Non-linear Equations

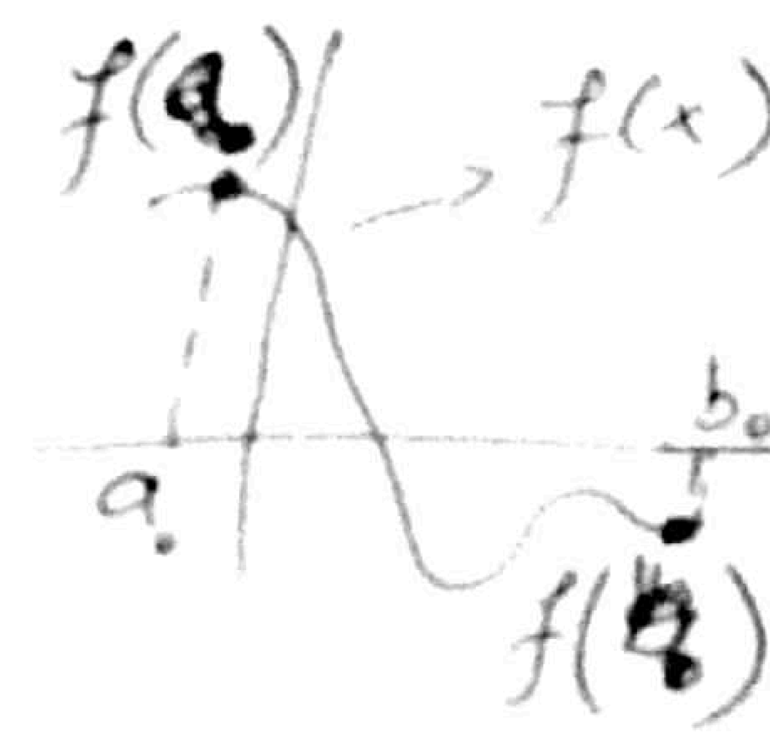
1. Root Finding:

Let $g(x) = h(x)$ be given [Find values of x when g and h are equal]

