left ($\frac{1}{2}f(x + at)$). Both waves travel with speed a and have the same basic shape as the initial displacement $f(x)$. The form of $u(x, t)$ given in (13) is called **d'Alembert's solution.**

In Problems 15–18 use d'Alembert's solution (13) to solve the initial-value problem in Problem 14 subject to the given initial conditions.

15. $f(x) = \sin x, \quad g(x) = 1$

16. $f(x) = \sin x, \quad g(x) = \cos x$

17. $f(x) = 0, \quad g(x) = \sin 2x$

18. $f(x) = e^{-x^2}, \quad g(x) = 0$

Computer Lab Assignments

19. **(a)** Use a CAS to plot d'Alembert's solution in Problem 18 on the interval $[-5, 5]$ at the times $t = 0$, $t = 1$, $t = 2$, $t = 3$, and $t = 4$. Superimpose the graphs on one coordinate system. Assume that $a = 1$.

(b) Use the 3D-plot application of your CAS to plot d'Alembert's solution $u(x, t)$ in Problem 18 for $-5 \le x \le 5$, $0 \le t \le 4$. Experiment with various three-dimensional viewing perspectives of this surface. Choose the perspective of the surface for which you feel the graphs in part (a) are most apparent.

20. A model for an infinitely long string that is initially held at the three points $(-1, 0)$, $(1, 0)$, and $(0, 1)$ and then simultaneously released at all three points at time $t = 0$ is given by (12) with

$$f(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \quad \text{and} \quad g(x) = 0.$$

(a) Plot the initial position of the string on the interval $[-6, 6]$.

(b) Use a CAS to plot d'Alembert's solution (13) on $[-6, 6]$ for $t = 0.2k$, $k = 0, 1, 2, \ldots, 25$. Assume that $a = 1$.

(c) Use the animation feature of your computer algebra system to make a movie of the solution. Describe the motion of the string over time.

21. An infinitely long string coinciding with the x-axis is struck at the origin with a hammer whose head is 0.2 inch in diameter. A model for the motion of the string is given by (12) with

$$f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} 1, & |x| \le 0.1 \\ 0, & |x| \ge 0.1. \end{cases}$$

(a) Use a CAS to plot d'Alembert's solution (13) on $[-6, 6]$ for $t = 0.2k$, $k = 0, 1, 2, \ldots, 25$. Assume that $a = 1$.

(b) Use the animation feature of your computer algebra system to make a movie of the solution. Describe the motion of the string over time.

22. The model of the vibrating string in Problem 7 is called the **plucked string.** The string is tied to the x-axis at $x = 0$ and $x = L$ and is held at $x = L/2$ at h units above the x-axis. See Figure 12.2.4. Starting at $t = 0$ the string is released from rest.

(a) Use a CAS to plot the partial sum $S_6(x, t)$—that is, the first six nonzero terms of your solution—for $t = 0.1k$, $k = 0, 1, 2, \ldots, 20$. Assume that $a = 1$, $h = 1$, and $L = \pi$.

(b) Use the animation feature of your computer algebra system to make a movie of the solution to Problem 7.

12.5 LAPLACE'S EQUATION

REVIEW MATERIAL

- Reread page 438 of Section 12.2 and Example 1 in Section 11.4.

INTRODUCTION Suppose we wish to find the steady-state temperature $u(x, y)$ in a rectangular plate whose vertical edges $x = 0$ and $x = a$ are insulated, as shown in Figure 12.5.1. When no heat escapes from the lateral faces of the plate, we solve the following boundary-value problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b \tag{1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0, \quad 0 < y < b \tag{2}$$

$$u(x, 0) = 0, \quad u(x, b) = f(x), \quad 0 < x < a. \tag{3}$$

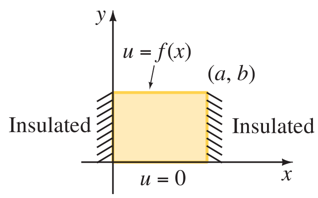


FIGURE 12.5.1 Steady-state temperatures in a rectangular plate

SOLUTION OF THE BVP With $u(x, y) = X(x)Y(y)$ separation of variables in (1) leads to

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$X'' + \lambda X = 0 \tag{4}$$

$$Y'' - \lambda Y = 0. \tag{5}$$

The three homogeneous boundary conditions in (2) and (3) translate into $X'(0) = 0$, $X'(a) = 0$, and $Y(0) = 0$. The Sturm-Liouville problem associated with the equation in (4) is then

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(a) = 0. \tag{6}$$

Examination of the cases corresponding to $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$, has already been carried out in Example 1 in Section 11.4.* Here is a brief summary of that analysis.

