This snowboarder flying through the air shows an example of motion in two dimensions. In the absence of air resistance, the path would be a perfect parabola. The gold arrow represents the downward acceleration of gravity, $\vec{g}$. Galileo analyzed the motion of objects in 2 dimensions under the action of gravity near the Earth's surface (now called "projectile motion") into its horizontal and vertical components.

We will discuss how to manipulate vectors and how to add them. Besides analyzing projectile motion, we will also see how to work with relative velocity.

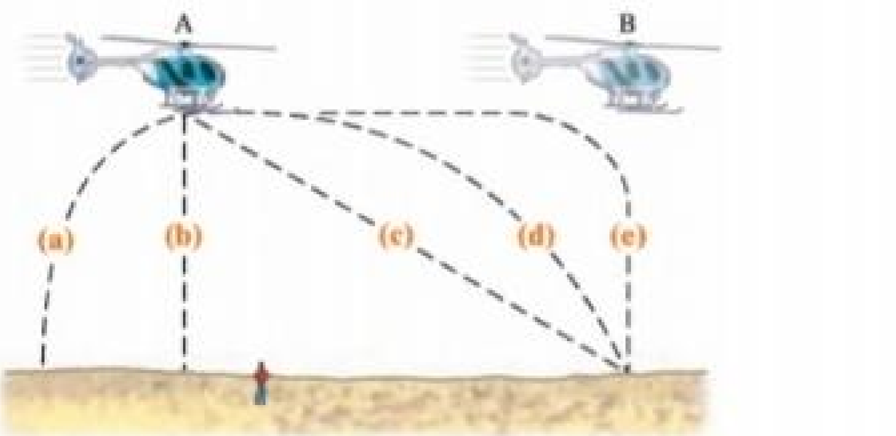
CHAPTER 3

Kinematics in Two or Three Dimensions; Vectors

CHAPTER-OPENING QUESTION—Guess now!

[Don't worry about getting the right answer now—you will get another chance later in the Chapter. See also p. 1 of Chapter 1 for more explanation.]

A small heavy box of emergency supplies is dropped from a moving helicopter at point A as it flies along in a horizontal direction. Which path in the drawing below best describes the path of the box (neglecting air resistance) as seen by a person standing on the ground?



In Chapter 2 we dealt with motion along a straight line. We now consider the description of the motion of objects that move in paths in two (or three) dimensions. To do so, we first need to discuss vectors and how they are added. We will examine the description of motion in general, followed by an interesting special case, the motion of projectiles near the Earth's surface. We also discuss how to determine the relative velocity of an object as measured in different reference frames.

CONTENTS

3–1 Vectors and Scalars

We mentioned in Chapter 2 that the term *velocity* refers not only to how fast an object is moving but also to its direction. A quantity such as velocity, which has *direction* as well as *magnitude*, is a **vector** quantity. Other quantities that are also vectors are displacement, force, and momentum. However, many quantities have no direction associated with them, such as mass, time, and temperature. They are specified completely by a number and units. Such quantities are called **scalar** quantities.

Drawing a diagram of a particular physical situation is always helpful in physics, and this is especially true when dealing with vectors. On a diagram, each vector is represented by an arrow. The arrow is always drawn so that it points in the direction of the vector quantity it represents. The length of the arrow is drawn proportional to the magnitude of the vector quantity. For example, in Fig. 3–1, green arrows have been drawn representing the velocity of a car at various places as it rounds a curve. The magnitude of the velocity at each point can be read off Fig. 3–1 by measuring the length of the corresponding arrow and using the scale shown (1 cm = 90 km/h).

When we write the symbol for a vector, we will always use boldface type, with a tiny arrow over the symbol. Thus for velocity we write $\vec{\mathbf{v}}$. If we are concerned only with the magnitude of the vector, we will write simply v, in italics, as we do for other symbols.

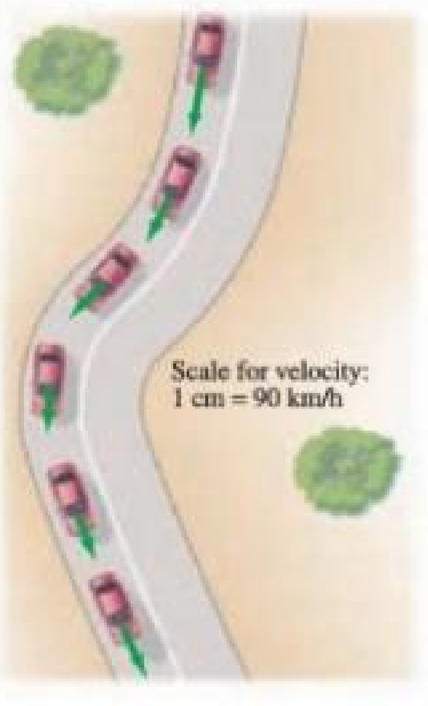


FIGURE 3–1 Car traveling on a road, slowing down to round the curve. The green arrows represent the velocity vector at each position.

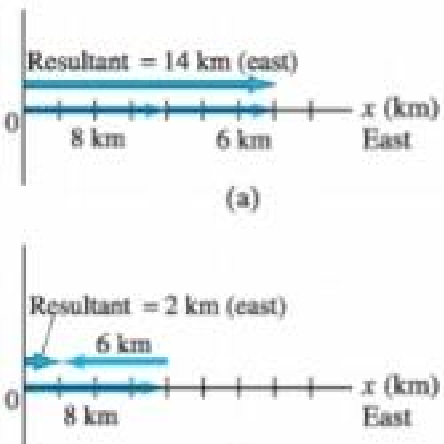
3–2 Addition of Vectors—Graphical Methods

Because vectors are quantities that have direction as well as magnitude, they must be added in a special way. In this Chapter, we will deal mainly with displacement vectors, for which we now use the symbol $\vec{\mathbf{D}}$, and velocity vectors, $\vec{\mathbf{v}}$. But the results will apply for other vectors we encounter later.

We use simple arithmetic for adding scalars. Simple arithmetic can also be used for adding vectors if they are in the same direction. For example, if a person walks 8 km east one day, and 6 km east the next day, the person will be 8 km + 6 km = 14 km east of the point of origin. We say that the *net* or *resultant* displacement is 14 km to the east (Fig. 3–2a). If, on the other hand, the person walks 8 km east on the first day, and 6 km west (in the reverse direction) on the second day, then the person will end up 2 km from the origin (Fig. 3–2b), so the resultant displacement is 2 km to the east. In this case, the resultant displacement is obtained by subtraction: 8 km − 6 km = 2 km.

But simple arithmetic cannot be used if the two vectors are not along the same line. For example, suppose a person walks 10.0 km east and then walks 5.0 km north. These displacements can be represented on a graph in which the positive y axis points north and the positive x axis points east, Fig. 3–3. On this graph, we draw an arrow, labeled $\vec{\mathbf{D}}_1$, to represent the 10.0-km displacement to the east. Then we draw a second arrow, $\vec{\mathbf{D}}_2$, to represent the 5.0-km displacement to the north. Both vectors are drawn to scale, as in Fig. 3–3.

FIGURE 3–2 Combining vectors in one dimension.



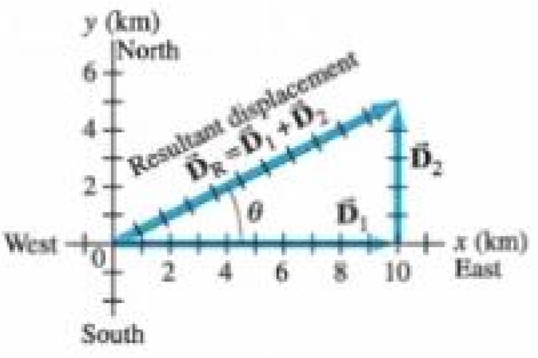


FIGURE 3–3 A person walks 10.0 km east and then 5.0 km north. These two displacements are represented by the vectors $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$, which are shown as arrows. The resultant displacement vector, $\vec{\mathbf{D}}_R$, which is the vector sum of $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$, is also shown. Measurement on the graph with ruler and protractor shows that $\vec{\mathbf{D}}_R$ has a magnitude of 11.2 km and points at an angle $\theta = 27°$ north of east.

After taking this walk, the person is now 10.0 km east and 5.0 km north of the

After taking this walk, the person is now 10.0 km east and 5.0 km north of the point of origin. The **resultant displacement** is represented by the arrow labeled $\vec{\mathbf{D}}_R$ in Fig. 3–3. Using a ruler and a protractor, you can measure on this diagram that the person is 11.2 km from the origin at an angle $\theta = 27°$ north of east. In other words, the resultant displacement vector has a magnitude of 11.2 km and makes an angle $\theta = 27°$ with the positive x axis. The magnitude (length) of $\vec{\mathbf{D}}_R$ can also be obtained using the theorem of Pythagoras in this case, since D_1, D_2, and D_R form a right triangle with D_R as the hypotenuse. Thus

$$D_R = \sqrt{D_1^2 + D_2^2} = \sqrt{(10.0\text{ km})^2 + (5.0\text{ km})^2}$$
$$= \sqrt{125\text{ km}^2} = 11.2\text{ km}.$$

You can use the Pythagorean theorem, of course, only when the vectors are *perpendicular* to each other.

The resultant displacement vector, $\vec{\mathbf{D}}_R$, is the sum of the vectors $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$. That is,

$$\vec{\mathbf{D}}_R = \vec{\mathbf{D}}_1 + \vec{\mathbf{D}}_2.$$

This is a *vector* equation. An important feature of adding two vectors that are not along the same line is that the magnitude of the resultant vector is not equal to the sum of the magnitudes of the two separate vectors, but is smaller than their sum. That is,

$$D_R \le D_1 + D_2,$$

where the equals sign applies only if the two vectors point in the same direction. In our example (Fig. 3–3), $D_R = 11.2$ km, whereas $D_1 + D_2$ equals 15 km, which is the total distance traveled. Note also that we cannot set $\vec{\mathbf{D}}_R$ equal to 11.2 km, because we have a vector equation and 11.2 km is only a part of the resultant vector, its magnitude. We could write something like this, though: $\vec{\mathbf{D}}_R = \vec{\mathbf{D}}_1 + \vec{\mathbf{D}}_2 = (11.2\text{ km}, 27° \text{ N of E})$.

EXERCISE A Under what conditions can the magnitude of the resultant vector above be $D_R = D_1 + D_2$?

Figure 3–3 illustrates the general rules for graphically adding two vectors together, no matter what angles they make, to get their sum. The rules are as follows:

1. On a diagram, draw one of the vectors—call it $\vec{\mathbf{D}}_1$—to scale.
2. Next draw the second vector, $\vec{\mathbf{D}}_2$, to scale, placing its tail at the tip of the first vector and being sure its direction is correct.
3. The arrow drawn from the tail of the first vector to the tip of the second vector represents the *sum*, or **resultant**, of the two vectors.

The length of the resultant vector represents its magnitude. Note that vectors can be translated parallel to themselves (maintaining the same length and angle) to accomplish these manipulations. The length of the resultant can be measured with a ruler and compared to the scale. Angles can be measured with a protractor. This method is known as the **tail-to-tip method of adding vectors**.

The resultant is not affected by the order in which the vectors are added. For example, a displacement of 5.0 km north, to which is added a displacement of 10.0 km east, yields a resultant of 11.2 km and angle $\theta = 27°$ (see Fig. 3–4), the same as when they were added in reverse order (Fig. 3–3). That is, now using $\vec{\mathbf{V}}$ to represent any type of vector,

$$\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2 = \vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_1, \qquad \text{[commutative property]} \quad \textbf{(3–1a)}$$

which is known as the *commutative* property of vector addition.

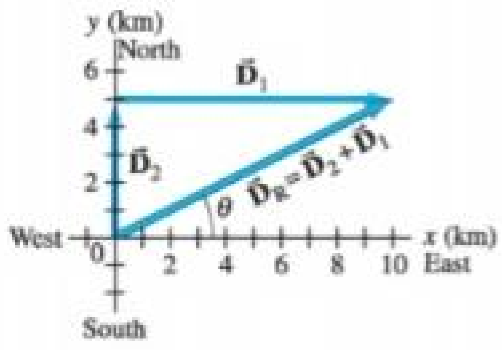


FIGURE 3–4 If the vectors are added in reverse order, the resultant is the same. (Compare to Fig. 3–3.)

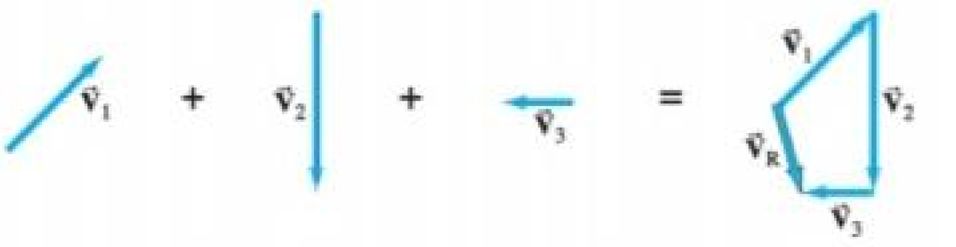


FIGURE 3–5 The resultant of three vectors: $\vec{\mathbf{V}}_R = \vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_3$.

The tail-to-tip method of adding vectors can be extended to three or more vectors. The resultant is drawn from the tail of the first vector to the tip of the last one added. An example is shown in Fig. 3–5; the three vectors could represent displacements (northeast, south, west) or perhaps three forces. Check for yourself that you get the same resultant no matter in which order you add the three vectors; that is,

$$(\vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2) + \vec{\mathbf{V}}_3 = \vec{\mathbf{V}}_1 + (\vec{\mathbf{V}}_2 + \vec{\mathbf{V}}_3), \qquad \text{[associative property]} \quad \textbf{(3–1b)}$$

which is known as the *associative* property of vector addition.

A second way to add two vectors is the **parallelogram method**. It is fully equivalent to the tail-to-tip method. In this method, the two vectors are drawn starting from a common origin, and a parallelogram is constructed using these two vectors as adjacent sides as shown in Fig. 3–6b. The resultant is the diagonal drawn from the common origin. In Fig. 3–6a, the tail-to-tip method is shown, and it is clear that both methods yield the same result.

