8. The initial temperature in a rod of unit length is $f(x)$ throughout. There is heat transfer from both ends, $x = 0$ and $x = 1$, into a surrounding medium kept at a constant temperature zero. Show that

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-k\alpha_n^2 t}(\alpha_n \cos \alpha_n x + h \sin \alpha_n x),$$

where

$$A_n = \frac{2}{(\alpha_n^2 + 2h + h^2)} \int_0^1 f(x)(\alpha_n \cos \alpha_n x + h \sin \alpha_n x)\, dx.$$

The eigenvalues are $\lambda_n = \alpha_n^2$, $n = 1, 2, 3, \ldots$, where the α_n are the consecutive positive roots of $\tan \alpha = 2\alpha h/(\alpha^2 - h^2)$.

9. Use Method 2 of Section 12.6 to solve the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} + xe^{-2t} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = -u(1, t), \quad t > 0$$

$$u(x, 0) = 0, \quad 0 < x < 1.$$

Computer Lab Assignments

10. A vibrating cantilever beam is embedded at its left end ($x = 0$) and free at its right end ($x = 1$). See Figure 12.7.4. The transverse displacement $u(x, t)$ of the beam is determined from the boundary-value problem

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < 1, \quad t > 0$$

$$u(0, t) = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad t > 0$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=1} = 0, \quad \left.\frac{\partial^3 u}{\partial x^3}\right|_{x=1} = 0, \quad t > 0$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < 1.$$

Use a CAS to find approximations to the first two positive eigenvalues of the problem. [*Hint*: See Problems 11 and 12 in Exercises 12.4.]

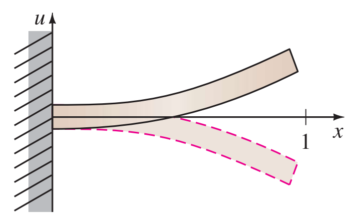


FIGURE 12.7.4 Vibrating cantilever beam in Problem 10

11. **(a)** Find an equation that defines the eigenvalues when the ends of the beam in Problem 10 are embedded at $x = 0$ and $x = 1$.

(b) Use a CAS to find approximations to the first two positive eigenvalues.

12.8 HIGHER-DIMENSIONAL PROBLEMS

REVIEW MATERIAL

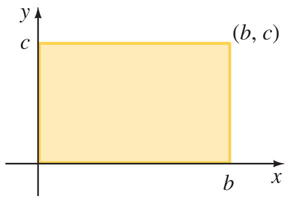
- Sections 12.3 and 12.4

INTRODUCTION Up to now we have solved boundary-value problems involving the one-dimensional heat and wave equations. In this section we show how to extend the method of separation of variables to problems involving the two-dimensional versions of these partial differential equations.

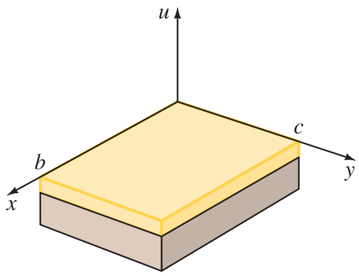
HEAT AND WAVE EQUATIONS IN TWO DIMENSIONS Suppose the rectangular region in Figure 12.8.1(a) is a thin plate in which the temperature u is a function of time t and position (x, y). Then, under suitable conditions, $u(x, y, t)$ can be shown to satisfy the **two-dimensional heat equation**

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}. \quad (1)$$

On the other hand, suppose Figure 12.8.1(b) represents a rectangular frame over which a thin flexible membrane has been stretched (a rectangular drum). If the membrane is set in motion, then its displacement u, measured from the xy-plane



(a)



(b)

FIGURE 12.8.1 (a) Rectangular plate and (b) rectangular membrane

(transverse vibrations), is also a function of t and position (x, y). When the vibrations are small, free, and undamped, $u(x, y, t)$ satisfies the **two-dimensional wave equation**

$$a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial t^2}. \tag{2}$$

To separate variables in (1) and (2), we assume a product solution of the form $u(x, y, t) = X(x)Y(y)T(t)$. We note that

$$\frac{\partial^2 u}{\partial x^2} = X''YT, \qquad \frac{\partial^2 u}{\partial y^2} = XY''T, \qquad \text{and} \qquad \frac{\partial u}{\partial t} = XYT'.$$

As we see next, with appropriate boundary conditions, boundary-value problems involving (1) and (2) lead to the concept of Fourier series in two variables.

EXAMPLE 1 Temperatures in a Plate

Find the temperature $u(x, y, t)$ in the plate shown in Figure 12.8.1(a) if the initial temperature is $f(x, y)$ throughout and if the boundaries are held at temperature zero for time $t > 0$.