For $\lambda = 0$, (6) becomes

$$X'' = 0, \quad X'(0) = 0, \quad X'(a) = 0.$$

The solution of the DE is $X = c_1 + c_2 x$. The boundary conditions imply $X = c_1$. By imposing $c_1 \neq 0$, this problem possesses a nontrivial solution. For $\lambda = -\alpha^2 < 0$, (6) possesses only the trivial solution. For $\lambda = \alpha^2 > 0$, (6) becomes

$$X'' + \alpha^2 X = 0, \quad X'(0) = 0, \quad X'(a) = 0.$$

The solution of the DE in this problem is $X = c_1 \cos \alpha x + c_2 \sin \alpha x$. The boundary condition $X'(0) = 0$ implies that $c_2 = 0$, so $X = c_1 \cos \alpha x$. Differentiating this last expression and then setting $x = a$ gives $-c_1 \sin \alpha a = 0$. Since we have assumed that $\alpha > 0$, this last condition is satisfied when $\alpha a = n\pi$ or $\alpha = n\pi/a$, $n = 1, 2, \ldots$. The eigenvalues of (6) are then $\lambda_0 = 0$ and $\lambda_n = \alpha_n^2 = n^2\pi^2/a^2, n = 1, 2, \ldots$. If we correspond $\lambda_0 = 0$ with $n = 0$, the eigenfunctions of (6) are

$$X = c_1, \qquad n = 0, \qquad \text{and} \qquad X = c_1 \cos \frac{n\pi}{a} x, \qquad n = 1, 2, \ldots.$$

We now solve equation (5) subject to the single homogeneous boundary condition $Y(0) = 0$. There are two cases. For $\lambda_0 = 0$, equation (5) is simply $Y'' = 0$; therefore its solution is $Y = c_3 + c_4 y$. But $Y(0) = 0$ implies that $c_3 = 0$, so $Y = c_4 y$. For $\lambda_n = n^2\pi^2/a^2$, (5) is $Y'' - \dfrac{n^2\pi^2}{a^2} Y = 0$. Because $0 < y < b$ defines a finite interval, we use (according to the informal rule indicated on pages 135–136) the hyperbolic form of the general solution:

$$Y = c_3 \cosh (n\pi y/a) + c_4 \sinh (n\pi y/a).$$

$Y(0) = 0$ again implies that $c_3 = 0$, so we are left with $Y = c_4 \sinh (n\pi y/a)$.

Thus product solutions $u_n = X(x)Y(y)$ that satisfy the Laplace's equation (1) and the three homogeneous boundary conditions in (2) and (3) are

$$A_0 y, \quad n = 0, \qquad \text{and} \qquad A_n \sinh \frac{n\pi}{a} y \cos \frac{n\pi}{a} x, \qquad n = 1, 2, \ldots,$$

where we have rewritten $c_1 c_4$ as A_0 for $n = 0$ and as A_n for $n = 1, 2, \ldots$.

*In that example the symbols y and L play the part of X and a in the current discussion.

The superposition principle yields another solution:

$$u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \cos \frac{n\pi}{a} x. \tag{7}$$

We are now in a position to use the last boundary condition in (3). Substituting $x = b$ in (7) gives

$$u(x, b) = f(x) = A_0 b + \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi}{a} b \right) \cos \frac{n\pi}{a} x,$$

which is a half-range expansion of f in a cosine series. If we make the identifications $A_0 b = a_0/2$ and $A_n \sinh (n\pi b/a) = a_n$, $n = 1, 2, 3, \ldots$, it follows from (2) and (3) of Section 11.3 that

$$2A_0 b = \frac{2}{a} \int_0^a f(x)\, dx$$

$$A_0 = \frac{1}{ab} \int_0^a f(x)\, dx \tag{8}$$

and

$$A_n \sinh \frac{n\pi}{a} b = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi}{a} x\, dx$$

$$A_n = \frac{2}{a \sinh \frac{n\pi}{a} b} \int_0^a f(x) \cos \frac{n\pi}{a} x\, dx. \tag{9}$$

The solution of the boundary-value problem (1)–(3) consists of the series in (7), with coefficients A_0 and A_n defined in (8) and (9), respectively.