define $f(x) = g(x) - h(x)$

Target: Find roots of $f(x)$

Method 1: Bisection Method



Take Two points a_0 and b_0, such that

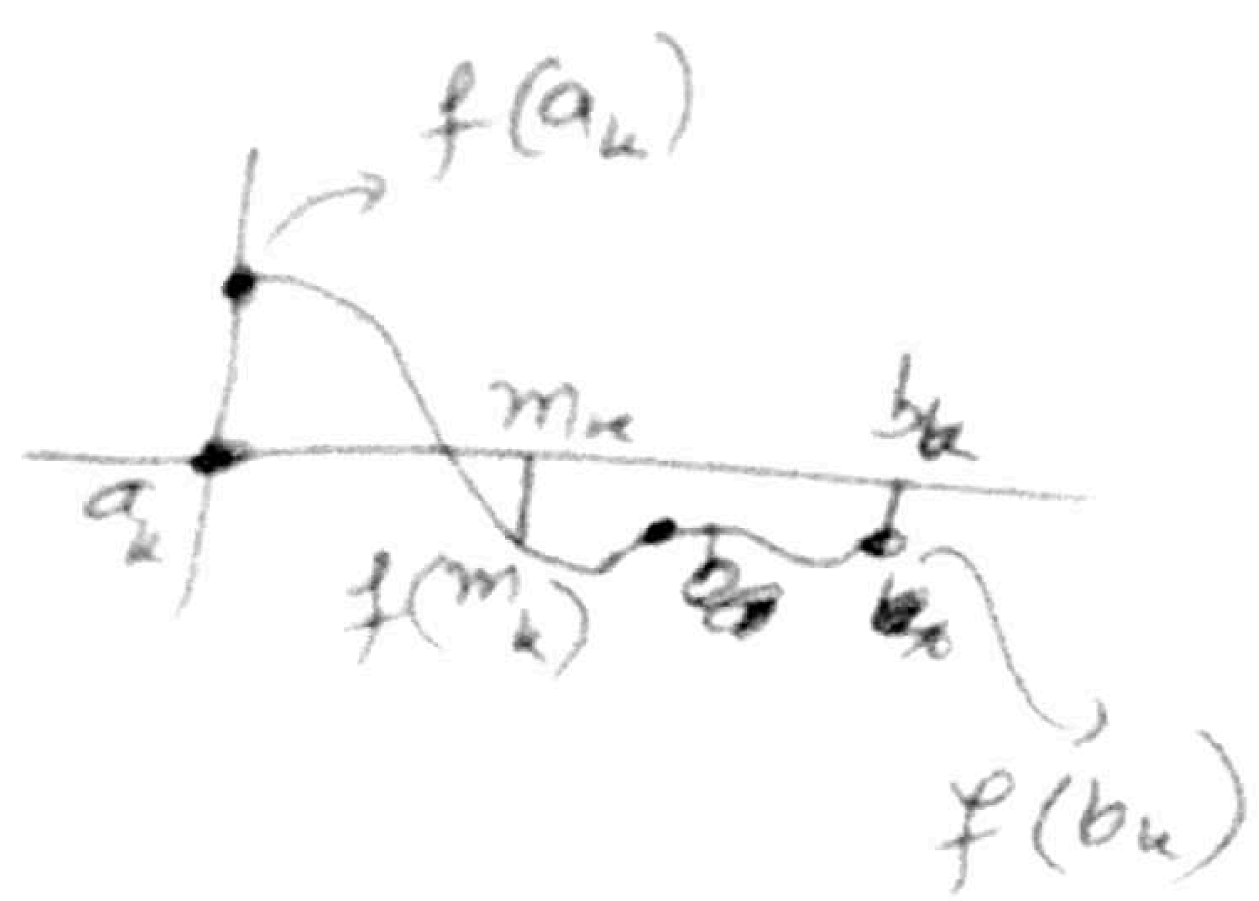
$f(a_0) > 0$, and $f(b_0) < 0$

So there is a point b/w a_0 and b_0 where f crosses the horizontal axis $\rightarrow [a_0, b_0]$

Let us consider $m_k = \frac{a_k + b_k}{2}$ [at stage k]

1. If $f(m_k) < 0 \Rightarrow$

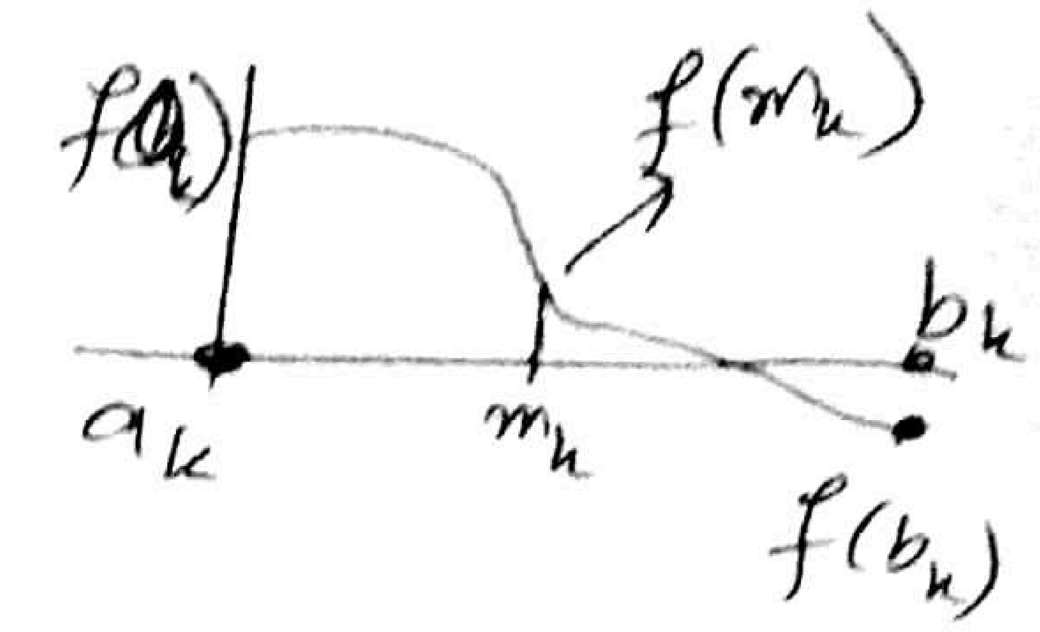
$\Rightarrow$ root is in $[a_k, m_k]$



So $a_{k+1} = a_k$

$b_{k+1} = m_k$

2. If $f(m_k) > 0$



$\Rightarrow$ Root is in $[m_k, b_k]$

so $a_{k+1} = m_k$, $b_{k+1} = b_k$

3. If $f(m_k) = 0$

Then m_k is the root. STOP.

(This rarely happens).