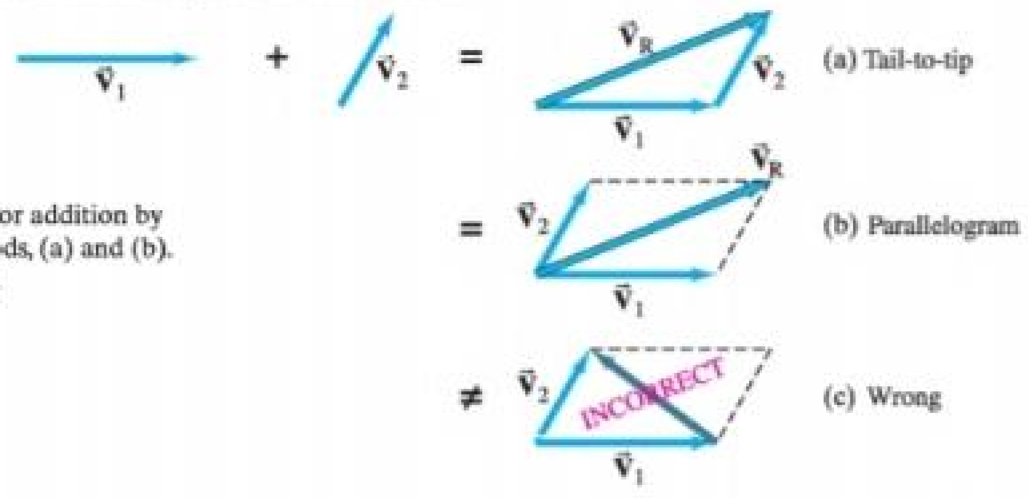


FIGURE 3–6 Vector addition by two different methods, (a) and (b). Part (c) is incorrect.

CAUTION
Be sure to use the correct diagonal on parallelogram to get the resultant

It is a common error to draw the sum vector as the diagonal running between the tips of the two vectors, as in Fig. 3–6c. *This is incorrect*: it does not represent the sum of the two vectors. (In fact, it represents their difference, $\vec{\mathbf{V}}_2 - \vec{\mathbf{V}}_1$, as we will see in the next Section.)

CONCEPTUAL EXAMPLE 3–1 **Range of vector lengths.** Suppose two vectors each have length 3.0 units. What is the range of possible lengths for the vector representing the sum of the two?

RESPONSE The sum can take on any value from 6.0 (= 3.0 + 3.0) where the vectors point in the same direction, to 0 (= 3.0 − 3.0) when the vectors are antiparallel.

EXERCISE B If the two vectors of Example 3–1 are perpendicular to each other, what is the resultant vector length?

3–3 Subtraction of Vectors, and Multiplication of a Vector by a Scalar

Given a vector $\vec{\mathbf{V}}$, we define the *negative* of this vector $(-\vec{\mathbf{V}})$ to be a vector with the same magnitude as $\vec{\mathbf{V}}$ but opposite in direction, Fig. 3–7. Note, however, that no vector is ever negative in the sense of its magnitude: the magnitude of every vector is positive. Rather, a minus sign tells us about its direction.

FIGURE 3–7 The negative of a vector is a vector having the same length but opposite direction.

FIGURE 3–8 Subtracting two vectors: $\vec{V}_2 - \vec{V}_1$.

We can now define the subtraction of one vector from another: the difference between two vectors $\vec{V}_2 - \vec{V}_1$ is defined as

$$\vec{V}_2 - \vec{V}_1 = \vec{V}_2 + (-\vec{V}_1).$$

That is, the difference between two vectors is equal to the sum of the first plus the negative of the second. Thus our rules for addition of vectors can be applied as shown in Fig. 3–8 using the tail-to-tip method.

A vector $\vec{V}$ can be multiplied by a scalar c. We define their product so that $c\vec{V}$ has the same direction as $\vec{V}$ and has magnitude cV. That is, multiplication of a vector by a positive scalar c changes the magnitude of the vector by a factor c but doesn't alter the direction. If c is a negative scalar, the magnitude of the product $c\vec{V}$ is still $|c|V$ (where $|c|$ means the magnitude of c), but the direction is precisely opposite to that of $\vec{V}$. See Fig. 3–9.

FIGURE 3–9 Multiplying a vector $\vec{V}$ by a scalar c gives a vector whose magnitude is c times greater and in the same direction as $\vec{V}$ (or opposite direction if c is negative).

EXERCISE C What does the "incorrect" vector in Fig. 3–6c represent? (a) $\vec{V}_2 - \vec{V}_1$, (b) $\vec{V}_1 - \vec{V}_2$, (c) something else (specify).

3–4 Adding Vectors by Components

Adding vectors graphically using a ruler and protractor is often not sufficiently accurate and is not useful for vectors in three dimensions. We discuss now a more powerful and precise method for adding vectors. But do not forget graphical methods—they are useful for visualizing, for checking your math, and thus for getting the correct result.

Consider first a vector $\vec{V}$ that lies in a particular plane. It can be expressed as the sum of two other vectors, called the **components** of the original vector. The components are usually chosen to be along two perpendicular directions, such as the x and y axes. The process of finding the components is known as **resolving the vector into its components**. An example is shown in Fig. 3–10; the vector $\vec{V}$ could be a displacement vector that points at an angle $\theta = 30°$ north of east, where we have chosen the positive x axis to be to the east and the positive y axis north. This vector $\vec{V}$ is resolved into its x and y components by drawing dashed lines out from the tip (A) of the vector (lines AB and AC) making them perpendicular to the x and y axes. Then the lines OB and OC represent the x and y components of $\vec{V}$, respectively, as shown in Fig. 3–10b. These *vector components* are written $\vec{V}_x$ and $\vec{V}_y$. We generally show vector components as arrows, like vectors, but dashed. The *scalar components*, V_x and V_y, are the magnitudes of the vector components, with units, accompanied by a positive or negative sign depending on whether they point along the positive or negative x or y axis. As can be seen in Fig. 3–10, $\vec{V}_x + \vec{V}_y = \vec{V}$ by the parallelogram method of adding vectors.

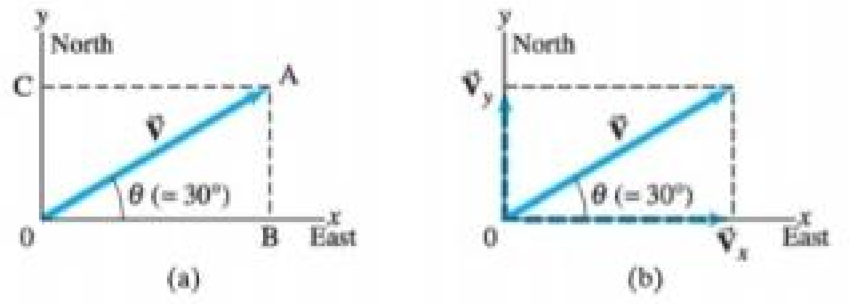
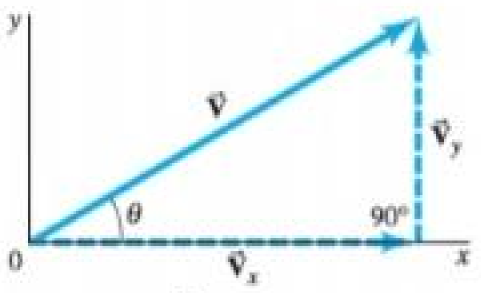


FIGURE 3–10 Resolving a vector $\vec{V}$ into its components along an arbitrarily chosen set of x and y axes. The components, once found, themselves represent the vector. That is, the components contain as much information as the vector itself.



$\sin\theta = \frac{V_y}{V}$

$\cos\theta = \frac{V_x}{V}$

$\tan\theta = \frac{V_y}{V_x}$

$V^2 = V_x^2 + V_y^2$

FIGURE 3–11 Finding the components of a vector using trigonometric functions.

Space is made up of three dimensions, and sometimes it is necessary to resolve a vector into components along three mutually perpendicular directions. In rectangular coordinates the components are $\vec{V}_x$, $\vec{V}_y$, and $\vec{V}_z$. Resolution of a vector in three dimensions is merely an extension of the above technique.

The use of trigonometric functions for finding the components of a vector is illustrated in Fig. 3–11, where a vector and its two components are thought of as making up a right triangle. (See also Appendix A for other details on trigonometric functions and identities.) We then see that the sine, cosine, and tangent are as given in Fig. 3–11. If we multiply the definition of $\sin\theta = V_y/V$ by V on both sides, we get

$$V_y = V\sin\theta. \tag{3–2a}$$

Similarly, from the definition of $\cos\theta$, we obtain

$$V_x = V\cos\theta. \tag{3–2b}$$

Note that θ is chosen (by convention) to be the angle that the vector makes with the positive x axis, measured positive counterclockwise.

The components of a given vector will be different for different choices of coordinate axes. It is therefore crucial to specify the choice of coordinate system when giving the components.

There are two ways to specify a vector in a given coordinate system:

1. We can give its components, V_x and V_y.
2. We can give its magnitude V and the angle θ it makes with the positive x axis.

We can shift from one description to the other using Eqs. 3–2, and, for the reverse, by using the theorem of Pythagoras[†] and the definition of tangent:

$$V = \sqrt{V_x^2 + V_y^2} \tag{3–3a}$$

$$\tan\theta = \frac{V_y}{V_x} \tag{3–3b}$$

as can be seen in Fig. 3–11.

We can now discuss how to add vectors using components. The first step is to resolve each vector into its components. Next we can see, using Fig. 3–12, that the addition of any two vectors $\vec{V}_1$ and $\vec{V}_2$ to give a resultant, $\vec{V} = \vec{V}_1 + \vec{V}_2$, implies that

$$\begin{aligned} V_x &= V_{1x} + V_{2x} \\ V_y &= V_{1y} + V_{2y}. \end{aligned} \tag{3–4}$$

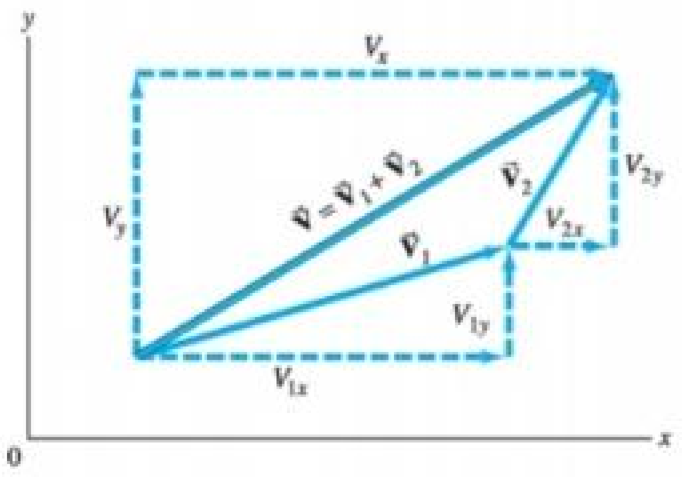
That is, the sum of the x components equals the x component of the resultant, and the sum of the y components equals the y component of the resultant, as can be verified by a careful examination of Fig. 3–12. Note that we do *not* add x components to y components.

[†]In three dimensions, the theorem of Pythagoras becomes $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$, where V_z is the component along the third, or z, axis.

FIGURE 3–12 The components of $\vec{V} = \vec{V}_1 + \vec{V}_2$ are

$V_x = V_{1x} + V_{2x}$

$V_y = V_{1y} + V_{2y}$.



If the magnitude and direction of the resultant vector are desired, they can be obtained using Eqs. 3–3.

The components of a given vector depend on the choice of coordinate axes. You can often reduce the work involved in adding vectors by a good choice of axes—for example, by choosing one of the axes to be in the same direction as one of the vectors. Then that vector will have only one nonzero component.

EXAMPLE 3–2 Mail carrier's displacement. A rural mail carrier leaves the post office and drives 22.0 km in a northerly direction. She then drives in a direction 60.0° south of east for 47.0 km (Fig. 3–13a). What is her displacement from the post office?

APPROACH We choose the positive x axis to be east and the positive y axis to be north, since those are the compass directions used on most maps. The origin of the xy coordinate system is at the post office. We resolve each vector into its x and y components. We add the x components together, and then the y components together, giving us the x and y components of the resultant.

SOLUTION Resolve each displacement vector into its components, as shown in Fig. 3–13b. Since $\vec{\mathbf{D}}_1$ has magnitude 22.0 km and points north, it has only a y component:

$$D_{1x} = 0, \quad D_{1y} = 22.0\text{ km}.$$

$\vec{\mathbf{D}}_2$ has both x and y components:

$$D_{2x} = +(47.0\text{ km})(\cos 60°) = +(47.0\text{ km})(0.500) = +23.5\text{ km}$$
$$D_{2y} = -(47.0\text{ km})(\sin 60°) = -(47.0\text{ km})(0.866) = -40.7\text{ km}.$$

Notice that D_{2y} is negative because this vector component points along the negative y axis. The resultant vector, $\vec{\mathbf{D}}$, has components:

$$D_x = D_{1x} + D_{2x} = 0\text{ km} + 23.5\text{ km} = +23.5\text{ km}$$
$$D_y = D_{1y} + D_{2y} = 22.0\text{ km} + (-40.7\text{ km}) = -18.7\text{ km}.$$

This specifies the resultant vector completely:

$$D_x = 23.5\text{ km}, \quad D_y = -18.7\text{ km}.$$

We can also specify the resultant vector by giving its magnitude and angle using Eqs. 3–3:

$$D = \sqrt{D_x^2 + D_y^2} = \sqrt{(23.5\text{ km})^2 + (-18.7\text{ km})^2} = 30.0\text{ km}$$
$$\tan\theta = \frac{D_y}{D_x} = \frac{-18.7\text{ km}}{23.5\text{ km}} = -0.796.$$

A calculator with an INV TAN, an ARC TAN, or a TAN^{-1} key gives $\theta = \tan^{-1}(-0.796) = -38.5°$. The negative sign means $\theta = 38.5°$ below the x axis, Fig. 3–13c. So, the resultant displacement is 30.0 km directed at 38.5° in a southeasterly direction.

NOTE Always be attentive about the quadrant in which the resultant vector lies. An electronic calculator does not fully give this information, but a good diagram does.