DIRICHLET PROBLEM A boundary-value problem in which we seek a solution of an elliptic partial differential equation such as Laplace's equation $\nabla^2 u = 0$, within a bounded region R (in the plane or 3-space) such that u takes on prescribed values on the entire boundary of the region is called a **Dirichlet problem.** In Problem 1 in Exercises 12.5 you are asked to show that the solution of the Dirichlet problem for a rectangular region

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b$$

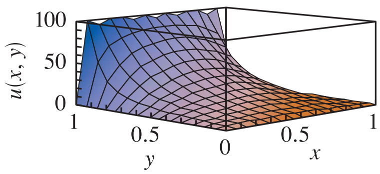
$$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 < y < b$$

$$u(x, 0) = 0, \quad u(x, b) = f(x), \quad 0 < x < a$$

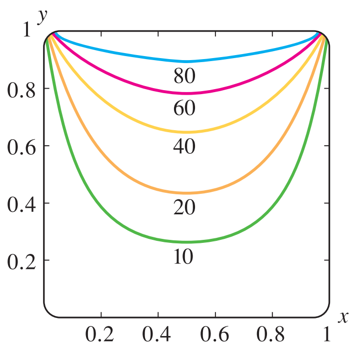
is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi}{a} y \sin \frac{n\pi}{a} x, \qquad \text{where} \qquad A_n = \frac{2}{a \sinh \frac{n\pi}{a} b} \int_0^a f(x) \sin \frac{n\pi}{a} x\, dx. \tag{10}$$

In the special case when $f(x) = 100$, $a = 1$, $b = 1$, the coefficients A_n in (10) are given by $A_n = 200 \dfrac{1 - (-1)^n}{n\pi \sinh n\pi}$. With the help of a CAS we plotted the surface defined by $u(x, y)$ over the region R: $0 \le x \le 1$, $0 \le y \le 1$, in Figure 12.5.2(a). You can see in the figure that the boundary conditions are satisfied; especially note that along $y = 1$, $u = 100$ for $0 \le x \le 1$. The **isotherms,** or curves in the rectangular region along which the temperature $u(x, y)$ is constant, can be obtained by using the contour plotting capabilities of a CAS and are illustrated in



(a) Surface



(b) Isotherms

FIGURE 12.5.2 Surface is graph of partial sums when $f(x) = 100$ and $a = b = 1$ in (10)

Figure 12.5.2(b). The isotherms can also be visualized as the curves of intersection (projected into the xy-plane) of horizontal planes $u = 80$, $u = 60$, and so on, with the surface in Figure 12.5.2(a). Notice that throughout the region the maximum temperature is $u = 100$ and occurs on the portion of the boundary corresponding to $y = 1$. This is no coincidence. There is a **maximum principle** that states a solution u of Laplace's equation within a bounded region R with boundary B (such as a rectangle, circle, sphere, and so on) takes on its maximum and minimum values on B. In addition, it can be proved that u can have no relative extrema (maxima or minima) in the interior of R. This last statement is clearly borne out by the surface shown in Figure 12.5.2(a).

SUPERPOSITION PRINCIPLE A Dirichlet problem for a rectangle can be readily solved by separation of variables when homogeneous boundary conditions are specified on two *parallel* boundaries. However, the method of separation of variables is not applicable to a Dirichlet problem when the boundary conditions on all four sides of the rectangle are nonhomogeneous. To get around this difficulty, we break the problem

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b \\
&u(0, y) = F(y), \quad u(a, y) = G(y), \quad 0 < y < b \qquad (11) \\
&u(x, 0) = f(x), \quad u(x, b) = g(x), \quad 0 < x < a
\end{aligned}$$

into two problems, each of which has homogeneous boundary conditions on parallel boundaries, as shown:

Problem 1

$$\begin{aligned}
&\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b \\
&u_1(0, y) = 0, \quad u_1(a, y) = 0, \quad 0 < y < b \\
&u_1(x, 0) = f(x), \quad u_1(x, b) = g(x), \quad 0 < x < a
\end{aligned}$$

Problem 2

$$\begin{aligned}
&\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b \\
&u_2(0, y) = F(y), \quad u_2(a, y) = G(y), \quad 0 < y < b \\
&u_2(x, 0) = 0, \quad u_2(x, b) = 0, \quad 0 < x < a
\end{aligned}$$

