12.3 HEAT EQUATION

REVIEW MATERIAL

- Section 12.1
- A rereading of Example 2 in Section 5.2 and Example 1 of Section 11.4 is recommended.

INTRODUCTION Consider a thin rod of length L with an initial temperature $f(x)$ throughout and whose ends are held at temperature zero for all time $t > 0$. If the rod shown in Figure 12.3.1 satisfies the assumptions given on page 438, then the temperature $u(x, t)$ in the rod is determined from the boundary-value problem

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < L, \quad t > 0 \tag{1}$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \tag{2}$$

$$u(x, 0) = f(x), \quad 0 < x < L. \tag{3}$$

In this section we shall solve this BVP.

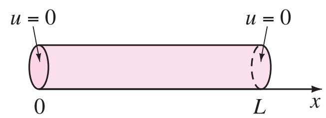


FIGURE 12.3.1 Temperatures in a rod of length L

SOLUTION OF THE BVP To start, we use the product $u(x, t) = X(x)T(t)$ to separate variables in (1). Then, if $-\lambda$ is the separation constant, the two equalities

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \tag{4}$$

lead to the two ordinary differential equations

$$X'' + \lambda X = 0 \tag{5}$$

$$T' + k\lambda T = 0. \tag{6}$$

Before solving (5), note that the boundary conditions (2) applied to $u(x, t) = X(x)T(t)$ are

$$u(0, t) = X(0)T(t) = 0 \qquad \text{and} \qquad u(L, t) = X(L)T(t) = 0.$$

Since it makes sense to expect that $T(t) \neq 0$ for all t, the foregoing equalities hold only if $X(0) = 0$ and $X(L) = 0$. These homogeneous boundary conditions together with the homogeneous DE (5) constitute a regular Sturm-Liouville problem:

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \tag{7}$$

The solution of this BVP was discussed thoroughly in Example 2 of Section 5.2. In that example we considered three possible cases for the parameter λ: zero, negative, or positive. The corresponding solutions of the DEs are, in turn, given by

$$X(x) = c_1 + c_2 x, \qquad \lambda = 0 \tag{8}$$

$$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x, \qquad \lambda = -\alpha^2 < 0 \tag{9}$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x, \qquad \lambda = \alpha^2 > 0. \tag{10}$$

When the boundary conditions $X(0) = 0$ and $X(L) = 0$ are applied to (8) and (9), these solutions yield only $X(x) = 0$, and so we would have to conclude that $u = 0$. But when $X(0) = 0$ is applied to (10), we find that $c_1 = 0$ and $X(x) = c_2 \sin \alpha x$. The second boundary condition then implies that $X(L) = c_2 \sin \alpha L = 0$. To obtain a nontrivial solution, we must have $c_2 \neq 0$ and $\sin \alpha L = 0$. The last equation is satisfied when $\alpha L = n\pi$ or $\alpha = n\pi/L$. Hence (7) possesses nontrivial solutions when

$\lambda_n = \alpha_n^2 = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$. These values of λ are the **eigenvalues** of the problem; the **eigenfunctions** are

$$X(x) = c_2 \sin\frac{n\pi}{L}x, \qquad n = 1, 2, 3, \ldots . \tag{11}$$

From (6) we have $T(t) = c_3 e^{-k(n^2\pi^2/L^2)t}$, so

$$u_n = X(x)T(t) = A_n e^{-k(n^2\pi^2/L^2)t} \sin\frac{n\pi}{L}x, \tag{12}$$

where we have replaced the constant c_2c_3 by A_n. Each of the product functions $u_n(x, t)$ given in (12) is a particular solution of the partial differential equation (1), and each $u_n(x, t)$ satisfies both boundary conditions (2) as well. However, for (12) to satisfy the initial condition (3), we would have to choose the coefficient A_n in such a manner that

$$u_n(x, 0) = f(x) = A_n \sin\frac{n\pi}{L}x. \tag{13}$$

In general, we would not expect condition (13) to be satisfied for an arbitrary but reasonable choice of f. Therefore we are forced to admit that *$u_n(x, t)$ is not a solution of the given problem*. Now by the superposition principle (Theorem 12.1.1) the function $u(x, t) = \sum_{n=1}^{\infty} u_n$ or

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin\frac{n\pi}{L}x \tag{14}$$

must also, although formally, satisfy equation (1) and the conditions in (2). Substituting $t = 0$ into (14) implies that

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x.$$

This last expression is recognized as a half-range expansion of f in a sine series. If we make the identification $A_n = b_n$, $n = 1, 2, 3, \ldots$, it follows from (5) of Section 11.3 that

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L}x\,dx. \tag{15}$$

We conclude that a solution of the boundary-value problem described in (1), (2), and (3) is given by the infinite series

$$u(x, t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x) \sin\frac{n\pi}{L}x\,dx\right)e^{-k(n^2\pi^2/L^2)t} \sin\frac{n\pi}{L}x. \tag{16}$$