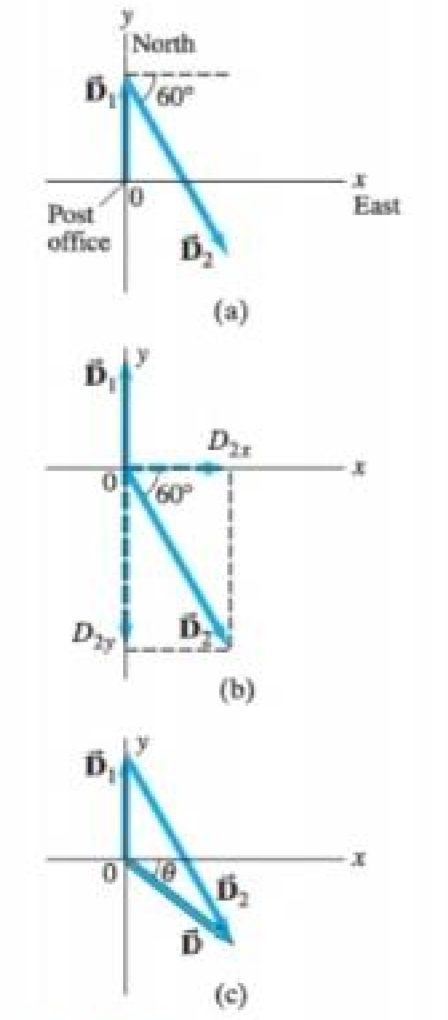


FIGURE 3–13 Example 3–2. (a) The two displacement vectors, $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$. (b) $\vec{\mathbf{D}}_2$ is resolved into its components. (c) $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$ are added graphically to obtain the resultant $\vec{\mathbf{D}}$. The component method of adding the vectors is explained in the Example.

The signs of trigonometric functions depend on which "quadrant" the angle falls in: for example, the tangent is positive in the first and third quadrants (from 0° to 90°, and 180° to 270°), but negative in the second and fourth quadrants; see Appendix A. The best way to keep track of angles, and to check any vector result, is always to draw a vector diagram. A vector diagram gives you something tangible to look at when analyzing a problem, and provides a check on the results.

PROBLEM SOLVING
Identify the correct quadrant by drawing a careful diagram

The following Problem Solving Strategy should not be considered a prescription. Rather it is a summary of things to do to get you thinking and involved in the problem at hand.

PROBLEM SOLVING

Adding Vectors

Here is a brief summary of how to add two or more vectors using components:

1. **Draw a diagram,** adding the vectors graphically by either the parallelogram or tail-to-tip method.
2. **Choose** x and y **axes.** Choose them in a way, if possible, that will make your work easier. (For example, choose one axis along the direction of one of the vectors so that vector will have only one component.)
3. **Resolve** each vector into its x and y **components,** showing each component along its appropriate (x or y) axis as a (dashed) arrow.
4. **Calculate each component** (when not given) using sines and cosines. If θ_1 is the angle that vector $\vec{\mathbf{V}}_1$ makes with the positive x axis, then:
$$V_{1x} = V_1\cos\theta_1, \quad V_{1y} = V_1\sin\theta_1.$$
Pay careful attention to **signs:** any component that points along the negative x or y axis gets a minus sign.
5. **Add the** x **components** together to get the x component of the resultant. Ditto for y:
$$V_x = V_{1x} + V_{2x} + \text{any others}$$
$$V_y = V_{1y} + V_{2y} + \text{any others}.$$
This is the answer: the components of the resultant vector. Check signs to see if they fit the quadrant shown in your diagram (point 1 above).
6. If you want to know the **magnitude and direction** of the resultant vector, use Eqs. 3–3:
$$V = \sqrt{V_x^2 + V_y^2}, \quad \tan\theta = \frac{V_y}{V_x}.$$
The vector diagram you already drew helps to obtain the correct position (quadrant) of the angle θ.

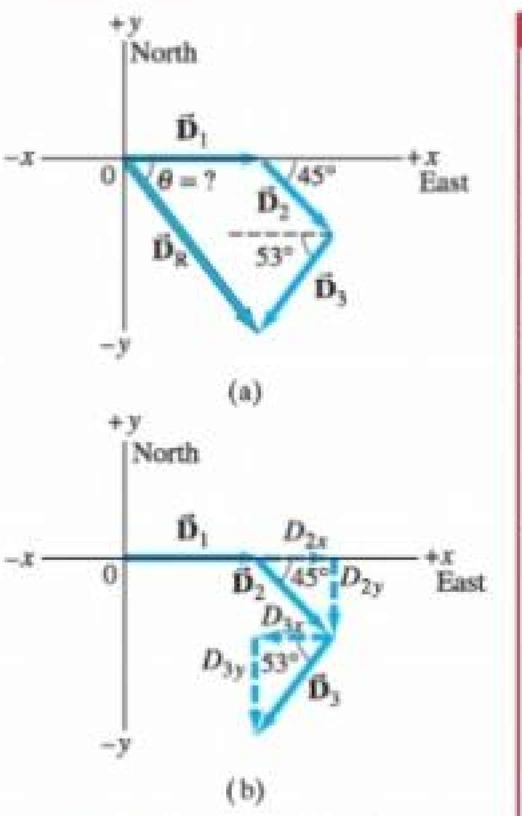


FIGURE 3–14 Example 3–3.

Vector	Components x (km)	y (km)
$\vec{\mathbf{D}}_1$	620	0
$\vec{\mathbf{D}}_2$	311	−311
$\vec{\mathbf{D}}_3$	−331	−439
$\vec{\mathbf{D}}_R$	600	−750

EXAMPLE 3–3 Three short trips. An airplane trip involves three legs, with two stopovers, as shown in Fig. 3–14a. The first leg is due east for 620 km; the second leg is southeast (45°) for 440 km; and the third leg is at 53° south of west, for 550 km, as shown. What is the plane's total displacement?

APPROACH We follow the steps in the Problem Solving Strategy above.

SOLUTION

1. **Draw a diagram** such as Fig. 3–14a, where $\vec{\mathbf{D}}_1$, $\vec{\mathbf{D}}_2$, and $\vec{\mathbf{D}}_3$ represent the three legs of the trip, and $\vec{\mathbf{D}}_R$ is the plane's total displacement.
2. **Choose axes:** Axes are also shown in Fig. 3–14a: x is east, y north.
3. **Resolve components:** It is imperative to draw a good diagram. The components are drawn in Fig. 3–14b. Instead of drawing all the vectors starting from a common origin, as we did in Fig. 3–13b, here we draw them "tail-to-tip" style, which is just as valid and may make it easier to see.
4. **Calculate the components:**
$$\vec{\mathbf{D}}_1: D_{1x} = +D_1\cos 0° = D_1 = 620\text{ km}$$
$$D_{1y} = +D_1\sin 0° = 0\text{ km}$$
$$\vec{\mathbf{D}}_2: D_{2x} = +D_2\cos 45° = +(440\text{ km})(0.707) = +311\text{ km}$$
$$D_{2y} = -D_2\sin 45° = -(440\text{ km})(0.707) = -311\text{ km}$$
$$\vec{\mathbf{D}}_3: D_{3x} = -D_3\cos 53° = -(550\text{ km})(0.602) = -331\text{ km}$$
$$D_{3y} = -D_3\sin 53° = -(550\text{ km})(0.799) = -439\text{ km}.$$
We have given a minus sign to each component that in Fig. 3–14b points in the $-x$ or $-y$ direction. The components are shown in the Table in the margin.
5. **Add the components:** We add the x components together, and we add the y components together to obtain the x and y components of the resultant:
$$D_x = D_{1x} + D_{2x} + D_{3x} = 620\text{ km} + 311\text{ km} - 331\text{ km} = 600\text{ km}$$
$$D_y = D_{1y} + D_{2y} + D_{3y} = 0\text{ km} - 311\text{ km} - 439\text{ km} = -750\text{ km}.$$
The x and y components are 600 km and −750 km, and point respectively to the east and south. This is one way to give the answer.
6. **Magnitude and direction:** We can also give the answer as
$$D_R = \sqrt{D_x^2 + D_y^2} = \sqrt{(600)^2 + (-750)^2}\text{ km} = 960\text{ km}$$
$$\tan\theta = \frac{D_y}{D_x} = \frac{-750\text{ km}}{600\text{ km}} = -1.25, \quad \text{so } \theta = -51°.$$
Thus, the total displacement has magnitude 960 km and points 51° below the x axis (south of east), as was shown in our original sketch, Fig. 3–14a.

3–5 Unit Vectors

Vectors can be conveniently written in terms of *unit vectors*. A **unit vector** is defined to have a magnitude exactly equal to one (1). It is useful to define unit vectors that point along coordinate axes, and in an x, y, z rectangular coordinate system these unit vectors are called $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. They point, respectively, along the positive x, y, and z axes as shown in Fig. 3–15. Like other vectors, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ do not have to be placed at the origin, but can be placed elsewhere as long as the direction and unit length remain unchanged. It is common to write unit vectors with a "hat": $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ (and we will do so in this book) as a reminder that each is a unit vector.

Because of the definition of multiplication of a vector by a scalar (Section 3–3), the components of a vector $\vec{\mathbf{V}}$ can be written $\vec{\mathbf{V}}_x = V_x\hat{\mathbf{i}}$, $\vec{\mathbf{V}}_y = V_y\hat{\mathbf{j}}$, and $\vec{\mathbf{V}}_z = V_z\hat{\mathbf{k}}$. Hence any vector $\vec{\mathbf{V}}$ can be written in terms of its components as

$$\vec{\mathbf{V}} = V_x\hat{\mathbf{i}} + V_y\hat{\mathbf{j}} + V_z\hat{\mathbf{k}}. \tag{3–5}$$

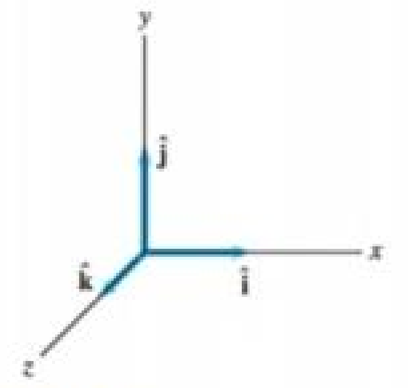


FIGURE 3–15 Unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ along the x, y, and z axes.

Unit vectors are helpful when adding vectors analytically by components. For example, Eq. 3–4 can be seen to be true by using unit vector notation for each vector (which we write for the two-dimensional case, with the extension to three dimensions being straightforward):

$$\begin{aligned}\vec{\mathbf{V}} &= (V_x)\hat{\mathbf{i}} + (V_y)\hat{\mathbf{j}} = \vec{\mathbf{V}}_1 + \vec{\mathbf{V}}_2 \\ &= (V_{1x}\hat{\mathbf{i}} + V_{1y}\hat{\mathbf{j}}) + (V_{2x}\hat{\mathbf{i}} + V_{2y}\hat{\mathbf{j}}) \\ &= (V_{1x} + V_{2x})\hat{\mathbf{i}} + (V_{1y} + V_{2y})\hat{\mathbf{j}}.\end{aligned}$$

Comparing the first line to the third line, we get Eq. 3–4.

EXAMPLE 3–4 **Using unit vectors.** Write the vectors of Example 3–2 in unit vector notation, and perform the addition.

APPROACH We use the components we found in Example 3–2,

$$D_{1x} = 0, \quad D_{1y} = 22.0\text{ km}, \text{ and } \quad D_{2x} = 23.5\text{ km}, \quad D_{2y} = -40.7\text{ km},$$

and we now write them in the form of Eq. 3–5.

SOLUTION We have

$$\begin{aligned}\vec{\mathbf{D}}_1 &= 0\hat{\mathbf{i}} + 22.0\text{ km}\,\hat{\mathbf{j}} \\ \vec{\mathbf{D}}_2 &= 23.5\text{ km}\,\hat{\mathbf{i}} - 40.7\text{ km}\,\hat{\mathbf{j}}.\end{aligned}$$

Then

$$\begin{aligned}\vec{\mathbf{D}} = \vec{\mathbf{D}}_1 + \vec{\mathbf{D}}_2 &= (0 + 23.5)\text{ km}\,\hat{\mathbf{i}} + (22.0 - 40.7)\text{ km}\,\hat{\mathbf{j}} \\ &= 23.5\text{ km}\,\hat{\mathbf{i}} - 18.7\text{ km}\,\hat{\mathbf{j}}.\end{aligned}$$

The components of the resultant displacement, $\vec{\mathbf{D}}$, are $D_x = 23.5$ km and $D_y = -18.7$ km. The magnitude of $\vec{\mathbf{D}}$ is $D = \sqrt{(23.5\text{ km})^2 + (18.7\text{ km})^2} = 30.0$ km, just as in Example 3–2.

3–6 Vector Kinematics

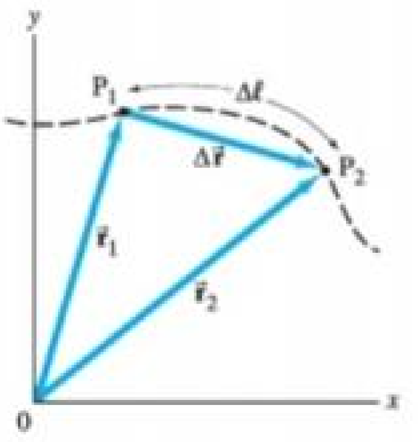
We can now extend our definitions of velocity and acceleration in a formal way to two- and three-dimensional motion. Suppose a particle follows a path in the xy plane as shown in Fig. 3–16. At time t_1, the particle is at point P_1, and at time t_2, it is at point P_2. The vector $\vec{\mathbf{r}}_1$ is the position vector of the particle at time t_1 (it represents the displacement of the particle from the origin of the coordinate system). And $\vec{\mathbf{r}}_2$ is the position vector at time t_2.

In one dimension, we defined displacement as the *change in position* of the particle. In the more general case of two or three dimensions, the **displacement vector** is defined as the vector representing change in position. We call it $\Delta\vec{\mathbf{r}}$,[†] where

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1.$$

This represents the displacement during the time interval $\Delta t = t_2 - t_1$.

FIGURE 3–16 Path of a particle in the xy plane. At time t_1 the particle is at point P_1 given by the position vector $\vec{\mathbf{r}}_1$; at t_2 the particle is at point P_2 given by the position vector $\vec{\mathbf{r}}_2$. The displacement vector for the time interval $t_2 - t_1$ is $\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$.



[†]We used $\vec{\mathbf{D}}$ for the displacement vector earlier in the Chapter for illustrating vector addition. The new notation here, $\Delta\vec{\mathbf{r}}$, emphasizes that it is the difference between two position vectors.