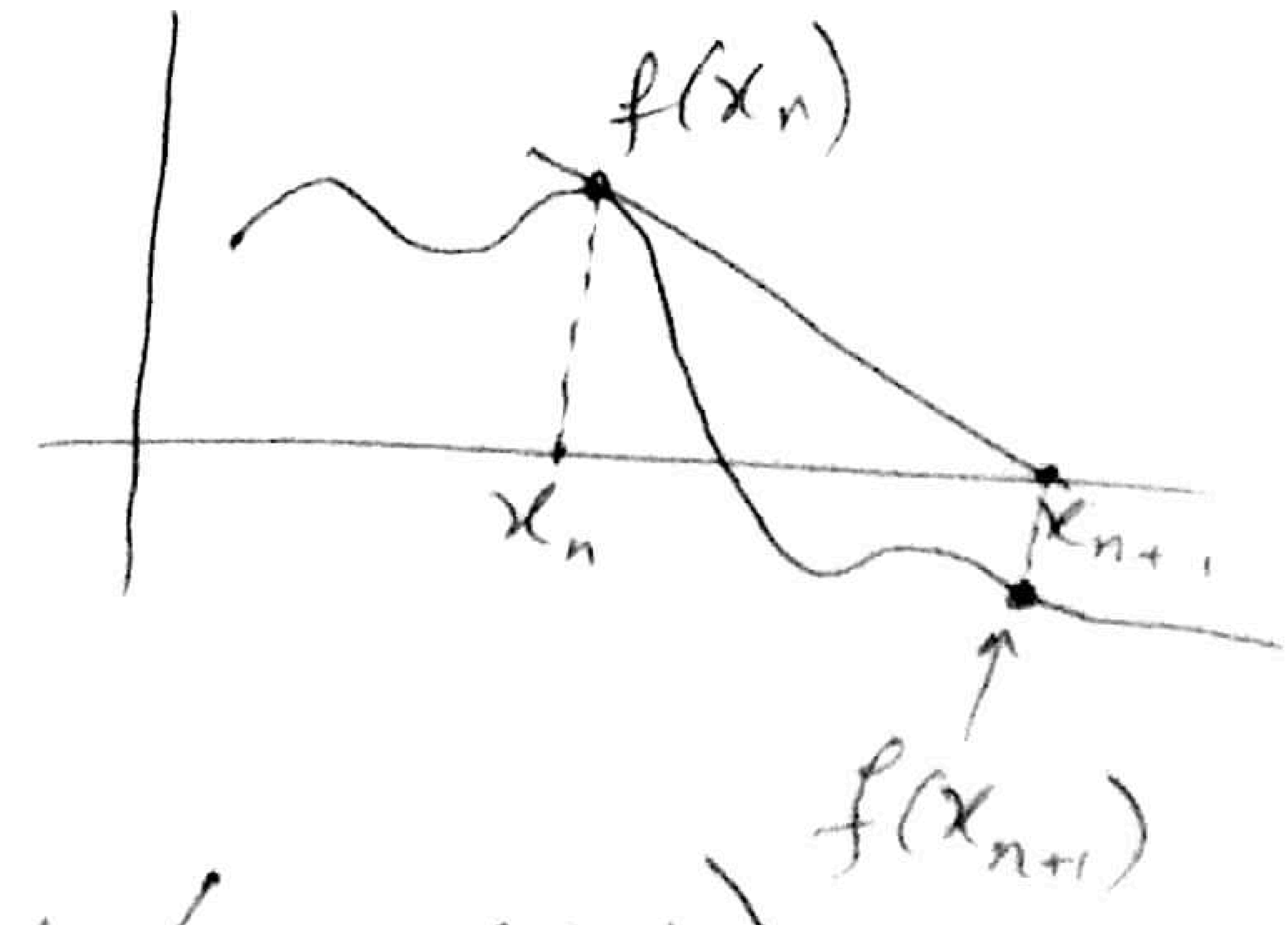
So, stopping criteria is $|b_k - a_k|$ is very small.

like $|b_k - a_k| = 2^{-k} |b_0 - a_0|$.

- Algo is guaranteed to converge.
- Algo working time may be large.
- If there are many roots, it will give only one of them.

Method 2: Newton-Raphson:

To make faster, we look for tangent crossing the horizontal axis



As tangent is a line, so eq of line is

$y - y_n = m(x - x_n)$

(eq of Tangent passing through $(x_n, f(x_n))$).

new points is $\left(x_{n+1}, \underbrace{f(x_{n+1})}_{y_{n+1}}\right)$

slope is the derivative at $\left(x_n, \underbrace{f(x_n)}_{y_n}\right)$

so $\quad m = f'(x_n)$.

$\Rightarrow$ Eq of Tangent becomes:

$$y_{n+1} - y_n = f'(x_n)\,(x_{n+1} - x_n)$$

We want $y_{n+1} = 0$ (to get the root).

$$\cancel{-y_n = f'(x_n}$$

$$-f(x_n) = f'(x_n)\,(x_{n+1} - x_n)$$

$$-\frac{f(x_n)}{f'(x_n)} = x_{n+1} - x_n$$

$$x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Ex: You want to find $\sqrt{2}$.

Let $x = \sqrt{2}$ or $x^2 = 2$

$f(x) = x^2 - 2$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

$\boxed{\sqrt{2} = 1.4142135}$

Let $x_0 = 1$

$x_1 = \frac{3}{2} = 1.5$

$x_2 = \frac{17}{12} = 1.4167\ldots$

$x_3 = \frac{577}{408} = 1.4142157\ldots$

We need derivative for this method.

Method 3: Secant Method.

As
$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{f(x_n)}{\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}$$

$$\boxed{x_{n+1} = x_n - \frac{(x_n - x_{n-1})\, f(x_n)}{f(x_n) - f(x_{n-1})}}$$

LHS involves x_{n+1}

RHS " x_n, x_{n-1}

$\Rightarrow$ We need two previous points.

Above three methods work for single variable (uni-variate or single dimension).