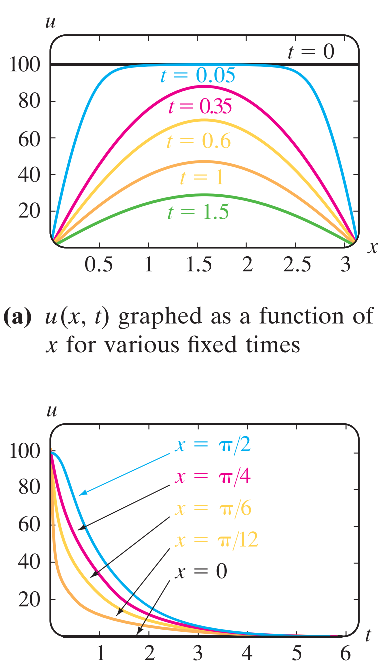
In the special case when the initial temperature is $u(x, 0) = 100$, $L = \pi$, and $k = 1$, you should verify that the coefficients (15) are given by

$$A_n = \frac{200}{\pi}\left[\frac{1-(-1)^n}{n}\right]$$

and that (16) is

$$u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]e^{-n^2t} \sin nx. \tag{17}$$

USE OF COMPUTERS Since u is a function of two variables, the graph of the solution (17) is a surface in 3-space. We could use the 3D-plot application of a computer algebra system to approximate this surface by graphing partial sums $S_n(x, t)$ over a rectangular region defined by $0 \le x \le \pi$, $0 \le t \le T$. Alternatively, with the aid of the 2D-plot application of a CAS we can plot the solution $u(x, t)$ on the x-interval $[0, \pi]$ for increasing values of time t. See Figure 12.3.2(a). In Figure 12.3.2(b) the solution $u(x, t)$ is graphed on the t-interval [0, 6] for increasing values of x ($x = 0$ is the left end and $x = \pi/2$ is the midpoint of the rod of length $L = \pi$.) Both sets of graphs verify what is apparent in (17)—namely, $u(x, t) \to 0$ as $t \to \infty$.



(a) $u(x, t)$ graphed as a function of x for various fixed times

(b) $u(x, t)$ graphed as a function of t for various fixed positions

FIGURE 12.3.2 Graphs of (17) when one variable is held fixed

EXERCISES 12.3

Answers to selected odd-numbered problems begin on page ANS-20.

In Problems 1 and 2 solve the heat equation (1) subject to the given conditions. Assume a rod of length L.

1. $u(0, t) = 0, \quad u(L, t) = 0$

$$u(x, 0) = \begin{cases} 1, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

2. $u(0, t) = 0, \quad u(L, t) = 0$

$u(x, 0) = x(L - x)$

3. Find the temperature $u(x, t)$ in a rod of length L if the initial temperature is $f(x)$ throughout and if the ends $x = 0$ and $x = L$ are insulated.

4. Solve Problem 3 if $L = 2$ and

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

5. Suppose heat is lost from the lateral surface of a thin rod of length L into a surrounding medium at temperature zero. If the linear law of heat transfer applies, then the heat equation takes on the form

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t},$$

$0 < x < L$, $t > 0$, h a constant. Find the temperature $u(x, t)$ if the initial temperature is $f(x)$ throughout and the ends $x = 0$ and $x = L$ are insulated. See Figure 12.3.3.

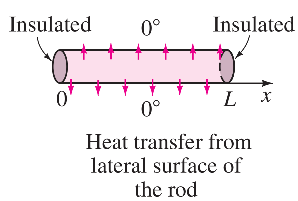


FIGURE 12.3.3 Rod losing heat in Problem 5

6. Solve Problem 5 if the ends $x = 0$ and $x = L$ are held at temperature zero.

Discussion Problems

7. Figure 12.3.2(b) shows the graphs of $u(x, t)$ for $0 \le t \le 6$ for $x = 0$, $x = \pi/12$, $x = \pi/6$, $x = \pi/4$, and $x = \pi/2$. Describe or sketch the graphs of $u(x, t)$ on the same time interval but for the fixed values $x = 3\pi/4$, $x = 5\pi/6$, $x = 11\pi/12$, and $x = \pi$.

8. Find the solution of the boundary-value problem given in (1)–(3) when $f(x) = 10 \sin(5\pi x/L)$.

Computer Lab Assignments

9. **(a)** Solve the heat equation (1) subject to

$$u(0, t) = 0, \quad u(100, t) = 0, \quad t > 0$$

$$u(x, 0) = \begin{cases} 0.8x, & 0 \le x \le 50 \\ 0.8(100 - x), & 50 < x \le 100. \end{cases}$$

(b) Use the 3D-plot application of your CAS to graph the partial sum $S_5(x, t)$ consisting of the first five nonzero terms of the solution in part (a) for $0 \le x \le 100$, $0 \le t \le 200$. Assume that $k = 1.6352$. Experiment with various three-dimensional viewing perspectives of the surface (called the **ViewPoint** option in *Mathematica*).

12.4 WAVE EQUATION

REVIEW MATERIAL

- Reread pages 439–441 of Section 12.2.

INTRODUCTION We are now in a position to solve the boundary-value problem (11) that was discussed in Section 12.2. The vertical displacement $u(x, t)$ of the vibrating string of length L shown in Figure 12.2.2(a) is determined from

$$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < x < L, \quad t > 0 \tag{1}$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \tag{2}$$

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad 0 < x < L. \tag{3}$$