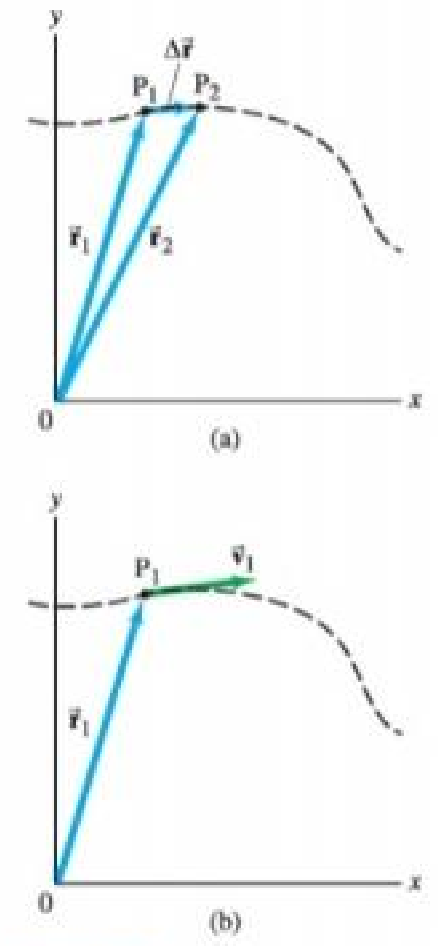
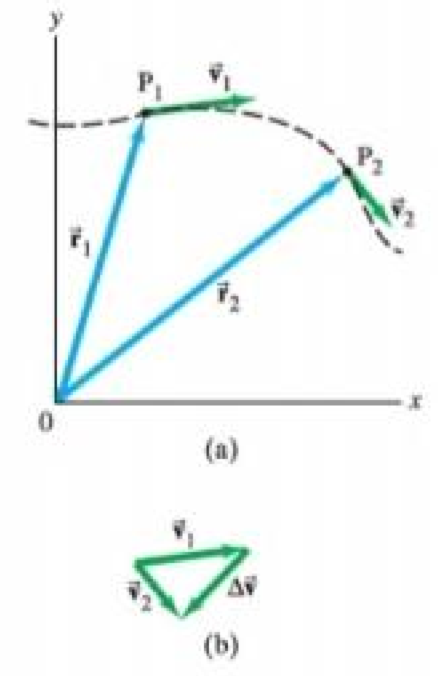


FIGURE 3–17 (a) As we take Δt and $\Delta\vec{\mathbf{r}}$ smaller and smaller [compare to Fig. 3–16] we see that the direction of $\Delta\mathbf{r}$ and of the instantaneous velocity ($\Delta\vec{\mathbf{r}}/\Delta t$, where $\Delta t \to 0$) is (b) tangent to the curve at P_1.

FIGURE 3–18 (a) Velocity vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ at instants t_1 and t_2 for a particle at points P_1 and P_2, as in Fig. 3–16. (b) The direction of the average acceleration is in the direction of $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1$.



In unit vector notation, we can write

$$\vec{\mathbf{r}}_1 = x_1\hat{\mathbf{i}} + y_1\hat{\mathbf{j}} + z_1\hat{\mathbf{k}}, \tag{3–6a}$$

where x_1, y_1, and z_1 are the coordinates of point P_1. Similarly,

$$\vec{\mathbf{r}}_2 = x_2\hat{\mathbf{i}} + y_2\hat{\mathbf{j}} + z_2\hat{\mathbf{k}}.$$

Hence

$$\Delta\vec{\mathbf{r}} = (x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}} + (z_2 - z_1)\hat{\mathbf{k}}. \tag{3–6b}$$

If the motion is along the x axis only, then $y_2 - y_1 = 0$, $z_2 - z_1 = 0$, and the magnitude of the displacement is $\Delta r = x_2 - x_1$, which is consistent with our earlier one-dimensional equation (Section 2–1). Even in one dimension, displacement is a vector, as are velocity and acceleration.

The **average velocity vector** over the time interval $\Delta t = t_2 - t_1$ is defined as

$$\text{average velocity} = \frac{\Delta\vec{\mathbf{r}}}{\Delta t}. \tag{3–7}$$

Now let us consider shorter and shorter time intervals—that is, we let Δt approach zero so that the distance between points P_2 and P_1 also approaches zero, Fig. 3–17. We define the **instantaneous velocity vector** as the limit of the average velocity as Δt approaches zero:

$$\vec{\mathbf{v}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{r}}}{\Delta t} = \frac{d\vec{\mathbf{r}}}{dt}. \tag{3–8}$$

The direction of $\vec{\mathbf{v}}$ at any moment is along the line tangent to the path at that moment (Fig. 3–17).

Note that the magnitude of the average velocity in Fig. 3–16 is not equal to the average speed, which is the actual distance traveled along the path, $\Delta\ell$, divided by Δt. In some special cases, the average speed and average velocity are equal (such as motion along a straight line in one direction), but in general they are not. However, in the limit $\Delta t \to 0$, Δr always approaches $\Delta\ell$, so the instantaneous speed *always* equals the magnitude of the instantaneous velocity at any time.

The instantaneous velocity (Eq. 3–8) is equal to the derivative of the position vector with respect to time. Equation 3–8 can be written in terms of components starting with Eq. 3–6a as:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}, \tag{3–9}$$

where $v_x = dx/dt$, $v_y = dy/dt$, $v_z = dz/dt$ are the x, y, and z components of the velocity. Note that $d\hat{\mathbf{i}}/dt = d\hat{\mathbf{j}}/dt = d\hat{\mathbf{k}}/dt = 0$ since these unit vectors are constant in both magnitude and direction.

Acceleration in two or three dimensions is treated in a similar way. The **average acceleration vector**, over a time interval $\Delta t = t_2 - t_1$ is defined as

$$\text{average acceleration} = \frac{\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{\vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1}{t_2 - t_1}, \tag{3–10}$$

where $\Delta\vec{\mathbf{v}}$ is the change in the instantaneous velocity vector during that time interval: $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}_2 - \vec{\mathbf{v}}_1$. Note that $\vec{\mathbf{v}}_2$ in many cases, such as in Fig. 3–18a, may not be in the same direction as $\vec{\mathbf{v}}_1$. Hence the average acceleration vector may be in a different direction from either $\vec{\mathbf{v}}_1$ or $\vec{\mathbf{v}}_2$ (Fig. 3–18b). Furthermore, $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_1$ may have the same magnitude but different directions, and the difference of two such vectors will not be zero. Hence acceleration can result from either a change in the magnitude of the velocity, or from a change in direction of the velocity, or from a change in both.

The **instantaneous acceleration vector** is defined as the limit of the average acceleration vector as the time interval Δt is allowed to approach zero:

$$\vec{\mathbf{a}} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\mathbf{v}}}{\Delta t} = \frac{d\vec{\mathbf{v}}}{dt}, \tag{3–11}$$

and is thus the derivative of $\vec{\mathbf{v}}$ with respect to t.

We can write $\vec{\mathbf{a}}$ using components:

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}}$$
$$= a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}, \qquad \textbf{(3–12)}$$

where $a_x = dv_x/dt$, etc. Because $v_x = dx/dt$, then $a_x = dv_x/dt = d^2x/dt^2$, as we saw in Section 2–4. Thus we can also write the acceleration as

$$\vec{\mathbf{a}} = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} + \frac{d^2z}{dt^2}\hat{\mathbf{k}}. \qquad \textbf{(3–12c)}$$

The instantaneous acceleration will be nonzero not only when the magnitude of the velocity changes but also if its direction changes. For example, a person riding in a car traveling at constant speed around a curve, or a child riding on a merry-go-round, will both experience an acceleration because of a change in the direction of the velocity, even though the speed may be constant. (More on this in Chapter 5.)

In general, we will use the terms "velocity" and "acceleration" to mean the instantaneous values. If we want to discuss average values, we will use the word "average."

EXAMPLE 3–5 **Position given as a function of time.** The position of a particle as a function of time is given by

$$\vec{\mathbf{r}} = [(5.0\,\mathrm{m/s})t + (6.0\,\mathrm{m/s^2})t^2]\hat{\mathbf{i}} + [(7.0\,\mathrm{m}) - (3.0\,\mathrm{m/s^3})t^3]\hat{\mathbf{j}},$$

where r is in meters and t is in seconds. (*a*) What is the particle's displacement between $t_1 = 2.0\,\mathrm{s}$ and $t_2 = 3.0\,\mathrm{s}$? (*b*) Determine the particle's instantaneous velocity and acceleration as a function of time. (*c*) Evaluate $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$ at $t = 3.0\,\mathrm{s}$.

APPROACH For (*a*), we find $\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$, inserting $t_1 = 2.0\,\mathrm{s}$ for finding $\vec{\mathbf{r}}_1$, and $t_2 = 3.0\,\mathrm{s}$ for $\vec{\mathbf{r}}_2$. For (*b*), we take derivatives (Eqs. 3–9 and 3–11), and for (*c*) we substitute $t = 3.0\,\mathrm{s}$ into our results in (*b*).

SOLUTION (*a*) At $t_1 = 2.0\,\mathrm{s}$,

$$\vec{\mathbf{r}}_1 = [(5.0\,\mathrm{m/s})(2.0\,\mathrm{s}) + (6.0\,\mathrm{m/s^2})(2.0\,\mathrm{s})^2]\hat{\mathbf{i}} + [(7.0\,\mathrm{m}) - (3.0\,\mathrm{m/s^3})(2.0\,\mathrm{s})^3]\hat{\mathbf{j}}$$
$$= (34\,\mathrm{m})\hat{\mathbf{i}} - (17\,\mathrm{m})\hat{\mathbf{j}}.$$

Similarly, at $t_2 = 3.0\,\mathrm{s}$,

$$\vec{\mathbf{r}}_2 = (15\,\mathrm{m} + 54\,\mathrm{m})\hat{\mathbf{i}} + (7.0\,\mathrm{m} - 81\,\mathrm{m})\hat{\mathbf{j}} = (69\,\mathrm{m})\hat{\mathbf{i}} - (74\,\mathrm{m})\hat{\mathbf{j}}.$$

Thus

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 = (69\,\mathrm{m} - 34\,\mathrm{m})\hat{\mathbf{i}} + (-74\,\mathrm{m} + 17\,\mathrm{m})\hat{\mathbf{j}} = (35\,\mathrm{m})\hat{\mathbf{i}} - (57\,\mathrm{m})\hat{\mathbf{j}}.$$

That is, $\Delta x = 35\,\mathrm{m}$, and $\Delta y = -57\,\mathrm{m}$.

(*b*) To find velocity, we take the derivative of the given $\vec{\mathbf{r}}$ with respect to time, noting (Appendix B–2) that $d(t^2)/dt = 2t$, and $d(t^3)/dt = 3t^2$:

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = [5.0\,\mathrm{m/s} + (12\,\mathrm{m/s^2})t]\hat{\mathbf{i}} + [0 - (9.0\,\mathrm{m/s^3})t^2]\hat{\mathbf{j}}.$$

The acceleration is (keeping only two significant figures):

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = (12\,\mathrm{m/s^2})\hat{\mathbf{i}} - (18\,\mathrm{m/s^3})t\hat{\mathbf{j}}.$$

Thus $a_x = 12\,\mathrm{m/s^2}$ is constant; but $a_y = -(18\,\mathrm{m/s^3})t$ depends linearly on time, increasing in magnitude with time in the negative y direction.

(*c*) We substitute $t = 3.0\,\mathrm{s}$ into the equations we just derived for $\vec{\mathbf{v}}$ and $\vec{\mathbf{a}}$:

$$\vec{\mathbf{v}} = (5.0\,\mathrm{m/s} + 36\,\mathrm{m/s})\hat{\mathbf{i}} - (81\,\mathrm{m/s})\hat{\mathbf{j}} = (41\,\mathrm{m/s})\hat{\mathbf{i}} - (81\,\mathrm{m/s})\hat{\mathbf{j}}$$
$$\vec{\mathbf{a}} = (12\,\mathrm{m/s^2})\hat{\mathbf{i}} - (54\,\mathrm{m/s^2})\hat{\mathbf{j}}.$$

Their magnitudes at $t = 3.0\,\mathrm{s}$ are $v = \sqrt{(41\,\mathrm{m/s})^2 + (81\,\mathrm{m/s})^2} = 91\,\mathrm{m/s}$, and $a = \sqrt{(12\,\mathrm{m/s^2})^2 + (54\,\mathrm{m/s^2})^2} = 55\,\mathrm{m/s^2}$.

Constant Acceleration

In Chapter 2 we studied the important case of one-dimensional motion for which the acceleration is constant. In two or three dimensions, if the acceleration vector, $\vec{\mathbf{a}}$, is constant in magnitude and direction, then a_x = constant, a_y = constant, a_z = constant. The average acceleration in this case is equal to the instantaneous acceleration at any moment. The equations we derived in Chapter 2 for one dimension, Eqs. 2–12a, b, and c, apply separately to each perpendicular component of two- or three-dimensional motion. In two dimensions we let $\vec{\mathbf{v}}_0 = v_{x0}\hat{\mathbf{i}} + v_{y0}\hat{\mathbf{j}}$ be the initial velocity, and we apply Eqs. 3–6a, 3–9, and 3–12b for the position vector, $\vec{\mathbf{r}}$, velocity, $\vec{\mathbf{v}}$, and acceleration, $\vec{\mathbf{a}}$. We can then write Eqs. 2–12a, b, and c, for two dimensions as shown in Table 3–1.

TABLE 3–1 Kinematic Equations for Constant Acceleration in 2 Dimensions

x Component (horizontal)		y Component (vertical)
$v_x = v_{x0} + a_x t$	(Eq. 2–12a)	$v_y = v_{y0} + a_y t$
$x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2$	(Eq. 2–12b)	$y = y_0 + v_{y0}t + \frac{1}{2}a_y t^2$
$v_x^2 = v_{x0}^2 + 2a_x(x - x_0)$	(Eq. 2–12c)	$v_y^2 = v_{y0}^2 + 2a_y(y - y_0)$

The first two of the equations in Table 3–1 can be written more formally in vector notation.

