A sequence is a possibly infinite collection of numbers in order

$$a_1, a_2, a_3 \cdots$$

Sum is

$$a_1 + a_2 + a_3 + \cdots$$

<u>Def 1:</u> A sequence $\{a_j\}_{j=0}^{\infty}$ is said to be ϵ-close to a number b if there exists $N \geq 0$, such that for all $n \geq N$,

$$|a_j - b| \leq \epsilon$$

$\{a_j\}_{j=0}^{\infty}$ is said to converge to b, $\forall \epsilon > 0$.

i.e., $\quad a_j \to b \quad$ or $\quad \lim_{j \to \infty} a_j = b$

$e^{-n} \to 0$ as $n \to \infty$ (very fast} convergence

$\frac{n}{n+2} \to 1$ as $n \to \infty$ (slower convergence)

$(-1)^n$ is bounded but does not converge.

$\log n \to \infty$ as $n \to \infty$ diverge

Def: Consider $\{a_j\}_{j=0}^{\infty}$. We define Nth partial sum S

as $S_N = a_0 + a_1 + \cdots + a_N = \sum_{j=0}^{N} a_j$

$\{a_j\}_{j=0}^{N}$ $\sum_{j=0}^{N} a_j$ Converges, if $S_N \to b$ as $N \to \infty$, i.e.,

$$\sum_{j=0}^{\infty} a_j = b.$$

Ex: $\sum_{j=0}^{\infty} 2^{-j}$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$$

Ex: $\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$ (see $x = \frac{1}{2}$ in the above example)

Converges if $|x| < 1$

Ex: Harmonic Series: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$$S_N = \sum_{j=1}^{N} \frac{(-1)^j}{j}$$ (conditionally convergent)

↳ terms are rearranged so convergence to a different limit.

1) Assum N is even, group 2 each

$$\frac{1}{j} - \frac{1}{j+1} = \frac{1}{j(j+1)} \le \frac{1}{j^2}$$

so $S_N \le \sum_{j=1,3,5\ldots}^{N-1} \frac{1}{j^2}$ Now it converges.

2) If N is odd.

$$S_N = S_{N-1} + \frac{1}{N+1}$$

Both converge, so sum converges

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots = \log(2)$$

[Try to write Taylor series of $\log(1+x)$ about $x=0$]

Alternating Series Test: $\sum_{j=0}^{\infty} (-1)^j a_j \qquad a_j > 0$

pairs

If $\{a_j\}$ converges to zero, then the series is convergent

Absolutely Convergent:

Let $f: \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ be a bijection

(You rearrange)

Consider $\sum_{j=0}^{\infty} a_j$,

If $\sum_{j=0}^{\infty} a_j = \sum_{j=0}^{\infty} a_{f(j)}$

Then series is absolutely convergent

For absolutely convergent series

1. $$\left| \sum_{j=0}^{\infty} a_j \right| \le \sum_{j=0}^{\infty} |a_j|$$

2. $$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{j,k} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{j,k}$$

<u>Def</u>: Consider two non-zero sequences f_n and g_n for $n=0,1,2,\ldots$

$\cancel{f}$ $f_n = O(g_n)$ when $|f_n| \le C|g_n|$, where $C > 0$.

$f_n = o(g_n)$ when $\frac{f_n}{g_n} \to 0$ as $n \to \infty$

$f_n = \Theta(g_n)$ when $f_n = O(g_n)$ and $g_n = O(f_n)$

<u>Ex</u>: $f_n = n^2$, $g_n = n^3$

$n^2 = O(n^3)$, $n^2 = o(n^3)$ but $n^2 \neq \Theta(n^3)$

<u>Ex</u>: $f_n = \frac{n}{n+2}$, $g_n = \frac{n}{n-3}$

$f_n = O(g_n)$, $f_n = \Theta(g_n)$, but $f_n \neq o(g_n)$.

<u>Ex</u>: $f_n = n^a$, $g_n = e^{bn}$ $a>0, b>0$

$\cancel{g_n}$ $f_n = o(g_n)$ conversely, $e^{-bn} = o(n^{-a})$