Suppose u_1 and u_2 are the solutions of Problems 1 and 2, respectively. If we define $u(x, y) = u_1(x, y) + u_2(x, y)$, it is seen that u satisfies all boundary conditions in the original problem (11). For example,

$$\begin{aligned}
u(0, y) &= u_1(0, y) + u_2(0, y) = 0 + F(y) = F(y), \\
u(x, b) &= u_1(x, b) + u_2(x, b) = g(x) + 0 = g(x),
\end{aligned}$$

and so on. Furthermore, u is a solution of Laplace's equation by Theorem 12.1.1. In other words, by solving Problems 1 and 2 and adding their solutions, we have solved the original problem. This additive property of solutions is known as the **superposition principle.** See Figure 12.5.3.

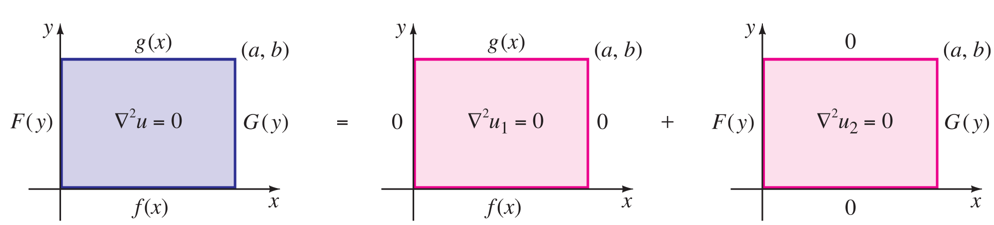


FIGURE 12.5.3 Solution u = Solution u_1 of Problem 1 + Solution u_2 of Problem 2

We leave as exercises (see Problems 13 and 14 in Exercises 12.5) to show that a solution of Problem 1 is

$$u_1(x, y) = \sum_{n=1}^{\infty} \left\{ A_n \cosh \frac{n\pi}{a} y + B_n \sinh \frac{n\pi}{a} y \right\} \sin \frac{n\pi}{a} x,$$

where
$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx$$

$$B_n = \frac{1}{\sinh \frac{n\pi}{a} b} \left(\frac{2}{a} \int_0^a g(x) \sin \frac{n\pi}{a} x \, dx - A_n \cosh \frac{n\pi}{a} b \right),$$

and that a solution of Problem 2 is

$$u_2(x, y) = \sum_{n=1}^{\infty} \left\{ A_n \cosh \frac{n\pi}{b} x + B_n \sinh \frac{n\pi}{b} x \right\} \sin \frac{n\pi}{b} y,$$

where
$$A_n = \frac{2}{b} \int_0^b F(y) \sin \frac{n\pi}{b} y \, dy$$

$$B_n = \frac{1}{\sinh \frac{n\pi}{b} a} \left(\frac{2}{b} \int_0^b G(y) \sin \frac{n\pi}{b} y \, dy - A_n \cosh \frac{n\pi}{b} a \right).$$

EXERCISES 12.5

Answers to selected odd-numbered problems begin on page ANS-21.

In Problems 1–10 solve Laplace's equation (1) for a rectangular plate subject to the given boundary conditions.

1. $u(0, y) = 0, \quad u(a, y) = 0$
$u(x, 0) = 0, \quad u(x, b) = f(x)$

2. $u(0, y) = 0, \quad u(a, y) = 0$
$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad u(x, b) = f(x)$

3. $u(0, y) = 0, \quad u(a, y) = 0$
$u(x, 0) = f(x), \quad u(x, b) = 0$

4. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$
$u(x, 0) = x, \quad u(x, b) = 0$

5. $u(0, y) = 0, \quad u(1, y) = 1 - y$
$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=1} = 0$

6. $u(0, y) = g(y), \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = 0$
$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=\pi} = 0$

7. $\left.\frac{\partial u}{\partial x}\right|_{x=0} = u(0, y), \quad u(\pi, y) = 1$
$u(x, 0) = 0, \quad u(x, \pi) = 0$

8. $u(0, y) = 0, \quad u(1, y) = 0$
$\left.\frac{\partial u}{\partial y}\right|_{y=0} = u(x, 0), \quad u(x, 1) = f(x)$

9. $u(0, y) = 0, \quad u(1, y) = 0$
$u(x, 0) = 100, \quad u(x, 1) = 200$

10. $u(0, y) = 10y, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = -1$
$u(x, 0) = 0, \quad u(x, 1) = 0$

In Problems 11 and 12 solve Laplace's equation (1) for the given semi-infinite plate extending in the positive y-direction. In each case assume that $u(x, y)$ is bounded as $y \to \infty$.