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}_0 + \vec{\mathbf{a}}t \qquad [\vec{\mathbf{a}} = \text{constant}] \quad \textbf{(3–13a)}$$
$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + \vec{\mathbf{v}}_0 t + \tfrac{1}{2}\vec{\mathbf{a}}t^2. \qquad [\vec{\mathbf{a}} = \text{constant}] \quad \textbf{(3–13b)}$$

Here, $\vec{\mathbf{r}}$ is the position vector at any time, and $\vec{\mathbf{r}}_0$ is the position vector at $t = 0$. These equations are the vector equivalent of Eqs. 2–12a and b. In practical situations, we usually use the component form given in Table 3–1.

3–7 Projectile Motion

In Chapter 2, we studied one-dimensional motion of an object in terms of displacement, velocity, and acceleration, including purely vertical motion of a falling object undergoing acceleration due to gravity. Now we examine the more general translational motion of objects moving through the air in two dimensions near the Earth's surface, such as a golf ball, a thrown or batted baseball, kicked footballs, and speeding bullets. These are all examples of **projectile motion** (see Fig. 3–19), which we can describe as taking place in two dimensions.

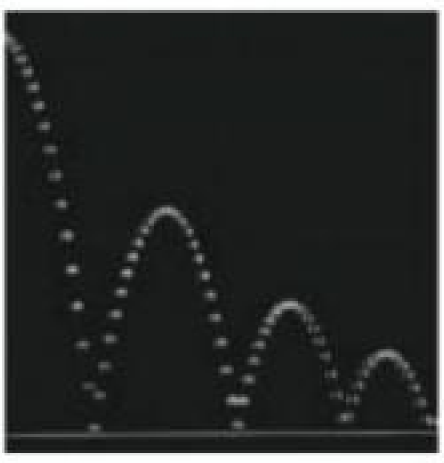


FIGURE 3–19 This strobe photograph of a ball making a series of bounces shows the characteristic "parabolic" path of projectile motion.

Although air resistance is often important, in many cases its effect can be ignored, and we will ignore it in the following analysis. We will not be concerned now with the process by which the object is thrown or projected. We consider only its motion *after* it has been projected, and *before* it lands or is caught—that is, we analyze our projected object only when it is moving freely through the air under the action of gravity alone. Then the acceleration of the object is that due to gravity, which acts downward with magnitude $g = 9.80\,\mathrm{m/s^2}$, and we assume it is constant.[†]

Galileo was the first to describe projectile motion accurately. He showed that it could be understood by analyzing the horizontal and vertical components of the motion separately. For convenience, we assume that the motion begins at time $t = 0$ at the origin of an xy coordinate system (so $x_0 = y_0 = 0$).

Let us look at a (tiny) ball rolling off the end of a horizontal table with an initial velocity in the horizontal (x) direction, v_{x0}. See Fig. 3–20, where an object falling vertically is also shown for comparison. The velocity vector $\vec{\mathbf{v}}$ at each instant points in the direction of the ball's motion at that instant and is always tangent to the path. Following Galileo's ideas, we treat the horizontal and vertical components of the velocity, v_x and v_y, separately, and we can apply the kinematic equations (Eqs. 2–12a through 2–12c) to the x and y components of the motion.

First we examine the vertical (y) component of the motion. At the instant the ball leaves the table's top ($t = 0$), it has only an x component of velocity. Once the

[†]This restricts us to objects whose distance traveled and maximum height above the Earth are small compared to the Earth's radius (6400 km).

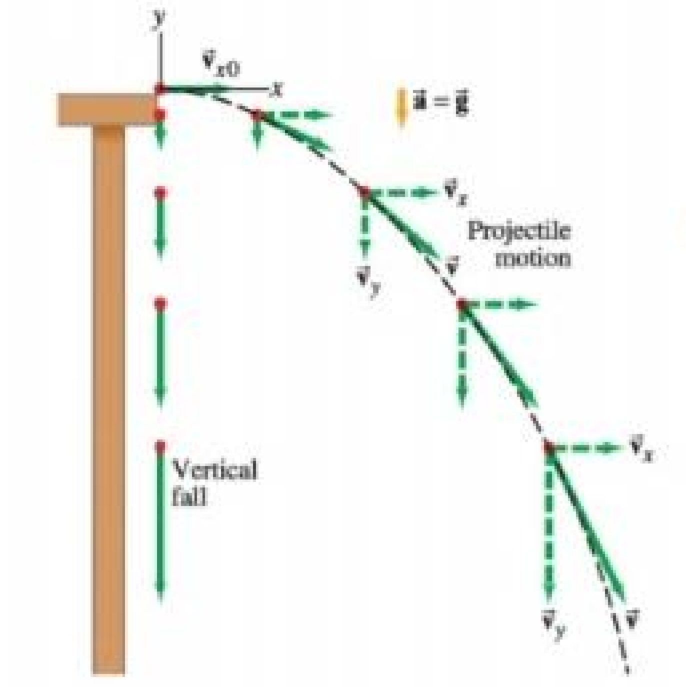


FIGURE 3–20 Projectile motion of a small ball projected horizontally. The dashed black line represents the path of the object. The velocity vector $\vec{v}$ at each point is in the direction of motion and thus is tangent to the path. The velocity vectors are green arrows, and velocity components are dashed. (A vertically falling object starting at the same point is shown at the left for comparison; v_y is the same for the falling object and the projectile.)

ball leaves the table (at $t = 0$), it experiences a vertically downward acceleration g, the acceleration due to gravity. Thus v_y is initially zero ($v_{y0} = 0$) but increases continually in the downward direction (until the ball hits the ground). Let us take y to be positive upward. Then $a_y = -g$, and from Eq. 2–12a we can write $v_y = -gt$ since we set $v_{y0} = 0$. The vertical displacement is given by $y = -\frac{1}{2}gt^2$.

In the horizontal direction, on the other hand, the acceleration is zero (we are ignoring air resistance). With $a_x = 0$, the horizontal component of velocity, v_x, remains constant, equal to its initial value, v_{x0}, and thus has the same magnitude at each point on the path. The horizontal displacement is then given by $x = v_{x0}t$. The two vector components, $\vec{v}_x$ and $\vec{v}_y$, can be added vectorially at any instant to obtain the velocity $\vec{v}$ at that time (that is, for each point on the path), as shown in Fig. 3–20.

One result of this analysis, which Galileo himself predicted, is that *an object projected horizontally will reach the ground in the same time as an object dropped vertically*. This is because the vertical motions are the same in both cases, as shown in Fig. 3–20. Figure 3–21 is a multiple-exposure photograph of an experiment that confirms this.

FIGURE 3–21 Multiple-exposure photograph showing positions of two balls at equal time intervals. One ball was dropped from rest at the same time the other was projected horizontally outward. The vertical position of each ball is seen to be the same at each instant.

EXERCISE D Return to the Chapter-Opening Question, page 51, and answer it again now. Try to explain why you may have answered differently the first time.

If an object is projected at an upward angle, as in Fig. 3–22, the analysis is similar, except that now there is an initial vertical component of velocity, v_{y0}. Because of the downward acceleration of gravity, the upward component of velocity v_y gradually decreases with time until the object reaches the highest point on its path, at which point $v_y = 0$. Subsequently the object moves downward (Fig. 3–22) and v_y increases in the downward direction, as shown (that is, becoming more negative). As before, v_x remains constant.

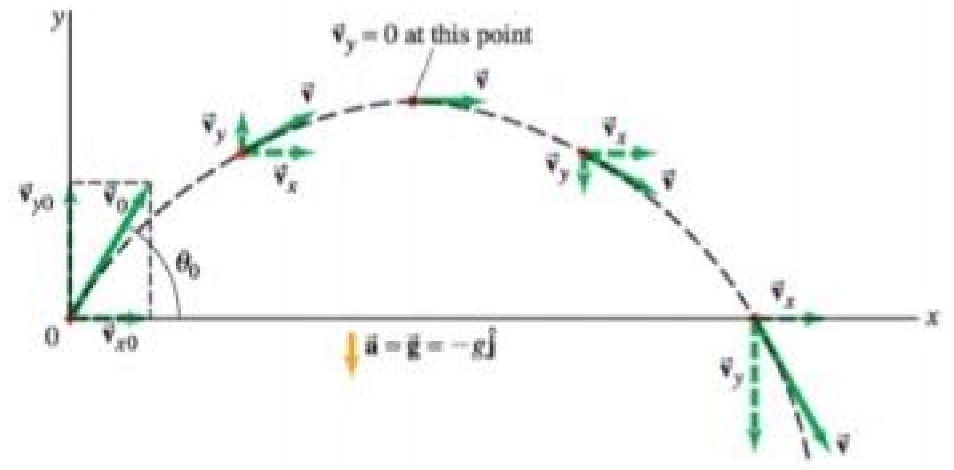


FIGURE 3–22 Path of a projectile fired with initial velocity $\vec{v}_0$ at angle θ_0 to the horizontal. Path is shown dashed in black, the velocity vectors are green arrows, and velocity components are dashed. The acceleration $\vec{a} = d\vec{v}/dt$ is downward. That is, $\vec{a} = \vec{g} = -g\hat{j}$ where $\hat{j}$ is the unit vector in the positive y direction.

3–8 Solving Problems Involving Projectile Motion

We now work through several Examples of projectile motion quantitatively.

We can simplify Eqs. 2–12 (Table 3–1) for the case of projectile motion because we can set $a_x = 0$. See Table 3–2, which assumes y is positive upward, so $a_y = -g = -9.80\,\text{m/s}^2$. Note that if θ is chosen relative to the $+x$ axis, as in Fig. 3–22, then

$$v_{x0} = v_0 \cos\theta_0,$$

$$v_{y0} = v_0 \sin\theta_0.$$

PROBLEM SOLVING
Choice of time interval

In doing problems involving projectile motion, we must consider a time interval for which our chosen object is in the air, influenced only by gravity. We do not consider the throwing (or projecting) process, nor the time after the object lands or is caught, because then other influences act on the object, and we can no longer set $\vec{a} = \vec{g}$.

TABLE 3–2 Kinematic Equations for Projectile Motion
(y positive upward; $a_x = 0$, $a_y = -g = -9.80\,\text{m/s}^2$)

Horizontal Motion ($a_x = 0$, v_x = constant)		Vertical Motion† ($a_y = -g$ = constant)
$v_x = v_{x0}$	(Eq. 2–12a)	$v_y = v_{y0} - gt$
$x = x_0 + v_{x0}t$	(Eq. 2–12b)	$y = y_0 + v_{y0}t - \frac{1}{2}gt^2$
	(Eq. 2–12c)	$v_y^2 = v_{y0}^2 - 2g(y - y_0)$

† If y is taken positive downward, the minus (−) signs in front of g become plus (+) signs.

PROBLEM SOLVING

Projectile Motion

Our approach to solving problems in Section 2–6 also applies here. Solving problems involving projectile motion can require creativity, and cannot be done just by following some rules. Certainly you must avoid just plugging numbers into equations that seem to "work."

1. As always, **read** carefully; **choose** the **object** (or objects) you are going to analyze.
2. **Draw** a careful **diagram** showing what is happening to the object.
3. **Choose** an origin and an xy **coordinate system**.
4. Decide on the **time interval**, which for projectile motion can only include motion under the effect of gravity alone, not throwing or landing. The time interval must be the same for the x and y analyses. The x and y motions are connected by the common time.
5. **Examine** the horizontal (x) and vertical (y) **motions** separately. If you are given the initial velocity, you may want to resolve it into its x and y components.
6. List the **known** and **unknown** quantities, choosing $a_x = 0$ and $a_y = -g$ or $+g$, where $g = 9.80\,\text{m/s}^2$, and using the + or − sign, depending on whether you choose y positive down or up. Remember that v_x never changes throughout the trajectory, and that $v_y = 0$ at the highest point of any trajectory that returns downward. The velocity just before landing is generally not zero.
7. Think for a minute before jumping into the equations. A little planning goes a long way. **Apply** the **relevant equations** (Table 3–2), combining equations if necessary. You may need to combine components of a vector to get magnitude and direction (Eqs. 3–3).

EXAMPLE 3–6 Driving off a cliff. A movie stunt driver on a motorcycle speeds horizontally off a 50.0-m-high cliff. How fast must the motorcycle leave the cliff top to land on level ground below, 90.0 m from the base of the cliff where the cameras are? Ignore air resistance.

APPROACH We explicitly follow the steps of the Problem Solving Strategy above.

SOLUTION

1. and 2. **Read, choose the object, and draw a diagram.** Our object is the motorcycle and driver, taken as a single unit. The diagram is shown in Fig. 3–23.

3. **Choose a coordinate system.** We choose the y direction to be positive upward, with the top of the cliff as $y_0 = 0$. The x direction is horizontal with $x_0 = 0$ at the point where the motorcycle leaves the cliff.

4. **Choose a time interval.** We choose our time interval to begin ($t = 0$) just as the motorcycle leaves the cliff top at position $x_0 = 0$, $y_0 = 0$; our time interval ends just before the motorcycle hits the ground below.

5. **Examine x and y motions.** In the horizontal (x) direction, the acceleration $a_x = 0$, so the velocity is constant. The value of x when the motorcycle reaches the ground is $x = +90.0$ m. In the vertical direction, the acceleration is the acceleration due to gravity, $a_y = -g = -9.80\text{ m/s}^2$. The value of y when the motorcycle reaches the ground is $y = -50.0$ m. The initial velocity is horizontal and is our unknown, v_{x0}; the initial vertical velocity is zero, $v_{y0} = 0$.

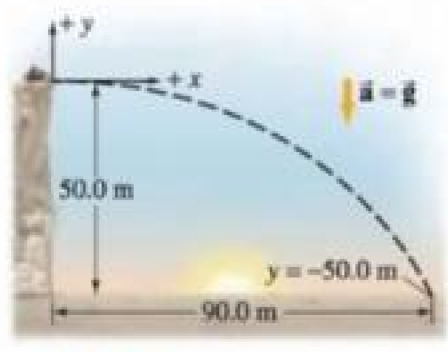


FIGURE 3–23 Example 3–6.

6. **List knowns and unknowns.** See the Table in the margin. Note that in addition to not knowing the initial horizontal velocity v_{x0} (which stays constant until landing), we also do not know the time t when the motorcycle reaches the ground.