SOLUTION We must solve

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t}, \qquad 0 < x < b, \quad 0 < y < c, \quad t > 0$$

subject to

$$u(0, y, t) = 0, \quad u(b, y, t) = 0, \quad 0 < y < c, \quad t > 0$$
$$u(x, 0, t) = 0, \quad u(x, c, t) = 0, \quad 0 < x < b, \quad t > 0$$
$$u(x, y, 0) = f(x, y), \quad 0 < x < b, \quad 0 < y < c.$$

Substituting $u(x, y, t) = X(x)Y(y)T(t)$, we get

$$k(X''YT + XY''T) = XYT' \qquad \text{or} \qquad \frac{X''}{X} = -\frac{Y''}{Y} + \frac{T'}{kT}. \tag{3}$$

Since the left-hand side of the last equation in (3) depends only on x and the right side depends only on y and t, we must have both sides equal to a constant $-\lambda$:

$$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{T'}{kT} = -\lambda$$

and so

$$X'' + \lambda X = 0 \tag{4}$$

$$\frac{Y''}{Y} = \frac{T'}{kT} + \lambda. \tag{5}$$

By the same reasoning, if we introduce another separation constant $-\mu$ in (5), then

$$\frac{Y''}{Y} = -\mu \qquad \text{and} \qquad \frac{T'}{kT} + \lambda = -\mu$$

yield

$$Y'' + \mu Y = 0 \qquad \text{and} \qquad T' + k(\lambda + \mu)T = 0. \tag{6}$$

Now the homogeneous boundary conditions

$$\left.\begin{array}{l} u(0, y, t) = 0, \quad u(b, y, t) = 0 \\ u(x, 0, t) = 0, \quad u(x, c, t) = 0 \end{array}\right\} \quad \text{imply that} \quad \left\{\begin{array}{l} X(0) = 0, \quad X(b) = 0 \\ Y(0) = 0, \quad Y(c) = 0. \end{array}\right.$$

Thus we have two Sturm-Liouville problems:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(b) = 0 \tag{7}$$

and

$$Y'' + \mu Y = 0, \quad Y(0) = 0, \quad Y(c) = 0. \tag{8}$$

The usual consideration of cases ($\lambda = 0$, $\lambda = \alpha^2 > 0$, $\lambda = -\alpha^2 < 0$, $\mu = 0$, and so on) leads to two independent sets of eigenvalues,

$$\lambda_m = \frac{m^2\pi^2}{b^2} \qquad \text{and} \qquad \mu_n = \frac{n^2\pi^2}{c^2}.$$

The corresponding eigenfunctions are

$$X(x) = c_2 \sin\frac{m\pi}{b}x, \quad m = 1, 2, 3 \ldots, \qquad \text{and} \qquad Y(y) = c_4 \sin\frac{n\pi}{c}y, \quad n = 1, 2, 3, \ldots. \tag{9}$$

After we substitute the known values of λ_n and μ_n in the first-order DE in (6), its general solution is found to be $T(t) = c_5 e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t}$. A product solution of the two-dimensional heat equation that satisfies the four homogeneous boundary conditions is then

$$u_{mn}(x, y, t) = A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y,$$

where A_{mn} is an arbitrary constant. Because we have two sets of eigenvalues, we are prompted to try the superposition principle in the form of a double sum

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y. \tag{10}$$

At $t = 0$ we must have

$$u(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y. \tag{11}$$

We can find the coefficients A_{mn} by multiplying the double sum (11) by the product $\sin(m\pi x/b)\sin(n\pi y/c)$ and integrating over the rectangle defined by the inequalities $0 \le x \le b$, $0 \le y \le c$. It follows that

$$A_{mn} = \frac{4}{bc}\int_0^c\int_0^b f(x, y) \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y\, dx\, dy. \tag{12}$$

Thus the solution of the BVP consists of (10) with the A_{mn} defined in (12). ■

The series (11) with coefficients (12) is called a **sine series in two variables** or a **double sine series.** We summarize next the **cosine series in two variables.**

