x_{j-1})\, x\Big|_{x_{j-1}}^{x_j} + f'(x_{j-1}) \int_{x_{j-1}}^{x_j} (x - x_{j-1})\,dx + O(h^2) \int_{x_{j-1}}^{x_j} dx.$$

$$= f(x_{j-1})\underbrace{(x_j - x_{j-1})}_{h} + f'(x_{j-1})\,\frac{1}{2}(x - x_{j-1})^2\Big|_{x_{j-1}}^{x_j} + O(h^2) \cdot \underbrace{(x_j - x_{j-1})}_{h}$$

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$$\int_{x_{j-1}}^{x_j} f(x)\,dx = h f(x_{j-1}) + \frac{h^2}{2} f'(x_{j-1}) + O(h^3)$$

↳ Note this is also area of the trapezoid.

So equating both forms of the area.

$$\underbrace{\frac{h}{2}\left[f(x_{j-1}) + f(x_j)\right]}_{A} = \underbrace{h f(x_{j-1}) + \frac{h^2}{2} f'(x_{j-1})}_{B} + \underbrace{O(h^3)}_{\text{Diff}}$$

equal

How

$$\frac{h}{2} f(x_{j-1}) + \frac{h}{2} f(x_j) = h f(x_{j-1}) - \frac{h}{2} f(x_{j-1}) + \frac{h}{2} f(x_j).$$

$$= h f(x_{j-1}) + \frac{h}{2}\left[f(x_j) - f(x_{j-1})\right]$$

$$= h f(x_{j-1}) + \frac{h^2}{2}\left[\frac{f(x_j) - f(x_{j-1})}{h}\right] \leftarrow f'(x_{j-1})$$

$$= h f(x_{j-1}) + \frac{h^2}{2} f'(x_{j-1})$$

Therefore,

$$\int_0^1 f(x)\,dx \simeq \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} f(x)\,dx$$

As A & B are equal, so add $O(h^3)$ to A

$$= \sum_{j=1}^{N} \left\{\frac{h}{2}\left[f(x_{j-1}) + f(x_j)\right] + O(h^3)\right\}$$

$$= \frac{h}{2}\sum_{j=1}^{N} f(x_{j-1}) + \frac{h}{2}\underbrace{\sum_{j=1}^{N} f(x_j)}_{} + \sum_{j=1}^{N} O(h^3)$$

Now separate first term ↳ separate last term.

$$\int_0^1 f(x)\,dx = \frac{h}{2} f(x_0) + \underbrace{\frac{h}{2} \sum_{j=2}^{N} f(x_{j-1})}_{\text{equal}} + \underbrace{\frac{h}{2} \sum_{j=1}^{N-1} f(x_j)} + \frac{h}{2} f(x_N) + O(h^2)$$

$$\boxed{\int_0^1 f(x)\,dx = \frac{h}{2} f(x_0) + h \sum_{j=1}^{N-1} f(x_j) + \frac{h}{2} f(x_N) + O(h^2)}$$

$$\sum_{j=2}^{N} f(x_{j-1}) = f(x_1) + f(x_2) + \cdots$$

As you see h^2 in O, this is second order accurate.

Ex: $f(x) = x^2$ in $[0,1]$.

Direct Method: $\int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = 1/3 = 0.333\ldots$

Numerical Method:

1) Rectangular Rule (first method) [Red box on 3b].

Set $N=4$, so $h = \frac{1}{N} = \frac{1}{4}$

$f(x) = x^2$ (Given)

grid points: $x = \underset{x_0}{0}, \underset{x_1}{\frac{1}{4}}, \underset{x_2}{\frac{1}{2}}, \underset{x_3}{\frac{3}{4}}, \underset{x_4}{1}$ $\quad f(x) = 0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}, 1$

$$\int_0^1 f(x)\,dx = h \sum_{j=1}^{N} f(x_{j-1})$$

$$= h\left[f(x_0) + f(x_1) + f(x_2) + f(x_3)\right]$$

$$= \frac{1}{4}\left[0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right] = 0.2188 \text{ (Compare with actual } 1/3)$$

2) Trapeziodal rule (Red box on 5b)

$N = 4 \qquad h = \frac{1}{N} = \frac{1}{4}$

grid points: $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

$f(x) = 0, \frac{1}{16}, \frac{1}{4}, \frac{9}{16}, 1$ (as $f(x) = x^2$) given

$$\int_0^1 f(x)\,dx = \frac{h}{2} f(x_0) + h\left[f(x_1) + f(x_2) + f(x_3)\right] + \frac{h}{2} f(x_4)$$

$$= \frac{1}{4}\cdot\frac{1}{2}\cdot 0 + \frac{1}{4}\left[\frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right] + \frac{1}{2}\cdot\frac{1}{4}\cdot 1$$

$$= \frac{1}{4}\cdot\frac{14}{16} + \frac{1}{8} = \frac{22}{64} = 0.3438\ldots$$ (again compair with actual)