Known	Unknown
$x_0 = y_0 = 0$	v_{x0}
$x = 90.0$ m	t
$y = -50.0$ m	
$a_x = 0$	
$a_y = -g = -9.80\text{ m/s}^2$	
$v_{y0} = 0$	

7. **Apply relevant equations.** The motorcycle maintains constant v_x as long as it is in the air. The time it stays in the air is determined by the y motion—when it hits the ground. So we first find the time using the y motion, and then use this time value in the x equations. To find out how long it takes the motorcycle to reach the ground below, we use Eq. 2–12b (Table 3–2) for the vertical (y) direction with $y_0 = 0$ and $v_{y0} = 0$:

$$y = y_0 + v_{y0}t + \tfrac{1}{2}a_y t^2$$
$$= 0 + 0 + \tfrac{1}{2}(-g)t^2$$

or

$$y = -\tfrac{1}{2}gt^2.$$

We solve for t and set $y = -50.0$ m:

$$t = \sqrt{\frac{2y}{-g}} = \sqrt{\frac{2(-50.0\text{ m})}{-9.80\text{ m/s}^2}} = 3.19\text{ s}.$$

To calculate the initial velocity, v_{x0}, we again use Eq. 2–12b, but this time for the horizontal (x) direction, with $a_x = 0$ and $x_0 = 0$:

$$x = x_0 + v_{x0}t + \tfrac{1}{2}a_x t^2$$
$$= 0 + v_{x0}t + 0$$

or

$$x = v_{x0}t.$$

Then

$$v_{x0} = \frac{x}{t} = \frac{90.0\text{ m}}{3.19\text{ s}} = 28.2\text{ m/s},$$

which is about 100 km/h (roughly 60 mi/h).

NOTE In the time interval of the projectile motion, the only acceleration is g in the negative y direction. The acceleration in the x direction is zero.

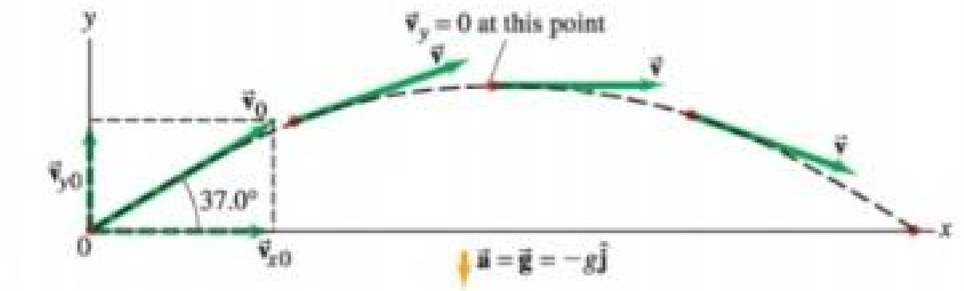


FIGURE 3–24 Example 3–7.

PHYSICS APPLIED
Sports

EXAMPLE 3–7 A kicked football. A football is kicked at an angle $\theta_0 = 37.0°$ with a velocity of 20.0 m/s, as shown in Fig. 3–24. Calculate (*a*) the maximum height, (*b*) the time of travel before the football hits the ground, (*c*) how far away it hits the ground, (*d*) the velocity vector at the maximum height, and (*e*) the acceleration vector at maximum height. Assume the ball leaves the foot at ground level, and ignore air resistance and rotation of the ball.

APPROACH This may seem difficult at first because there are so many questions. But we can deal with them one at a time. We take the y direction as positive upward, and treat the x and y motions separately. The total time in the air is again determined by the y motion. The x motion occurs at constant velocity. The y component of velocity varies, being positive (upward) initially, decreasing to zero at the highest point, and then becoming negative as the football falls.

SOLUTION We resolve the initial velocity into its components (Fig. 3–24):

$$v_{x0} = v_0 \cos 37.0° = (20.0\text{ m/s})(0.799) = 16.0\text{ m/s}$$
$$v_{y0} = v_0 \sin 37.0° = (20.0\text{ m/s})(0.602) = 12.0\text{ m/s}.$$

(*a*) We consider a time interval that begins just after the football loses contact with the foot until it reaches its maximum height. During this time interval, the acceleration is g downward. At the maximum height, the velocity is horizontal (Fig. 3–24), so $v_y = 0$; and this occurs at a time given by $v_y = v_{y0} - gt$ with $v_y = 0$ (see Eq. 2–12a in Table 3–2). Thus

$$t = \frac{v_{y0}}{g} = \frac{(12.0\text{ m/s})}{(9.80\text{ m/s}^2)} = 1.224\text{ s} \approx 1.22\text{ s}.$$

From Eq. 2–12b, with $y_0 = 0$, we have

$$y = v_{y0}t - \tfrac{1}{2}gt^2$$
$$= (12.0\text{ m/s})(1.224\text{ s}) - \tfrac{1}{2}(9.80\text{ m/s}^2)(1.224\text{ s})^2 = 7.35\text{ m}.$$

Alternatively, we could have used Eq. 2–12c, solved for y, and found

$$y = \frac{v_{y0}^2 - v_y^2}{2g} = \frac{(12.0\text{ m/s})^2 - (0\text{ m/s})^2}{2(9.80\text{ m/s}^2)} = 7.35\text{ m}.$$

The maximum height is 7.35 m.

(*b*) To find the time it takes for the ball to return to the ground, we consider a different time interval, starting at the moment the ball leaves the foot ($t = 0$, $y_0 = 0$) and ending just before the ball touches the ground ($y = 0$ again). We can use Eq. 2–12b with $y_0 = 0$ and also set $y = 0$ (ground level):

$$y = y_0 + v_{y0}t - \tfrac{1}{2}gt^2$$
$$0 = 0 + v_{y0}t - \tfrac{1}{2}gt^2.$$

This equation can be easily factored:

$$t\left(\tfrac{1}{2}gt - v_{y0}\right) = 0.$$

There are two solutions, $t = 0$ (which corresponds to the initial point, y_0), and

$$t = \frac{2v_{y0}}{g} = \frac{2(12.0\text{ m/s})}{(9.80\text{ m/s}^2)} = 2.45\text{ s},$$

which is the total travel time of the football.

NOTE The time needed for the whole trip, $t = 2v_{y0}/g = 2.45$ s, is double the

NOTE The time needed for the whole trip, $t = 2v_{y0}/g = 2.45\,\text{s}$, is double the time to reach the highest point, calculated in (*a*). That is, the time to go up equals the time to come back down to the same level (ignoring air resistance).

(*c*) The total distance traveled in the x direction is found by applying Eq. 2–12b with $x_0 = 0$, $a_x = 0$, $v_{x0} = 16.0\,\text{m/s}$:

$$x = v_{x0}t = (16.0\,\text{m/s})(2.45\,\text{s}) = 39.2\,\text{m}.$$

(*d*) At the highest point, there is no vertical component to the velocity. There is only the horizontal component (which remains constant throughout the flight), so $v = v_{x0} = v_0 \cos 37.0° = 16.0\,\text{m/s}$.

(*e*) The acceleration vector is the same at the highest point as it is throughout the flight, which is $9.80\,\text{m/s}^2$ downward.

NOTE We treated the football as if it were a particle, ignoring its rotation. We also ignored air resistance. Because air resistance is significant on a football, our results are only estimates.

EXERCISE E Two balls are thrown in the air at different angles, but each reaches the same height. Which ball remains in the air longer: the one thrown at the steeper angle or the one thrown at a shallower angle?

CONCEPTUAL EXAMPLE 3–8 Where does the apple land? A child sits upright in a wagon which is moving to the right at constant speed as shown in Fig. 3–25. The child extends her hand and throws an apple straight upward (from her own point of view, Fig. 3–25a), while the wagon continues to travel forward at constant speed. If air resistance is neglected, will the apple land (*a*) behind the wagon, (*b*) in the wagon, or (*c*) in front of the wagon?

RESPONSE The child throws the apple straight up from her own reference frame with initial velocity $\vec{v}_{y0}$ (Fig. 3–25a). But when viewed by someone on the ground, the apple also has an initial horizontal component of velocity equal to the speed of the wagon, $\vec{v}_{x0}$. Thus, to a person on the ground, the apple will follow the path of a projectile as shown in Fig. 3–25b. The apple experiences no horizontal acceleration, so $\vec{v}_{x0}$ will stay constant and equal to the speed of the wagon. As the apple follows its arc, the wagon will be directly under the apple at all times because they have the same horizontal velocity. When the apple comes down, it will drop right into the outstretched hand of the child. The answer is (*b*).

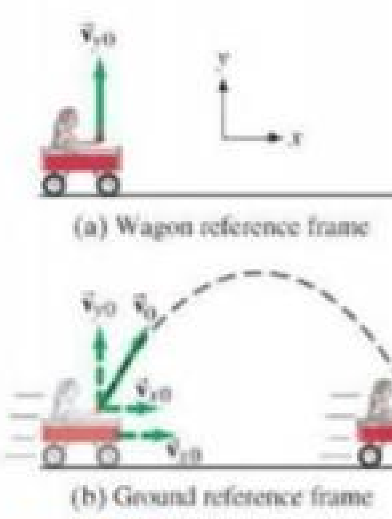


FIGURE 3–25 Example 3–8.

CONCEPTUAL EXAMPLE 3–9 The wrong strategy. A boy on a small hill aims his water-balloon slingshot horizontally, straight at a second boy hanging from a tree branch a distance d away, Fig. 3–26. At the instant the water balloon is released, the second boy lets go and falls from the tree, hoping to avoid being hit. Show that he made the wrong move. (He hadn't studied physics yet.) Ignore air resistance.

RESPONSE Both the water balloon and the boy in the tree start falling at the same instant, and in a time t they each fall the same vertical distance $y = \frac{1}{2}gt^2$, much like Fig. 3–21. In the time it takes the water balloon to travel the horizontal distance d, the balloon will have the same y position as the falling boy. Splat. If the boy had stayed in the tree, he would have avoided the humiliation.

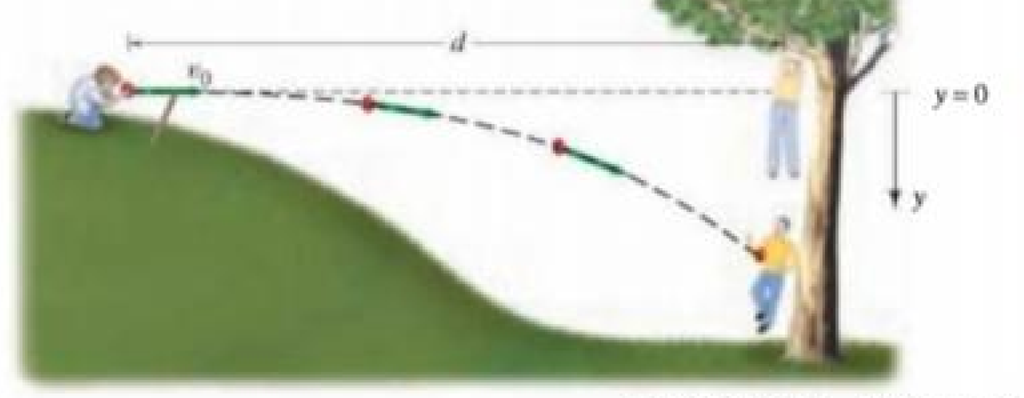


FIGURE 3–26 Example 3–9.

EXAMPLE 3–10 Level horizontal range. (*a*) Derive a formula for the horizontal range R of a projectile in terms of its initial speed v_0 and angle θ_0. The horizontal *range* is defined as the horizontal distance the projectile travels before returning to its original height (which is typically the ground); that is, $y\,(\text{final}) = y_0$. See Fig. 3–27a. (*b*) Suppose one of Napoleon's cannons had a muzzle speed, v_0, of 60.0 m/s. At what angle should it have been aimed (ignore air resistance) to strike a target 320 m away?

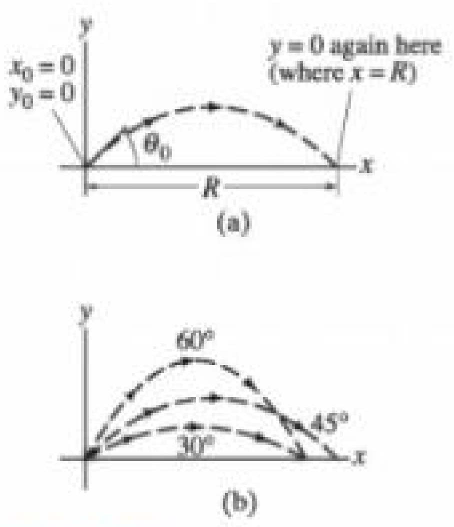


FIGURE 3–27 Example 3–10. (a) The range R of a projectile; (b) there are generally two angles θ_0 that will give the same range. Can you show that if one angle is θ_{01}, the other is $\theta_{02} = 90° - \theta_{01}$?

APPROACH The situation is the same as in Example 3–7, except we are now not given numbers in (*a*). We will algebraically manipulate equations to obtain our result.

SOLUTION (*a*) We set $x_0 = 0$ and $y_0 = 0$ at $t = 0$. After the projectile travels a horizontal distance R, it returns to the same level, $y = 0$, the final point. We choose our time interval to start ($t = 0$) just after the projectile is fired and to end when it returns to the same vertical height. To find a general expression for R, we set both $y = 0$ and $y_0 = 0$ in Eq. 2–12b for the vertical motion, and obtain

$$y = y_0 + v_{y0}t + \tfrac{1}{2}a_y t^2$$

so

$$0 = 0 + v_{y0}t - \tfrac{1}{2}gt^2.$$

We solve for t, which gives two solutions: $t = 0$ and $t = 2v_{y0}/g$. The first solution corresponds to the initial instant of projection and the second is the time when the projectile returns to $y = 0$. Then the range, R, will be equal to x at the moment t has this value, which we put into Eq. 2–12b for the *horizontal* motion ($x = v_{x0}t$, with $x_0 = 0$). Thus we have:

$$R = v_{x0}t = v_{x0}\left(\frac{2v_{y0}}{g}\right) = \frac{2v_{x0}v_{y0}}{g} = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g}, \qquad [y = y_0]$$

where we have written $v_{x0} = v_0 \cos\theta_0$ and $v_{y0} = v_0 \sin\theta_0$. This is the result we sought. It can be rewritten, using the trigonometric identity $2 \sin\theta \cos\theta = \sin 2\theta$ (Appendix A or inside the rear cover):

$$R = \frac{v_0^2 \sin 2\theta_0}{g}. \qquad [\text{only if } y\,(\text{final}) = y_0]$$

We see that the maximum range, for a given initial velocity v_0, is obtained when $\sin 2\theta$ takes on its maximum value of 1.0, which occurs for $2\theta_0 = 90°$; so

$$\theta_0 = 45° \text{ for maximum range, and } R_{max} = v_0^2/g.$$

[When air resistance is important, the range is less for a given v_0, and the maximum range is obtained at an angle smaller than 45°.]