The **double cosine series** of a function $f(x, y)$ defined over a rectangular region defined by $0 \le x \le b$, $0 \le y \le c$ is given by

$$f(x, y) = A_{00} + \sum_{m=1}^{\infty} A_{m0}\cos\frac{m\pi}{b}x + \sum_{n=1}^{\infty} A_{0n}\cos\frac{n\pi}{c}y + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos\frac{m\pi}{b}x\cos\frac{n\pi}{c}y,$$

where

$$A_{00} = \frac{1}{bc}\int_0^c\int_0^b f(x, y)\, dx\, dy$$

$$A_{m0} = \frac{2}{bc}\int_0^c\int_0^b f(x, y)\cos\frac{m\pi}{b}x\, dx\, dy$$

$$A_{0n} = \frac{2}{bc}\int_0^c\int_0^b f(x, y)\cos\frac{n\pi}{c}y\, dx\, dy$$

$$A_{mn} = \frac{4}{bc}\int_0^c\int_0^b f(x, y)\cos\frac{m\pi}{b}x\cos\frac{n\pi}{c}y\, dx\, dy.$$

For a problem leading to a double-cosine series see Problem 2 in Exercises 12.8.

EXERCISES 12.8

Answers to selected odd-numbered problems begin on page ANS-22.

In Problems 1 and 2 solve the heat equation (1) subject to the given conditions.

1. $u(0, y, t) = 0, \quad u(\pi, y, t) = 0$
$u(x, 0, t) = 0, \quad u(x, \pi, t) = 0$
$u(x, y, 0) = u_0$

2. $\left.\dfrac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\dfrac{\partial u}{\partial x}\right|_{x=1} = 0$
$\left.\dfrac{\partial u}{\partial y}\right|_{y=0} = 0, \quad \left.\dfrac{\partial u}{\partial y}\right|_{y=1} = 0$
$u(x, y, 0) = xy$

In Problems 3 and 4 solve the wave equation (2) subject to the given conditions.

3. $u(0, y, t) = 0, \quad u(\pi, y, t) = 0$
$u(x, 0, t) = 0, \quad u(x, \pi, t) = 0$
$u(x, y, 0) = xy(x - \pi)(y - \pi)$
$\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0$

4. $u(0, y, t) = 0, \quad u(b, y, t) = 0$
$u(x, 0, t) = 0, \quad u(x, c, t) = 0$
$u(x, y, 0) = f(x, y)$
$\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = g(x, y)$

The steady-state temperature $u(x, y, z)$ in the rectangular parallelepiped shown in Figure 12.8.2 satisfies Laplace's equation in three dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{13}$$

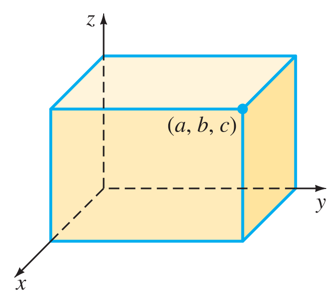


FIGURE 12.8.2 Rectangular parallelepiped in Problems 5 and 6

5. Solve Laplace's equation (13) if the top ($z = c$) of the parallelepiped is kept at temperature $f(x, y)$ and the remaining sides are kept at temperature zero.

6. Solve Laplace's equation (13) if the bottom ($z = 0$) of the parallelepiped is kept at temperature $f(x, y)$ and the remaining sides are kept at temperature zero.

CHAPTER 12 IN REVIEW

Answers to selected odd-numbered problems begin on page ANS-22.

1. Use separation of variables to find product solutions of

$$\frac{\partial^2 u}{\partial x\, \partial y} = u.$$

2. Use separation of variables to find product solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0.$$

Is it possible to choose a separation constant so that both X and Y are oscillatory functions?

3. Find a steady-state solution $\psi(x)$ of the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < \pi, \quad t > 0,$$

$$u(0, t) = u_0, \quad -\left.\frac{\partial u}{\partial x}\right|_{x=\pi} = u(\pi, t) - u_1, \quad t > 0$$

$$u(x, 0) = 0, \quad 0 < x < \pi.$$

4. Give a physical interpretation for the boundary conditions in Problem 3.

5. At $t = 0$ a string of unit length is stretched on the positive x-axis. The ends of the string $x = 0$ and $x = 1$ are secured on the x-axis for $t > 0$. Find the displacement $u(x, t)$ if the initial velocity $g(x)$ is as given in Figure 12.R.1.

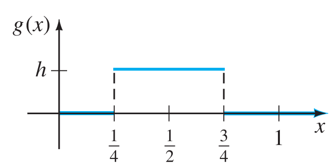


FIGURE 12.R.1 Initial velocity $g(x)$ in Problem 5

6. The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + x^2 = \frac{\partial^2 u}{\partial t^2}$$

is a form of the wave equation when an external vertical force proportional to the square of the horizontal distance from the left end is applied to the string. The string is secured at $x = 0$ one unit above the x-axis and on the x-axis at $x = 1$ for $t > 0$. Find the displacement $u(x, t)$ if the string starts from rest from the initial displacement $f(x)$.