11.

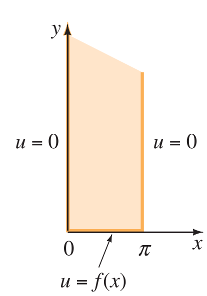


FIGURE 12.5.4 Plate in Problem 11

12.

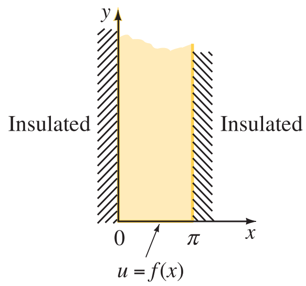


FIGURE 12.5.5 Plate in Problem 12

In Problems 13 and 14 solve Laplace's equation (1) for a rectangular plate subject to the given boundary conditions.

13. $u(0, y) = 0, \quad u(a, y) = 0$
$u(x, 0) = f(x), \quad u(x, b) = g(x)$

14. $u(0, y) = F(y), \quad u(a, y) = G(y)$
$u(x, 0) = 0, \quad u(x, b) = 0$

In Problems 15 and 16 use the superposition principle to solve Laplace's equation (1) for a square plate subject to the given boundary conditions.

15. $u(0, y) = 1, \quad u(\pi, y) = 1$
$u(x, 0) = 0, \quad u(x, \pi) = 1$

16. $u(0, y) = 0, \quad u(2, y) = y(2 - y)$

$$u(x, 0) = 0, \quad u(x, 2) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases}$$

Discussion Problems

17. **(a)** In Problem 1 suppose that $a = b = \pi$ and $f(x) = 100x(\pi - x)$. Without using the solution $u(x, y)$, sketch, by hand, what the surface would look like over the rectangular region defined by $0 \le x \le \pi, 0 \le y \le \pi$.

(b) What is the maximum value of the temperature u for $0 \le x \le \pi, 0 \le y \le \pi$?

(c) Use the information in part (a) to compute the coefficients for your answer in Problem 1. Then use the 3D-plot application of your CAS to graph the partial sum $S_5(x, y)$ consisting of the first five nonzero terms of the solution in part (a) for $0 \le x \le \pi$, $0 \le y \le \pi$. Use different perspectives and then compare with your sketch from part (a).

18. In Problem 16 what is the maximum value of the temperature u for $0 \le x \le 2, 0 \le y \le 2$?

Computer Lab Assignments

19. **(a)** Use the contour-plot application of your CAS to graph the isotherms $u = 170, 140, 110, 80, 60, 30$ for the solution of Problem 9. Use the partial sum $S_5(x, y)$ consisting of the first five nonzero terms of the solution.

(b) Use the 3D-plot application of your CAS to graph the partial sum $S_5(x, y)$.

20. Use the contour-plot application of your CAS to graph the isotherms $u = 2, 1, 0.5, 0.2, 0.1, 0.05, 0, -0.05$ for the solution of Problem 10. Use the partial sum $S_5(x, y)$ consisting of the first five nonzero terms of the solution.

12.6 NONHOMOGENEOUS BOUNDARY-VALUE PROBLEMS

REVIEW MATERIAL

- Sections 12.3–12.5

INTRODUCTION A boundary-value problem is said to be **nonhomogeneous** if either the partial differential equation or the boundary conditions are nonhomogeneous. The method of separation of variables that we employed in the preceding three sections may not be applicable to a nonhomogeneous boundary-value problem *directly*. However, in the first of the two techniques examined in this section we employ a change of variable that transforms a nonhomogeneous boundary-value problem into a two problems: one a relatively simple BVP for an ODE and the other a homogeneous BVP for a PDE. The latter problem is solvable by separation of variables. The second technique is basically a frontal attack on the BVP using orthogonal series expansions.

NONHOMOGENEOUS BVPs When heat is generated at a rate r within a rod of finite length, the heat equation takes on the form

$$k\frac{\partial^2 u}{\partial x^2} + r = \frac{\partial u}{\partial t}, \qquad 0 < x < L, \quad t > 0. \tag{1}$$