NOTE The maximum range increases by the square of v_0, so doubling the muzzle velocity of a cannon increases its maximum range by a factor of 4.

(*b*) We put $R = 320\,\text{m}$ into the equation we just derived, and (assuming, unrealistically, no air resistance) we solve it to find

$$\sin 2\theta_0 = \frac{Rg}{v_0^2} = \frac{(320\,\text{m})(9.80\,\text{m/s}^2)}{(60.0\,\text{m/s})^2} = 0.871.$$

We want to solve for an angle θ_0 that is between 0° and 90°, which means $2\theta_0$ in this equation can be as large as 180°. Thus, $2\theta_0 = 60.6°$ is a solution, but $2\theta_0 = 180° - 60.6° = 119.4°$ is also a solution (see Appendix A–9). In general we will have two solutions (see Fig. 3–27b), which in the present case are given by

$$\theta_0 = 30.3° \text{ or } 59.7°.$$

Either angle gives the same range. Only when $\sin 2\theta_0 = 1$ (so $\theta_0 = 45°$) is there a single solution (that is, both solutions are the same).

EXERCISE F The maximum range of a projectile is found to be 100 m. If the projectile strikes the ground a distance of 82 m away, what was the angle of launch? (*a*) 35° or 55°; (*b*) 30° or 60°; (*c*) 27.5° or 72.5°; (*d*) 13.75° or 76.25°.

The level range formula derived in Example 3–10 applies only if takeoff and landing are at the same height $(y = y_0)$. Example 3–11 below considers a case where they are not equal heights $(y \neq y_0)$.

PHYSICS APPLIED
Sports

EXAMPLE 3–11 **A punt.** Suppose the football in Example 3–7 was punted and left the punter's foot at a height of 1.00 m above the ground. How far did the football travel before hitting the ground? Set $x_0 = 0,\ y_0 = 0$.

PROBLEM SOLVING
Do not use any formula unless you are sure its range of validity fits the problem; the range formula does not apply here because $y \neq y_0$

APPROACH The x and y motions are again treated separately. But we cannot use the range formula from Example 3–10 because it is valid only if $y\,(\text{final}) = y_0$, which is not the case here. Now we have $y_0 = 0$, and the football hits the ground where $y = -1.00\text{ m}$ (see Fig. 3–28). We choose our time interval to start when the ball leaves his foot $(t = 0,\ y_0 = 0,\ x_0 = 0)$ and end just before the ball hits the ground $(y = -1.00\text{ m})$. We can get x from Eq. 2–12b, $x = v_{x0}t$, since we know that $v_{x0} = 16.0\text{ m/s}$ from Example 3–7. But first we must find t, the time at which the ball hits the ground, which we obtain from the y motion.

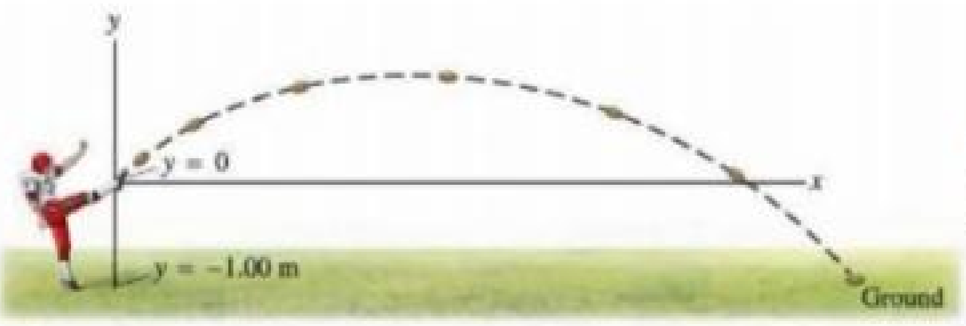


FIGURE 3–28 ... leaves the pun... the ground wh...

SOLUTION With $y = -1.00\text{ m}$ and $v_{y0} = 12.0\text{ m/s}$ (see Example 3–7), we use the equation

$$y = y_0 + v_{y0}t - \tfrac{1}{2}gt^2,$$

and obtain

$$-1.00\text{ m} = 0 + (12.0\text{ m/s})t - (4.90\text{ m/s}^2)t^2.$$

We rearrange this equation into standard form $(ax^2 + bx + c = 0)$ so we can use the quadratic formula:

$$(4.90\text{ m/s}^2)t^2 - (12.0\text{ m/s})t - (1.00\text{ m}) = 0.$$

The quadratic formula (Appendix A–1) gives

$$t = \frac{12.0\text{ m/s} \pm \sqrt{(-12.0\text{ m/s})^2 - 4(4.90\text{ m/s}^2)(-1.00\text{ m})}}{2(4.90\text{ m/s}^2)}$$

$$= 2.53\text{ s} \quad \text{or} \quad -0.081\text{ s}.$$

The second solution would correspond to a time prior to our chosen time interval that begins at the kick, so it doesn't apply. With $t = 2.53\text{ s}$ for the time at which the ball touches the ground, the horizontal distance the ball traveled is (using $v_{x0} = 16.0\text{ m/s}$ from Example 3–7):

$$x = v_{x0}t = (16.0\text{ m/s})(2.53\text{ s}) = 40.5\text{ m}.$$

Our assumption in Example 3–7 that the ball leaves the foot at ground level would result in an underestimate of about 1.3 m in the distance our punt traveled.

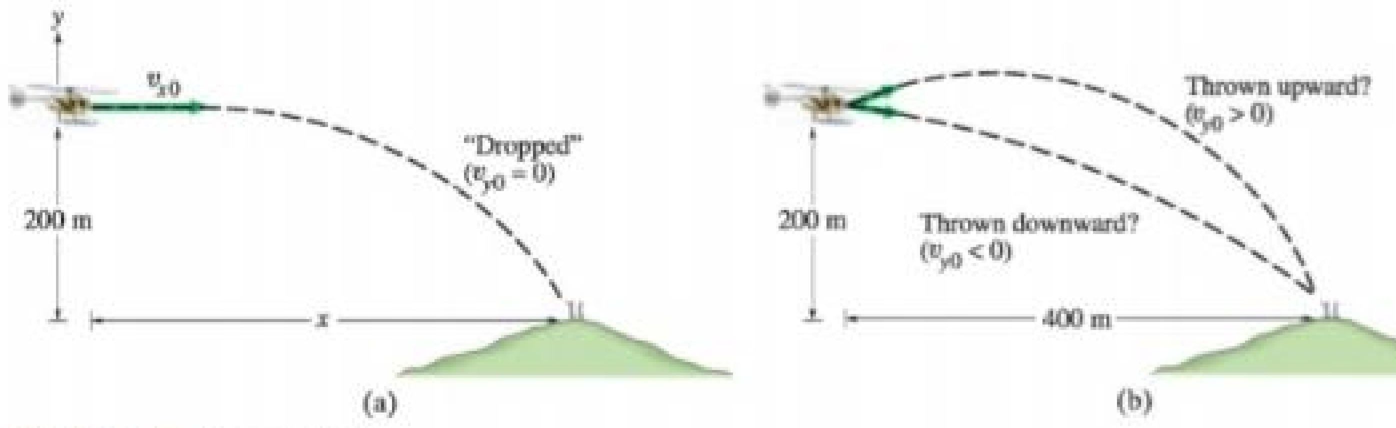


FIGURE 3–29 Example 3–12.

PHYSICS APPLIED
Reaching a target from a moving helicopter

EXAMPLE 3–12 **Rescue helicopter drops supplies.** A rescue helicopter wants to drop a package of supplies to isolated mountain climbers on a rocky ridge 200 m below. If the helicopter is traveling horizontally with a speed of 70 m/s (250 km/h), (*a*) how far in advance of the recipients (horizontal distance) must the package be dropped (Fig. 3–29a)? (*b*) Suppose, instead, that the helicopter releases the package a horizontal distance of 400 m in advance of the mountain climbers. What vertical velocity should the package be given (up or down) so that it arrives precisely at the climbers' position (Fig. 3–29b)? (*c*) With what speed does the package land in the latter case?

APPROACH We choose the origin of our xy coordinate system at the initial position of the helicopter, taking $+y$ upward, and use the kinematic equations (Table 3–2).

SOLUTION (*a*) We can find the time to reach the climbers using the vertical distance of 200 m. The package is "dropped" so initially it has the velocity of the helicopter, $v_{x0} = 70\text{ m/s}$, $v_{y0} = 0$. Then, since $y = -\tfrac{1}{2}gt^2$, we have

$$t = \sqrt{\frac{-2y}{g}} = \sqrt{\frac{-2(-200\text{ m})}{9.80\text{ m/s}^2}} = 6.39\text{ s}.$$

The horizontal motion of the falling package is at constant speed of 70 m/s. So

$$x = v_{x0}t = (70\text{ m/s})(6.39\text{ s}) = 447\text{ m} \approx 450\text{ m},$$

assuming the given numbers were good to two significant figures.

(*b*) We are given $x = 400\text{ m}$, $v_{x0} = 70\text{ m/s}$, $y = -200\text{ m}$, and we want to find v_{y0} (see Fig. 3–29b). Like most problems, this one can be approached in various ways. Instead of searching for a formula or two, let's try to reason it out in a simple way, based on what we did in part (*a*). If we know t, perhaps we can get v_{y0}. Since the horizontal motion of the package is at constant speed (once it is released we don't care what the helicopter does), we have $x = v_{x0}t$, so

$$t = \frac{x}{v_{x0}} = \frac{400\text{ m}}{70\text{ m/s}} = 5.71\text{ s}.$$

Now let's try to use the vertical motion to get v_{y0}: $y = y_0 + v_{y0}t - \tfrac{1}{2}gt^2$. Since $y_0 = 0$ and $y = -200\text{ m}$, we can solve for v_{y0}:

$$v_{y0} = \frac{y + \tfrac{1}{2}gt^2}{t} = \frac{-200\text{ m} + \tfrac{1}{2}(9.80\text{ m/s}^2)(5.71\text{ s})^2}{5.71\text{ s}} = -7.0\text{ m/s}.$$

Thus, in order to arrive at precisely the mountain climbers' position, the package must be thrown *downward* from the helicopter with a speed of 7.0 m/s.

(*c*) We want to know v of the package at $t = 5.71$ s. The components are:

$$v_x = v_{x0} = 70\text{ m/s}$$
$$v_y = v_{y0} - gt = -7.0\text{ m/s} - (9.80\text{ m/s}^2)(5.71\text{ s}) = -63\text{ m/s}.$$

So $v = \sqrt{(70\text{ m/s})^2 + (-63\text{ m/s})^2} = 94\text{ m/s}$. (Better not to release the package from such an altitude, or use a parachute.)

Projectile Motion Is Parabolic

We now show that the path followed by any projectile is a *parabola*, if we can ignore air resistance and can assume that $\vec{g}$ is constant. To do so, we need to find y as a function of x by eliminating t between the two equations for horizontal and vertical motion (Eq. 2–12b in Table 3–2), and for simplicity we set $x_0 = y_0 = 0$:

$$x = v_{x0}t$$
$$y = v_{y0}t - \tfrac{1}{2}gt^2$$

From the first equation, we have $t = x/v_{x0}$, and we substitute this into the second one to obtain

$$y = \left(\frac{v_{y0}}{v_{x0}}\right)x - \left(\frac{g}{2v_{x0}^2}\right)x^2. \qquad (3\text{–}14)$$

We see that y as a function of x has the form

$$y = Ax - Bx^2,$$

where A and B are constants for any specific projectile motion. This is the well-known equation for a parabola. See Figs. 3–19 and 3–30.

The idea that projectile motion is parabolic was, in Galileo's day, at the forefront of physics research. Today we discuss it in Chapter 3 of introductory physics!

FIGURE 3–30 Examples of projectile motion—sparks (small hot glowing pieces of metal), water, and fireworks. The parabolic path characteristic of projectile motion is affected by air resistance.

3–9 Relative Velocity

We now consider how observations made in different frames of reference are related to each other. For example, consider two trains approaching one another, each with a speed of 80 km/h with respect to the Earth. Observers on the Earth beside the train tracks will measure 80 km/hr for the speed of each of the trains. Observers on either one of the trains (a different frame of reference) will measure a speed of 160 km/h for the train approaching them.

Similarly, when one car traveling 90 km/h passes a second car traveling in the same direction at 75 km/h, the first car has a speed relative to the second car of 90 km/h − 75 km/h = 15 km/h.

When the velocities are along the same line, simple addition or subtraction is sufficient to obtain the relative velocity. But if they are not along the same line, we must make use of vector addition. We emphasize, as mentioned in Section 2–1, that when specifying a velocity, it is important to specify what the reference frame is.

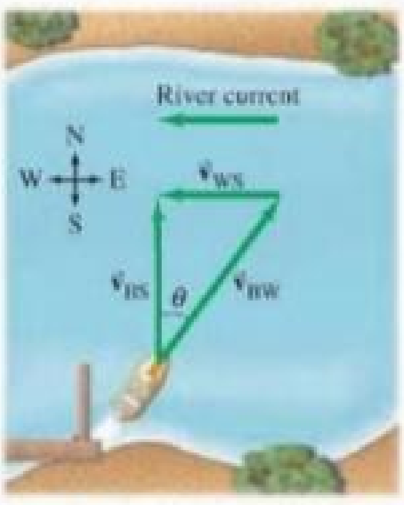


FIGURE 3–31 To move directly across the river, the boat must head upstream at an angle θ. Velocity vectors are shown as green arrows:
$\vec{v}_{BS}$ = velocity of **B**oat with respect to the **S**hore,
$\vec{v}_{BW}$ = velocity of **B**oat with respect to the **W**ater,
$\vec{v}_{WS}$ = velocity of the **W**ater with respect to the **S**hore (river current).

When determining relative velocity, it is easy to make a mistake by adding or subtracting the wrong velocities. It is important, therefore, to draw a diagram and use a careful labeling process. Each velocity is labeled by *two subscripts: the first refers to the object, the second to the reference frame in which it has this velocity*. For example, suppose a boat is to cross a river to the opposite side, as shown in Fig. 3–31. We let $\vec{v}_{BW}$ be the velocity of the **B**oat with respect to the **W**ater. (This is also what the boat's velocity would be relative to the shore if the water were still.) Similarly, $\vec{v}_{BS}$ is the velocity of the **B**oat with respect to the **S**hore, and $\vec{v}_{WS}$ is the velocity of the **W**ater with respect to the **S**hore (this is the river current). Note that $\vec{v}_{BW}$ is what the boat's motor produces (against the water), whereas $\vec{v}_{BS}$ is equal to $\vec{v}_{BW}$ plus the effect of the current, $\vec{v}_{WS}$. Therefore, the velocity of the boat relative to the shore is (see vector diagram, Fig. 3–31)

$$\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS}. \qquad (3\text{–}15)$$

By writing the subscripts using this convention, we see that the inner subscripts (the two W's) on the right-hand side of Eq. 3–15 are the same, whereas the outer subscripts on the right of Eq. 3–15 (the B and the S) are the same as the two subscripts for the sum vector on the left, $\vec{v}_{BS}$. By following this convention (first subscript for the object, second for the reference frame), you can write down the correct equation relating velocities in different reference frames.[†] Figure 3–32 gives a derivation of Eq. 3–15.

Equation 3–15 is valid in general and can be extended to three or more velocities. For example, if a fisherman on the boat walks with a velocity $\vec{v}_{FB}$ relative to the boat, his velocity relative to the shore is $\vec{v}_{FS} = \vec{v}_{FB} + \vec{v}_{BW} + \vec{v}_{WS}$. The equations involving relative velocity will be correct when adjacent inner subscripts are identical and when the outermost ones correspond exactly to the two on the velocity on the left of the equation. But this works only with plus signs (on the right), not minus signs.

It is often useful to remember that for any two objects or reference frames, A and B, the velocity of A relative to B has the same magnitude, but opposite direction, as the velocity of B relative to A:

$$\vec{v}_{BA} = -\vec{v}_{AB}. \qquad (3\text{–}16)$$

For example, if a train is traveling 100 km/h relative to the Earth in a certain direction, objects on the Earth (such as trees) appear to an observer on the train to be traveling 100 km/h in the opposite direction.

[†] We thus would know by inspection that (for example) the equation $\vec{v}_{BW} = \vec{v}_{BS} + \vec{v}_{WS}$ is wrong.

FIGURE 3–32 Derivation of relative velocity equation (Eq. 3–15), in this case for a person walking along the corridor in a train. We are looking down on the train and two reference frames are shown: xy on the Earth and $x'y'$ fixed on the train. We have:

$\vec{r}_{PT}$ = position vector of person (P) relative to train (T),
$\vec{r}_{PE}$ = position vector of person (P) relative to Earth (E),
$\vec{r}_{TE}$ = position vector of train's coordinate system (T) relative to Earth (E).

From the diagram we see that

$$\vec{r}_{PE} = \vec{r}_{PT} + \vec{r}_{TE}.$$

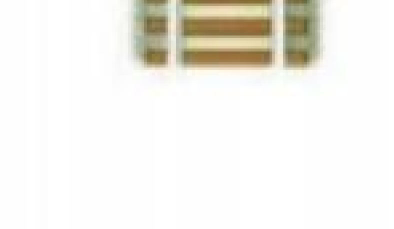
We take the derivative with respect to time to obtain

$$\frac{d}{dt}(\vec{r}_{PE}) = \frac{d}{dt}(\vec{r}_{PT}) + \frac{d}{dt}(\vec{r}_{TE}).$$

or, since $d\vec{r}/dt = \vec{v}$,

$$\vec{v}_{PE} = \vec{v}_{PT} + \vec{v}_{TE}.$$

This is the equivalent of Eq. 3–15 for the present situation (check the subscripts!).



CONCEPTUAL EXAMPLE 3–13 **Crossing a river.** A woman in a small motor boat is trying to cross a river that flows due west with a strong current. The woman starts on the south bank and is trying to reach the north bank directly north from her starting point. Should she (*a*) head due north, (*b*) head due west, (*c*) head in a northwesterly direction, (*d*) head in a northeasterly direction?

RESPONSE If the woman heads straight across the river, the current will drag the boat downstream (westward). To overcome the river's westward current, the boat must acquire an eastward component of velocity as well as a northward component. Thus the boat must (*d*) head in a northeasterly direction (see Fig. 3–33). The actual angle depends on the strength of the current and how fast the boat moves relative to the water. If the current is weak and the motor is strong, then the boat can head almost, but not quite, due north.

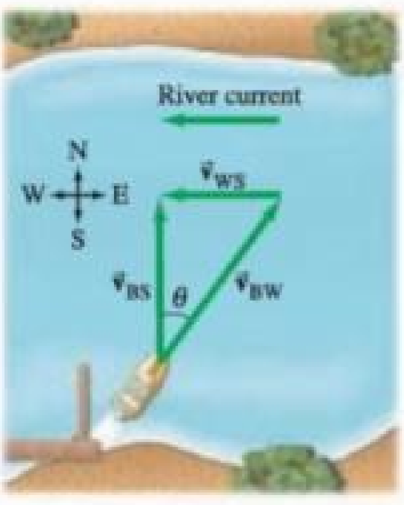


FIGURE 3–⁻⁻ ... and 3–⁾

EXAMPLE 3–14 **Heading upstream.** A boat's speed in still water is $v_{BW} = 1.85\,m/s$. If the boat is to travel directly across a river whose current has speed $v_{WS} = 1.20\,m/s$, at what upstream angle must the boat head? (See Fig. 3–33.)

APPROACH We reason as in Example 3–13, and use subscripts as in Eq. 3–15. Figure 3–33 has been drawn with $\vec{v}_{BS}$, the velocity of the **B**oat relative to the **S**hore, pointing directly across the river because this is how the boat is supposed to move. (Note that $\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS}$.) To accomplish this, the boat needs to head upstream to offset the current pulling it downstream.

SOLUTION Vector $\vec{v}_{BW}$ points upstream at an angle θ as shown. From the diagram,

$$\sin\theta = \frac{v_{WS}}{v_{BW}} = \frac{1.20\,m/s}{1.85\,m/s} = 0.6486.$$

Thus $\theta = 40.4°$, so the boat must head upstream at a 40.4° angle.

EXAMPLE 3–15 **Heading across the river.** The same boat ($v_{BW} = 1.85\,m/s$) now heads directly across the river whose current is still 1.20 m/s. (*a*) What is the velocity (magnitude and direction) of the boat relative to the shore? (*b*) If the river is 110 m wide, how long will it take to cross and how far downstream will the boat be then?

APPROACH The boat now heads directly across the river and is pulled downstream by the current, as shown in Fig. 3–34. The boat's velocity with respect to the shore, $\vec{v}_{BS}$, is the sum of its velocity with respect to the water, $\vec{v}_{BW}$, plus the velocity of the water with respect to the shore, $\vec{v}_{WS}$:

$$\vec{v}_{BS} = \vec{v}_{BW} + \vec{v}_{WS},$$

just as before.

SOLUTION (*a*) Since $\vec{v}_{BW}$ is perpendicular to $\vec{v}_{WS}$, we can get v_{BS} using the theorem of Pythagoras:

$$v_{BS} = \sqrt{v_{BW}^2 + v_{WS}^2} = \sqrt{(1.85\,m/s)^2 + (1.20\,m/s)^2} = 2.21\,m/s.$$

We can obtain the angle (note how θ is defined in the diagram) from:

$$\tan\theta = v_{WS}/v_{BW} = (1.20\,m/s)/(1.85\,m/s) = 0.6486.$$

Thus $\theta = \tan^{1}(0.6486) = 33.0°$. Note that this angle is not equal to the angle calculated in Example 3–14.

(*b*) The travel time for the boat is determined by the time it takes to cross the river. Given the river's width $D = 110\,m$, we can use the velocity component in the direction of D, $v_{BW} = D/t$. Solving for t, we get $t = 110\,m/1.85\,m/s = 59.5\,s$. The boat will have been carried downstream, in this time, a distance

$$d = v_{WS}t = (1.20\,m/s)(59.5\,s) = 71.4\,m \approx 71\,m.$$

NOTE There is no acceleration in this Example, so the motion involves only constant velocities (of the boat or of the river).

FIGURE 3–34 Example 3–15. A boat heading directly across a river whose current moves at 1.20 m/s.

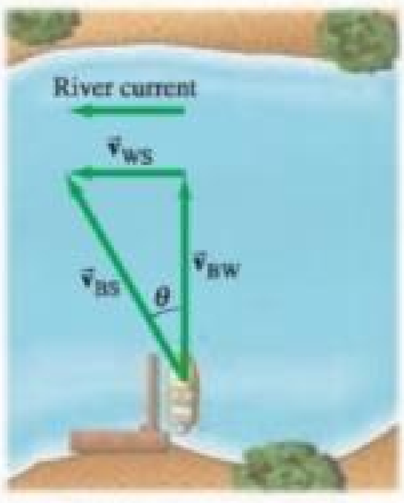
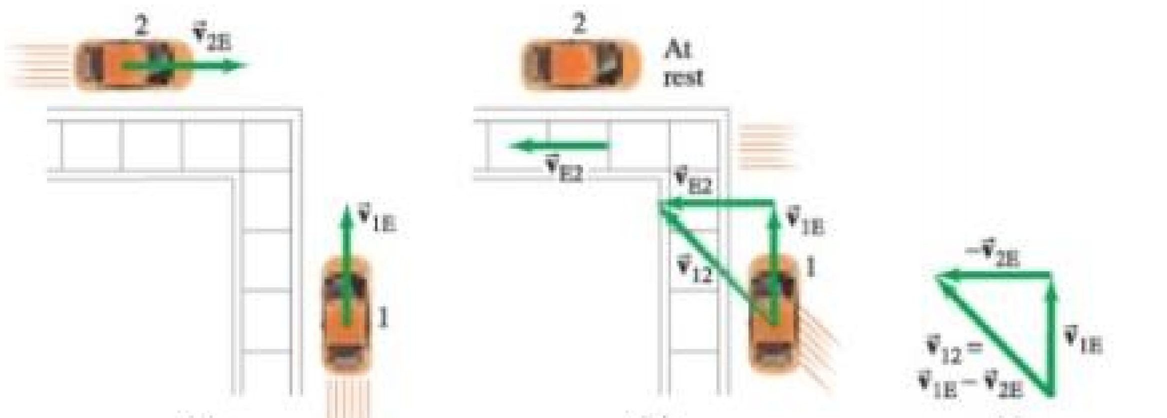


FIGURE 3–35 Example 3–16.



EXAMPLE 3–16 **Car velocities at 90°.** Two automobiles approach a street corner at right angles to each other with the same speed of 40.0 km/h (= 11.11 m/s), as shown in Fig. 3–35a. What is the relative velocity of one car with respect to the other? That is, determine the velocity of car 1 as seen by car 2.

APPROACH Figure 3–35a shows the situation in a reference frame fixed to the Earth. But we want to view the situation from a reference frame in which car 2 is at rest, and this is shown in Fig. 3–35b. In this reference frame (the world as seen by the driver of car 2), the Earth moves toward car 2 with velocity $\vec{v}_{E2}$ (speed of 40.0 km/h), which is of course equal and opposite to $\vec{v}_{2E}$, the velocity of car 2 with respect to the Earth (Eq. 3–16):

$$\vec{v}_{2E} = -\vec{v}_{E2}.$$

Then the velocity of car 1 as seen by car 2 is (see Eq. 3–15)

$$\vec{v}_{12} = \vec{v}_{1E} + \vec{v}_{E2}$$

SOLUTION Because $\vec{v}_{E2} = -\vec{v}_{2E}$, then

$$\vec{v}_{12} = \vec{v}_{1E} - \vec{v}_{2E}.$$

That is, the velocity of car 1 as seen by car 2 is the difference of their velocities, $\vec{v}_{1E} - \vec{v}_{2E}$, both measured relative to the Earth (see Fig. 3–35c). Since the magnitudes of $\vec{v}_{1E}$, $\vec{v}_{2E}$, and $\vec{v}_{E2}$ are equal (40.0 km/h = 11.11 m/s), we see (Fig. 3–35b) that $\vec{v}_{12}$ points at a 45° angle toward car 2; the speed is

$$v_{12} = \sqrt{(11.11\,m/s)^2 + (11.11\,m/s)^2} = 15.7\,m/s\ (= 56.6\,km/h).$$

Summary

A quantity that has both a magnitude and a direction is called a **vector**. A quantity that has only a magnitude is called a **scalar**.

Addition of vectors can be done graphically by placing the tail of each successive arrow (representing each vector) at the tip of the previous one. The sum, or **resultant vector**, is the arrow drawn from the tail of the first to the tip of the last. Two vectors can also be added using the parallelogram method.

Vectors can be added more accurately using the analytical method of adding their **components** along chosen axes with the aid of trigonometric functions. A vector of magnitude V making an angle θ with the x axis has components

$$V_x = V\cos\theta \qquad V_y = V\sin\theta. \qquad (3\text{–}2)$$

Given the components, we can find the magnitude and direction from

$$V = \sqrt{V_x^2 + V_y^2}, \qquad \tan\theta = \frac{V_y}{V_x}. \qquad (3\text{–}3)$$

It is often helpful to express a vector in terms of its components along chosen axes using **unit vectors**, which are vectors of unit length along the chosen coordinate axes; for Cartesian coordinates the unit vectors along the x, y, and z axes are called $\hat{i}$, $\hat{j}$, and $\hat{k}$.

The general definitions for the **instantaneous velocity**, $\vec{v}$, and **acceleration**, $\vec{a}$, of a particle (in one, two, or three dimensions) are

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad (3\text{–}8)$$

$$\vec{a} = \frac{d\vec{v}}{dt}, \qquad (3\text{–}11)$$

where $\vec{r}$ is the position vector of the particle. The kinematic equations for motion with constant acceleration can be written for each of the x, y, and z components of the motion and have the same form as for one-dimensional motion (Eqs. 2–12). Or they can be written in the more general vector form:

$$\vec{v} = \vec{v}_0 + \vec{a}t$$
$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \tfrac{1}{2}\vec{a}t^2 \qquad (3\text{–}13)$$

Projectile motion of an object moving in the air near the Earth's surface can be analyzed as two separate motions if air